V

QCD, hadronic structure and high temperature

13 Hadronic structure and quantum chromodynamics

13.1 Confined quarks in a cavity

A hadronic particle, according to section 3.1, is a quark-filled bubble, a 'swiss-cheese' hole, in the structured vacuum. The highly excited drop of QGP is indeed much akin to the picture of an individual, colorless hadron, except that it is the thermal pressure that acts against the vacuum pressure, not the quantum pressure. As a first step in a more detailed discussion of the QGP phase, we briefly discuss how this approach allows us to understand properties of individual hadrons.

In the quark-bag model of hadronic structure, colorless qqq baryons or $\bar{\mathrm{q}}\mathrm{q}$ mesons are embedded in the structured vacuum sea. In a calculational framework proposed by Bogoliubov [78], independent quarks confined by a static Lorentz-scalar potential with infinite walls were considered. This is ensuring permanent 'confinement' of the constituents within a given volume. The interest in this approach grew only after it was understood that the confining potential is not to be derived from quark–quark interactions, but that it arises from the repulsion of colored quarks by the structured QCD vacuum state.

The structure of hadrons emerges on considering a static spherical state in which residual quark–quark interactions are introduced. This MIT-bag model is able to capture most features of the hadron spectrum [92, 93, 99, 151]. Our limited objective is to extract from a study of the hadronic spectrum information about the latent heat of the vacuum $\mathcal{B}$, and the mass of the strange quark m_{s}. To accomplish this we will not need to introduce in this book improvements addressing the restoration of translational invariance, and the absence of chiral symmetry; see section 3.3. For further details on chiral symmetry and the bag model, we refer the interested reader to [94, 256, 258].

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Assuming that quarks are moving independently inside a region of space, the mass M_h of the bound-state system of quarks (bag) comprises the kinetic energy of confined quarks, as well as the volume energy of the disturbance in the vacuum,

$$M_\mathrm{h}(R_\mathrm{h}) = \frac{\sum_i \varepsilon_i}{R_\mathrm{h}} + \frac{4\pi}{3} R_\mathrm{h}^3 \mathcal{B} + \delta E_\mathrm{V} + \Delta M_\mathrm{mag}, \tag{13.1}$$

where ε_i is the (dimensionless) eigenvalue energy coefficient of the ith quark in a static cavity of radius R_h. Since we are considering the lowest possible hadronic states, this term becomes proportional to the number $n = 2$ or 3 of valence quarks and antiquarks in mesons or baryons, respectively:

$$\frac{\sum_i \varepsilon_i}{R_\mathrm{h}} \to n \frac{\varepsilon_0}{R_\mathrm{h}}. \tag{13.2}$$

We have further introduced, in Eq. (13.1), the finite-volume correction to the vacuum energy

$$\delta E_\mathrm{V} = \frac{z_0}{R_\mathrm{h}} \tag{13.3}$$

in $M_\mathrm{h}(R_\mathrm{h})$. Since this is a term independent of the number of quarks in the bag, not requiring a dimensioned constant, a judicious choice of the number z_0 allows for many other effects.

A residual interaction must be introduced in order to describe the energy splitting between the baryon octet with $j = \frac{1}{2}^+$ and the decuplet with $j = \frac{3}{2}^+$, see Fig. 2.1, and similarly between the nonets of pseudoscalar $j = 0^-$ and the vector mesons $j = 1^-$; see Fig. 2.2 on page 27. A Coulomb-like interaction could not accomplish this, since it can not distinguish among the different angular-state multiplets. Akin to the magnetic field which splits the spin states, an interaction of magnetic (hyperfine) type is needed,

$$\Delta M_\mathrm{mag} = \sum_{i>j} \left\langle \frac{\alpha_\mathrm{s}}{r_{ij}} t_i^a t_j^a \vec{\sigma}_i \cdot \vec{\sigma}_j \right\rangle = \frac{1}{R_\mathrm{h}} \sum_{i>j} c_{ij} h_{ij}. \tag{13.4}$$

This is the usual form with the spin-Pauli matrices $\vec{\sigma}_i$, Eq. (13.27), and specific to the color interaction, $SU(3)$ generators t^a, Eq. (13.58). The important feature of this color-magnetic hyperfine interaction is that it does reflect correctly the signs, and even the magnitude, of the splittings between various hadronic multiplets.

The coefficients c_{ij}, in Eq. (13.4), are found by evaluating

$$c_{ij} = \langle h \,|(t^a \vec{\sigma})_i \cdot (t^a \vec{\sigma})_j|\, h \rangle \,. \tag{13.5}$$

This is done by methods developed in the study of hyperfine QED interactions, and we defer this discussion. h_{ij}, in Eq. (13.4), are the transition-matrix elements of magnetic moments, which incorporate the coupling α_s, but where we have taken out the dominant dimensional factor $1/R_\mathrm{h}$.

The mass of the hadronic state is dependent on the size parameter R_h. In the absence of any other dimensioned constants (such as quark masses), but with a constant a_h containing quark and interaction contributions specific to each hadron state, we have

$$M_\mathrm{h}(R_\mathrm{h}) = \frac{4\pi}{3} R_\mathrm{h}^3 \mathcal{B} + \frac{a_\mathrm{h}}{R_\mathrm{h}}. \tag{13.6}$$

There is a clearly defined minimum in Eq. (13.1) as a function of R_h, at which the forces associated with the vacuum and quarks balance. The physical state has a mass associated with this minimum:

$$\frac{\partial M_\mathrm{h}}{\partial R_\mathrm{h}} = 0 \ . \tag{13.7}$$

This condition,

$$\frac{a_\mathrm{h}}{4\pi R_\mathrm{h}^4} - \mathcal{B} = 0, \tag{13.8}$$

is equivalent to the pressure equilibrium point between the internal Fermi pressure and the exterior vacuum pressure (negative pressure seen from the interior).

