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SELF–SPLITTING ABELIAN GROUPS

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G is a reduced torsion-free Abelian group such that for every direct sum $\oplus G$ of copies of G, $\operatorname{Ext}(\oplus G, \oplus G) = 0$ if and only if G is a free module over a rank 1 ring. For every direct product $\prod G$ of copies of G, $\operatorname{Ext}(\prod G, \prod G) = 0$ if and only if G is cotorsion.

This paper began as a Research Report of the Department of Mathematics of the University of Western Australia in 1988, and circulated among members of the Abelian group community. However, it was never submitted for publication. The results have been cited widely, and since copies of the original research report are no longer available, the paper is presented here in its original form in Sections 1 to 5. In Section 6, I survey the progress that has been made in the topic since 1988.

1. INTRODUCTION

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory in the category of Abelian groups, then $\mathcal{T}$ is closed under direct sums and extensions. Hence if $G \in \mathcal{T}$ is projective with respect to short exact sequences from $\mathcal{T}$ then $\operatorname{Ext}(G, \oplus G) = 0$ for any direct sum $\oplus G$ of copies of G. Dually, if $G \in \mathcal{F}$ is injective with respect to short exact sequences from $\mathcal{F}$, then $\operatorname{Ext}(\prod G, G) = 0$ for any direct product $\prod G$ of copies of G.

Call a group G a *self-splitting group* (*splitter* for short) if $\operatorname{Ext}(G, G) = 0$, a *$\oplus$–splitter* if every direct sum $\oplus G$ is a splitter, and a *$\prod$–splitter* if every direct product $\prod G$ is a splitter. Thus if G is projective [injective] as described above, then G is a $\oplus$–splitter [$\prod$–splitter]. In this paper I characterise $\oplus$–splitting and $\prod$–splitting Abelian groups, modulo some set theoretic axioms.

Splitters appear in various contexts besides those mentioned above. For example, they play a crucial rôle in the representation theory of finite dimensional algebras [**14**]. If $(\mathcal{F}, \mathcal{C})$ is a cotorsion theory [**19**], then $G \in \mathcal{F} \cap \mathcal{C}$ and only if G is a splitter. More generally, there is an extensive literature on the characterisation of pairs (X, Y) of groups for which $\operatorname{Ext}(X, Y) = 0$; for example see [**2, 3, 12, 6, 8, 20, 21, 22**].

The problem of finding all G for which $\bigoplus_\nu G$ or $\prod_\nu G$ are splitters for particular cardinals ν, and in particular of finding all splitters, seems to be difficult, and only

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fragmentary results are obtained in this paper. However the classification of such splitters for $\nu = \aleph_0$ has been published several times in various contexts. The original result, that $\aleph_0$–splitters are free modules over a rank 1 ring, and hence $\oplus$–splitters, is due to Hausen [**15**]. Other versions of this theorem occur in [**2, 12, 22, 21**].

I use the standard notation of Fuchs, [**4**] and [**5**] and in particular, 'group' means Abelian group.

2. Basic Results

The following results are immediate consequences of well known theorems found, for example, in [**4**, Section 51].

Lemma 2.1. *Let* $\operatorname{Ext}(A, B) = 0$. *If* C *is a subgroup of* A *and* D *is a factor group of* B, *then* $\operatorname{Ext}(C, D) = 0$.

Lemma 2.2. *If* G *is a splitter, then so is every endomorphic image of* G. *In particular, if* $\bigoplus_\nu G$ $\left[\prod_\nu G\right]$ *is a splitter, then so is* $\bigoplus_\mu G$ $\left[\prod_\mu G\right]$ *for every cardinal* $\mu \leqslant \nu$.

Lemma 2.3. $\bigoplus_\nu G$ *is a splitter if and only if* $\operatorname{Ext}\left(G, \bigoplus_\nu G\right) = 0$. $\prod_\nu G$ *is a splitter if and only if* $\operatorname{Ext}\left(\prod_\nu G, G\right) = 0$.

