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THE DEFINABILITY OF THE COMMUTATOR SUBGROUP IN A VARIETY GENERATED BY A FINITE GROUP

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ABSTRACT. In a variety generated by a finite non-Abelian group, the commutator subgroup is not definable by a first-order formula.

In [1], R. Lyndon discusses the question of when the product of two commutators in a group is a commutator. This apparently always occurs if the group is finite and simple and it is an open question whether this always occurs if the group is simple. Now it is clear that if the product of two commutators is necessarily a commutator in a group, then the commutator subgroup is first order definable in the group (it is just the set of commutators). In [2], the authors showed that free solvable groups of the same solvable length and different finite rank are not elementarily equivalent by showing that in the free solvable group the commutator subgroup is definable by a positive universal formula.

In this paper we ask the question: "If V is a variety of groups, when does there exist a formula $\phi(z)$ in one free variable z such that for each group $H \in V$, $\phi(a)$ holds in H if and only if $a \in H'$ (the commutator subgroup)?" If such a formula exists, we say that the commutator subgroup is defined in V.

An element of a group is in the commutator subgroup if and only if it is a product of commutators. Let $\phi_n(z)$ denote the formula

$$\exists x_1, x_2, \ldots, x_n, y_1, \ldots, y_n \left\{ z = \prod_{i=1}^{n} [x_i, y_i] \right\}.$$

Thus the formula $\phi(z)$ stating "z is in the commutator subgroup" is equivalent to the infinite disjunction

$$\bigvee_{n=1}^{\infty} \phi_n(z).$$

The following theorem is a straightforward application of the Compactness Theorem.

THEOREM 1. *Let $\mathscr{C}$ be a class of groups closed under ultraproducts. Then the commutator subgroup is definable in $\mathscr{C}$ if and only if there is a positive integer K such that for all groups in $\mathscr{C}$, a product of commutators is equal to a product of K commutators.*

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THEOREM 2. *Let V be a variety of groups which contains a finite non-abelian group. Then the commutator subgroup is not definable in V. If V contains only abelian groups, then, trivially, the commutator subgroup is definable in V. It is an open question whether a variety of groups which contains non-abelian groups must contain finite non-abelian groups.*

PROOF. First, if the variety consists of abelian groups, then the commutator subgroup is defined by the formula $\phi(z)$: $z = 1$.

Next suppose that the variety contains a finite non-abelian group G. We use induction on the order N of G; hence, we may assume, without loss of generality, that all the proper subgroups of G are abelian. It follows that G is metabelian ([3], p. 148).

Let H be a proper normal subgroup of G such that G/H is cyclic and suppose that the image of $a \in G$ generates G/H. Since G is not abelian, there is an element $b \in H$ which does not commute with a.

Suppose that the commutator subgroup is definable in V. By Theorem 1, there is a positive integer K such that for all groups in V, a product of commutators is equal to a product of K commutators. Let F be the relatively free group of the variety V with a basis $B = \{x_1, y_1, \ldots, x_m, y_m\}$ of $2m$ elements, for $m = N^{2K+1}$. Then, there must exist words $p_i = p_i(x_1, x_2, \ldots, x_m, y_1, \ldots, y_m)$ and $q_i = q_i(x_1, \ldots, x_m, y_1, \ldots, y_m)$, $i = 1, 2, \ldots, K$, such that

$$\prod_{i=1}^{m} [x_i, y_i] = \prod_{j=1}^{K} [p_j, q_j].$$

For $z \in B$ a generator and $t \in F$ a word, let $\deg_z t$ = sum of the exponents of z in the word t, and let $\overline{\deg_z t}$ be the image of $\deg_z t$ in the cyclic group $C = \mathbb{Z}_N$. Define the function

$$\phi : \{1, 2, \ldots, m\} \to C^{2K}$$

by

$$\phi(i) = (\overline{\deg_{x_i} p_1}, \overline{\deg_{x_i} q_1}, \overline{\deg_{x_i} p_2}, \ldots, \overline{\deg_{x_i} q_K}).$$

The number of distinct functions is N^{2K} and as $m = N^{2K+1}$ there must exist N distinct elements of $\{1, 2, \ldots, m\}$ which map to the same element under ϕ. By reordering if necessary, we may assume, without loss of generality, that $\phi(1) = \phi(2) = \ldots = \phi(N)$. Note that

$$\sum_{i=1}^{N} \deg_{x_i} p_j \equiv 0 \pmod{N}, \qquad j = 1, 2, \ldots, K,$$

and

$$\sum_{i=1}^{N} \deg_{x_i} q_j \equiv 0 \pmod{N}, \qquad j = 1, 2, \ldots, K.$$

We now define a map ψ from the free group F to G by sending

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$$x_i \to \begin{cases} a & \text{if } i \leq N \\ 1 & \text{if } i > N \end{cases}$$

and

$$y_i \to \begin{cases} b & \text{if } i = 1 \\ 1 & \text{if } i > 1 \end{cases}.$$

Under ψ, the words p_j and q_j are mapped into H; hence the right hand side of (*) is mapped to the identity element. The left hand side of (*) is mapped to $[a, b] \neq 1$, a contradiction. This completes the proof of the theorem.

References

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