The next proposition shows that we may limit consideration to reduced torsion-free splitters:

Proposition 2.4. *Let* $G = D \oplus R$, *where* D *is divisible and* R *is reduced.* G *is a splitter if and only if:*

1. R *is a torsion–free splitter;*
2. *If* D *is not torsion, then* R *is cotorsion;*
3. *If* $D_p \neq 0$, *then* $pR = R$.

Proof: $(\Rightarrow)$ 1. By Lemma 2.1, R is a splitter. If R is not torsion–free, then for some prime p, R has $\mathbb{Z}(p)$ as both subgroup and factor group, contradicting Lemma 2.1, since $\operatorname{Ext}\big(\mathbb{Z}(p), \mathbb{Z}(p)\big) \cong \mathbb{Z}(p)$.

2. If D is not torsion, then G has $\mathbb{Q}$ as a subgroup so by 2.1, $\operatorname{Ext}(\mathbb{Q}, R) = 0$. Hence R is cotorsion.

3. If $D_p \neq 0$, then G has $\mathbb{Z}(p^\infty)$ as a subgroup, so by 2.1, $\operatorname{Ext}\big(\mathbb{Z}(p^\infty), R\big) = 0$. By [**4**, Theorem 52.3], $pR = R$.

$(\Leftarrow)$ It suffices to show that $\operatorname{Ext}(D, R) = 0$. If D is not torsion, then $\operatorname{Ext}(\mathbb{Q}, R) = 0$ by 2; if $D_p \neq 0$, then $\operatorname{Ext}\big(\mathbb{Z}(p^\infty), R\big) = 0$ by 3. Since D is a direct sum of copies of $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$ for various primes p, $\operatorname{Ext}(D, R) = 0$. $\square$

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3. $\oplus$-SPLITTERS

This section begins with five technical lemmas which are the principal tools needed to determine the structure of $\oplus$-splitters.

LEMMA 3.1. *Let T be a torsion group and G a torsion-free group. If $k \in \mathbb{N}$ satisfies $k\operatorname{Ext}(T,G) = 0$, then for all primes p, either $kT_p = 0$ or $pG = G$.*

PROOF: Suppose $pG \neq G$. Since $\operatorname{Ext}(\mathbb{Z}(p^\infty), G)$ is torsion-free [4, Theorem 52.3], T_p is reduced. Let $U = \bigoplus_{i\in I} \mathbb{Z}(p^{n_i})$ be a basic subgroup of T_p, so $\operatorname{Ext}(U,G) \cong \prod_{i\in I} G/p^{n_i}G$.

Since $k\operatorname{Ext}(U,G) = 0$, $p^{n_i}|k$ for all $i \in I$, so U is bounded and hence $U = T_p$ and $kT_p = 0$. ∎

LEMMA 3.2. *Let A be a torsion-free group of finite rank and G a countable torsion-free group such that $\operatorname{Ext}(A,G) = 0$.*

If E is an essential subgroup of A, then A has a subgroup F containing E such that F/E is finite and $\operatorname{Ext}(A/F,G) = 0$.

PROOF: Since $\operatorname{Ext}(A,G) = 0$, there is an exact sequence

$$\operatorname{Hom}(A,G) \to \operatorname{Hom}(E,G) \twoheadrightarrow \operatorname{Ext}(A/E,G);$$

since E has finite rank and G is countable, $\operatorname{Hom}(E,G)$ and consequently $\operatorname{Ext}(A/E,G)$ are countable. Furthermore, $\operatorname{Ext}(A/E,G) \cong \prod_p \operatorname{Ext}\big((A/E)_p, G\big)$.

Let $S = \Big\{p : \operatorname{Ext}\big((A/E)_p, G\big) \neq 0\Big\}$. Then by [4, Theorem 52.3], S is finite and if $p \in S$, then $(A/E)_p$ is finite. Let $F \leqslant A$ be such that $F/E = \bigoplus_{p\in S} (A/E)_p$.

Then $\operatorname{Ext}(A/E,G) \cong \operatorname{Ext}(F/E,G)$ and $A/E \cong A/F \oplus F/E$, so $\operatorname{Ext}(A/F,G) = 0$. ∎

LEMMA 3.3. *Let A be a torsion-free group of finite rank and $\{G_i : i \in I\}$ a family of torsion-free groups.*

Then $\operatorname{Hom}\Big(A, \bigoplus_{i\in I} G_i\Big) \cong \bigoplus_{i\in I} \operatorname{Hom}(A, G_i)$.

PROOF: Let $f \mapsto (f_i)$ be the natural homomorphism from $\operatorname{Hom}\Big(A, \bigoplus_{i\in I} G_i\Big)$ to $\prod_{i\in I} \operatorname{Hom}(A,G_i)$, so for all $a \in A$, af_i is the i-component of af. We need to show that $f_i = 0$ for almost all $i \in I$.

Let E be a maximal independent set in A and let $f \in \operatorname{Hom}(A, \oplus G_i)$. For any $x \in E$, xf has i-component 0 for almost all i so $xf_i = 0$ for almost all i. Let $I_x = \{i : xf_i \neq 0\}$ and let $J = \bigcup_{x\in E} I_x$. Then J is finite and for all $i \notin J$, $Ef_i = 0$, so $Af_i = 0$. ∎

LEMMA 3.4. *Let A be a torsion-free group of finite rank and $\{G_i : i \in I\}$ a family of torsion-free groups satisfying $\operatorname{Ext}\Big(A, \bigoplus_{i\in I} G_i\Big) = 0$.*

If E is an essential subgroup of A, then there exists a subgroup F of A containing E such that F/E is finite and $\operatorname{Ext}(A/F, G_i) = 0$ for almost all $i \in I$.

PROOF: Write $B = \bigoplus_{i \in I} G_i$; since $\operatorname{Ext}(A, B) = 0$, there are exact sequences

$$\operatorname{Hom}(A, B) \to \operatorname{Hom}(E, B) \twoheadrightarrow \operatorname{Ext}(A/E, B) \text{ and}$$
$$\operatorname{Hom}(A, G_i) \to \operatorname{Hom}(E, G_i) \twoheadrightarrow \operatorname{Ext}(A/E, G_i) \text{ for each } i \in I.$$

By Lemma 3.3, $\operatorname{Hom}(A, B) \cong \bigoplus_{i \in I} \operatorname{Hom}(A, G_i)$ and $\operatorname{Hom}(E, B) \cong \bigoplus_{i \in I} \operatorname{Hom}(E, G_i)$, so $\operatorname{Ext}(A/E, B) \cong \bigoplus_{i \in I} \operatorname{Ext}(A/E, G_i)$.

Since $\operatorname{Ext}(A/E, B)$ is a reduced cotorsion group [**4**, Theorem 52.3], it follows from [**4**, Corollary 39.10] that there exists $k \in \mathbb{N}$ and a finite subset J of I such that $k \operatorname{Ext}(A/E, G_i) = 0$ for all $i \in I \setminus J$.

Let $S = \{p : k(A/E)_p = 0\}$ and let $F \leqslant A$ be such that $F/E = \bigoplus_{p \in S} (A/E)_p$. Thus $k(F/E) = 0$ and F/E is a finite group.

Suppose $i \in I \setminus J$. By Lemma 3.1, if $p \notin S$, then $pG_i = G_i$ so $\operatorname{Ext}\big((A/E)_p, G_i\big) = 0$. Hence $\operatorname{Ext}(A/F, G_i) = 0$. ∎

DEFINITION 3.5: Let G be a torsion-free group. The *nucleus* of G is the subring of $\mathbb{Q}$ generated by $\{p^{-1} : pG = G\}$.

Clearly the nucleus of G is the largest rank 1 ring Λ over which G is a Λ-module.

LEMMA 3.6. *Let G be a torsion-free group with nucleus Λ, and let A be a Λ-module of finite rank.*

If A has a full free submodule F such that $\operatorname{Ext}(A/F, G) = 0$, *then A is a free Λ-module.*

PROOF: If $(A/F)_p \neq 0$, then since $\operatorname{Ext}\big((A/F)_p, G\big) = 0$, $pG = G$, so $p\Lambda = \Lambda$. Hence $\operatorname{Ext}(A/F, \Lambda) = 0$ and from the exact sequence $\operatorname{Ext}(A/F, \Lambda) \to \operatorname{Ext}(A, \Lambda) \to \operatorname{Ext}(F, \Lambda) = 0$, $\operatorname{Ext}(A, \Lambda) = 0$. By [**4**, Section 99] applied to Λ-modules, A is free. ∎

We can now use these Lemmas to prove several structure theorems for groups satisfying various splitting conditions.

THEOREM 3.7. *Let G be a reduced torsion-free group with nucleus Λ and let A be a Λ-module.*

A is an ω_1-free Λ-module under either of the following conditions:

1. *G has a countable homomorphic image with nucleus Λ and* $\operatorname{Ext}(A, G) = 0$;
2. $\operatorname{Ext}\Big(A, \bigoplus_{\omega} G\Big) = 0$.

PROOF: In Case 1, let B be a countable homomorphic image of G with nucleus Λ so $\operatorname{Ext}(A, B) = 0$. By Lemma 3.2 in Case 1, and by Lemma 3.4 in Case 2, for each finite rank subgroup C of A and each essential subgroup E of C, C has a subgroup F such that F/E is finite and $\operatorname{Ext}(C/F, B) = 0$ in Case 1, and $\operatorname{Ext}(C/F, G) = 0$ in Case 2.

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By Lemma 3.6, each finite rank Λ-submodule C of A is free, so by Pontryagin's Criterion [4, Theorem 19.1], every countable submodule of A is free. $\square$

COROLLARY 3.8. (Hausen [15])

1. *If G is a countable splitter, then G is a free module over its nucleus, and hence G is a $\oplus$-splitter.*
2. *If $\bigoplus_{\omega} G$ is a splitter, then G is ω_1-free over its nucleus.*

Note that if G is free over its nucleus, then G satisfies both conditions of Theorem 3.7, with $A = G$. On the other hand, there are splitters, for example the p-adic integers, which satisfy neither of the conditions.

In order to replace "ω_1-free" by "free" in Theorem 3.7, we shall assume Gödel's Axiom of Constructibility $V = L$.

LEMMA 3.9. *Let G be a reduced torsion-free group with nucleus Λ. Then G has a factor group C with nucleus Λ such that $|C| \leqslant 2^{\omega}$.*

PROOF: Let $S = \{p : pG \neq G\}$. For each $p \in S$, there is a p-height preserving homomorphism $\phi_p : G \to I_p$, the p-adic integers. Let ϕ be the induced map $\phi : G \to \prod_{p \in S} I_p$, and take C to be the image of ϕ. $\square$

THEOREM 3.10. (V = L) *Let G be a reduced torsion-free group with nucleus Λ and A a torsion-free Λ-module such that either:*

1. $\operatorname{Ext}(A, G) = 0$ *and G has a countable homomorphic image with nucleus Λ; or*
2. $\operatorname{Ext}\left(A, \bigoplus_{\omega} G\right) = 0$ *and* $|G| \leqslant |A|$.

Then A is a free Λ-module.

PROOF: Let $|A| = \omega_m$; the proof is by induction on m. If $m = 0$ the theorem is true by Theorem 3.7. Let $m = 1$; by [1, Corollary 10.2], A is the union of a smooth chain $A = \bigcup_{\alpha<\omega_1} A_\alpha$, where each A_α is a submodule of A, $|A_\alpha| = \omega$ and, in Case 1, $\operatorname{Ext}(A_{\alpha+1}/A_\alpha, G) = 0$ while in Case 2, $\operatorname{Ext}\left(A_{\alpha+1}/A_\alpha, \bigoplus_{\omega} G\right) = 0$ for all $\alpha < \omega_1$.

By Theorem 3.7, A_0 and each $A_{\alpha+1}/A_\alpha$ are free Λ-modules, so A is a free Λ-module.

Suppose that the result is true for all ordinals $m < n$, where $n \geqslant 2$, and let $|A| = \omega_n$. In Case 1, G has a countable homomorphic image with nucleus Λ and in Case 2, by Lemma 3.9, G has a homomorphic image C with nucleus Λ such that $\left|\bigoplus_{\omega} C\right| \leqslant 2^{\omega} = \omega_1$. Hence without loss of generality we can assume $|G| \leqslant \omega_n$.

If E is a subgroup of A with $|E| < \omega_n$, then E is free by induction, so A is ω_n-free. If ω_n is singular, then A is free by [1, Theorem 20.9], so suppose ω_n is regular.

Then by [1, Corollary 10.2], A is the union of a smooth chain $A = \bigcup_{\alpha<\omega_n} A_\alpha$, where each A_α is a submodule of A, $|A_\alpha| < \omega_n$, and in Case 1, $\operatorname{Ext}(A_{\alpha+1}/A_\alpha, G) = 0$, while in

Case 2, $\operatorname{Ext}\left(A_{\alpha+1}/A_\alpha, \bigoplus_\omega G\right) = 0$ for all $\alpha < \omega_n$. By induction, A_0 and each $A_{\alpha+1}/A_\alpha$ are free, so A too is free. $\square$

COROLLARY 3.11. (V = L) *Let G be a reduced torsion-free group. If G has a countable homomorphic image with the same nucleus as G and G is a splitter, or if $\bigoplus_\omega G$ is a splitter, then G is a free module over its nucleus. Consequently G is an $\oplus$-splitter.*

Note that a weaker hypothesis than V = L probably suffices for Theorem 3.10, (see [**3**, Theorem 2.15]). This possibility has not been pursued, because it is by no means certain that any set theoretic hypothesis beyond ZFC is necessary.

4. $\prod$-SPLITTERS

Here the results are scantier but more clear cut: we show that if $\prod_\omega G$ is a splitter, then every torsion-free homomorphic image of G of cardinality $\leqslant 2^\omega$ is cotorsion; and if G is a $\prod$-splitter, then G is cotorsion. The results are based on the following theorem of Hunter:

LEMMA 4.1. [**16**, Proof of Theorem 4.2] *Let $|A| \leqslant 2^m$ for some cardinal $m < m^\omega$, and let $\operatorname{Ext}\left(\prod_m \mathbb{Z}, A\right) = 0$. Then A is cotorsion.*

Note that ω is a cardinal such that $\omega < \omega^\omega$, and Griffith [**13**, Lemma 3.1] has shown that for every cardinal n there is a cardinal $m \geqslant n$ satisfying $m < m^\omega$.

THEOREM 4.2. *Let A and B be reduced torsion-free groups with $|A| \doteq n$. If $n \leqslant 2^m$ where $m < m^\omega$ and $\operatorname{Ext}\left(\prod_m B, A\right) = 0$, then A is cotorsion.*

PROOF: Since $\prod_m \mathbb{Z}$ is a subgroup of $\prod_m B$, $\operatorname{Ext}\left(\prod_m \mathbb{Z}, A\right) = 0$ so by Lemma 4.1, A is cotorsion. $\square$

COROLLARY 4.3.

1. *If $\prod_\omega G$ is a splitter, then every reduced torsion-free homomorphic image of G of cardinality $\leqslant 2^\omega$ is cotorsion.*
2. *G is a $\prod$-splitter if and only if G is cotorsion.*

5. PROBLEMS

Some problems immediately suggest themselves:

1. Can the set theoretic axiom in Theorem 3.10 be weakened or removed?
2. If G is a reduced torsion-free splitter with nucleus Λ such that G has no countable homomorphic image with nucleus Λ, is G cotorsion?

3. If G is a torsion-free group such that every torsion-free reduced homomorphic image of cardinality $\leqslant 2^{\omega}$ is cotorsion, is G necessarily cotorsion?
4. Assuming $\mathrm{V} = \mathrm{L}$, is it true that every torsion-free reduced splitter is either free over its nucleus, or cotorsion?

6. Recent Progress

Since this paper first appeared as a UWA Research Report, the problems enumerated above have been solved. The first to fall was Problem 3. In [**7**], Göbel constructed some counterexamples to the conjecture that if every torsion-free reduced homomorphic image of G of cardinality $\leqslant 2^{\omega}$ is cotorsion, then G itself must be cotorsion. In fact he showed that there is no cardinal which is large enough to test cotorsion in this sense. That is, he showed that if $\kappa < \lambda$ are cardinals with $\kappa^{\omega} = \kappa$ and $\lambda^{\omega} = \lambda$ then there is a cotorsion-free group G of cardinality λ such that all torsion-free epimorphic images of cardinality $\leqslant \kappa$ are cotorsion. Here, 'G is cotorsion-free' means G has no cotorsion subgroups. The proof, which holds in ZFC set theory, is based on an infinite combinatorial argument known as Shelah's Black Box.

The next paper to attack the problems was Göbel and Shelah [**10**], although the objectives of this paper are much deeper. If a group G with nucleus R has countable R-submodules that are not free, then G has non-free submodules of some minimal rank $n + 1$. Göbel and Shelah call these 'n-free-by-1' R-modules. They show that certain systems of linear equations are always solvable in n-free-by-1 R-modules G if and only if G is a splitter.

Using Shelah's Black Box, they prove their Main Theorem, which is that for any ring A whose additive group is a free R-module, there is a splitter G whose endomorphism ring is isomorphic to A. Since the mutiplicative structure of A can be arbitrarily prescribed, it follows that splitters with all kinds of nasty algebraic properties can be constructed, for example having non-unique decompositions into indecomposables, having no indecomposable subgroups, failing Kaplansky's test problems and so on. This indicates that no classification of splitters is possible.

In particular, Göbel and Shelah settle Problem 2 in the negative by constructing cotorsion-free torsion-free splitters G with nucleus R such that G has no countable homomorphic image with nucleus R. They settle also Problem 4 by constructing torsion-free splitters which are neither cotorsion nor free. As a byproduct of these results, they are able to answer a problem of Salce [**19**], by showing that all rational cotorsion theories have enough injectives and enough projectives.

They also settle a stronger version of part of Problem 1, to find minimal set-theoretic hypotheses necessary to prove that if A and G are torsion-free groups with the same nucleus R and $\operatorname{Ext}\left(A, \bigoplus_{\omega} G\right) = 0$ with $|G| \leqslant |A|$ then A is a free R-module. They show that under the set-theoretic hypothesis $\diamondsuit_\kappa$, if $\mathrm{Ext}\left(A, \bigoplus_\omega G\right) = 0$ and $|A| \leqslant \kappa$, then A is a free R-module. It follows that under $ZFC + \diamondsuit_\kappa$, all ω-splitters of cardinality $\leqslant \kappa$ are free over their nucleus. For related results, see also [**9**].

One thing that has become evident over the past few years is that splitters are sufficiently important to have been introduced several times in various branches of algebra under different names. Several of these are listed in the 'Dictionary' in [**18**, p.351].

Apart from the questions raised in the original version of this paper, much progress has been made in constructing splitters and applying them to problems in algebra. For example, Hausen's result [**15**] stating that countable splitters are free R-modules, which was improved in Theorem 3.7 above, was further improved in [**10**] to state that if G is a splitter of cardinality $< 2^\omega$ the G is an ω_1-free R-module.

In [**11**], Göbel and Trlifaj describe the structure of tilting torsion classes of modules as the class of modules X such that $\mathrm{Ext}_R(M, X) = 0$ for a splitter M in Mod-R, and they similarly describe the dual class of cotilting torsion-free modules.

Generalising the notion of λ-free module (all submodules of cardinality $< \lambda$ are free), Pabst (now Wallutis), [**17**], calls an R-module λ-projective if every $< \lambda$-generated submodule is contained in a projective submodule. In general, non-projective λ-projective $\leqslant \lambda$-generated splitters do not exist, as shown in [**10**]. However, if R is a hereditary ring and λ a cardinal such that non-projective λ-projective $\leqslant \lambda$-generated R-modules do exist, Pabst constructs a non-projective λ-projective splitter of cardinality $|R|^\lambda$. In particular, for the case of Abelian groups, this implies that there is a non-free ω_1-free splitter of cardinality 2^{ω_1}, and, under $V = L$, if λ is a regular but not weakly compact cardinal then there exists a λ-free splitter of cardinality $|R|^\lambda$.

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