Reinserting the result of Eq. (13.7) into Eq. (13.6), we find that the volume and mass of a quark-bound state are related:

$$\boxed{M_\mathrm{h} = 4\mathcal{B}V_\mathrm{h}, \quad R_\mathrm{h}M_\mathrm{h} = \tfrac{4}{3}a_\mathrm{h}, \quad M_\mathrm{h} = \tfrac{4}{3}a_\mathrm{h}^{3/4}(4\pi\mathcal{B})^{1/4}.} \tag{13.9}$$

Aside from the bag constant, $\mathcal{B}$, the hadron-state-specific value of a_h determines the value of each hadron mass. To determine a_h in the study of hadronic spectra based on quark-cavity states, section 13.2, the five parameters ε_q and ε_s (the energies of light and strange quarks), and h_qq, h_qs, and h_ss (the (transition) magnetic moments seen in Eq. (13.1)) are set to the values expected from the structure of the unperturbed bag model, and the values of the elementary parameters α_s and m_s are fitted, along with z_0; thus one looks at four parameters. However, much more precise information on the magnitude of the bag constant, $\mathcal{B}$, is obtained in an approach in which one takes Eq. (13.1) as the starting point. In Eq. (13.1), aside from $\mathcal{B}$, six more parameters enter: z_0, ε_q, and ε_s; and h_qq, h_qs, and h_ss. One of the results of such an approach is the verification of how well hadronic structure is described by quark-cavity states.

Owing to the coincidence that the strange-quark mass and hadron radius have the same scale $m_\mathrm{s}R_\mathrm{h} \simeq 1$, there is considerable sensitivity to the exact value of the strange quark m_s, which determines the energy of the strange quark:

$$\varepsilon_\mathrm{s}^\mathrm{fit} = \sqrt{m_\mathrm{s}^2 + \frac{x_\mathrm{s}^2}{R_\mathrm{h}^2}}. \tag{13.10}$$

Considering the running of the mass of the strange quark predicted by QCD, see Fig. 17.4 on page 328, the effective mass of the strange quark should vary with the hadronic size, e.g., according to

$$m_\mathrm{s}(R_\mathrm{h}) \equiv m_\mathrm{s}^0 \ln(\pi R_\mathrm{h} \Lambda_\mathrm{h}). \tag{13.11}$$

The variation of the mass of the strange quark with the quark momentum introduces the eighth parameter Λ_h.

These considerations are quite successful at describing the hadronic spectrum [20]. Aside from the effect of the quark mass in the strange hadrons, the differences in mass between the various hadronic multiplets arise from the differences in the quantum numbers which influence the value of c_{ij}, Eq. (13.5). Thus, the value of a_h, Eq. (13.6), depends both on the quantum numbers of the multiplet and on the quark-flavor content.

A unique fit with a significant confidence level arises, yielding

$$\varepsilon_\mathrm{q}^\mathrm{fit} = 1.97 \pm 0.02. \tag{13.12}$$

The value of $\varepsilon_\mathrm{q}^\mathrm{fit}$ is close to the massless-quark value $x_0 = 2.04$ derived from solution of the Dirac equation in a cavity, see section 13.2. This result is thus providing an empirical foundation for the bag model of hadrons.

The values $m_\mathrm{s}^0 = 234 \pm 14\,\mathrm{MeV}$ and $\Lambda_\mathrm{h} = 240 \pm 20\,\mathrm{MeV}$ are found [20]. The effective mass of the strange quark $m_\mathrm{s}(R_\mathrm{h})$ varies between 170 MeV in kaons, which are relatively small ($R_\mathrm{K} = 0.5\,\mathrm{fm}$), and about 320 MeV in strange baryons. Using a fixed mass for the strange quark, $m_\mathrm{s} = 280\,\mathrm{MeV}$ [99], was obtained. The QCD-motivated variability with hadron size of m_s in each hadron leads to a much better fit to the hadronic spectrum, and yields for the value of the vacuum energy $\mathcal{B}$

$$\boxed{\mathcal{B} = (171\,\mathrm{MeV})^4, \quad 4\mathcal{B} = 0.45\,\mathrm{GeV\ fm}^{-3}.} \tag{13.13}$$

The bag constant is larger than the original MIT result, $\mathcal{B}^\mathrm{MIT} = (145\ \mathrm{MeV})^4$. The main reason for the difference from the MIT fit arises from the allowed variation in mass of the strange quark with the size of the hadron. There is a remaining systematic dependence of the results for $\mathcal{B}$ on the here assumed behavior, Eq. (13.11). However, Eq. (13.13) is in much better agreement with the value of $\mathcal{B}$ required to describe the lattice-pressure results, for which we need a still larger value of $\mathcal{B}$, see Eq. (16.12).

Since the matrix elements $h_{\rm qq}$, $h_{\rm qs}$, and $h_{\rm ss}$ comprise $\alpha_{\rm s}$, a value for $\alpha_{\rm s}$ is not determined within this procedure. We note that in the cavity bag model, $\alpha_{\rm s} = 0.55$ is found.*

13.2 Confined quark quantum states

Since the short-range interactions among quarks can be ignored in the first instance, the 'independent' quark wavefunction $\psi_{\rm q}$ obeys the Dirac equation,

$$i\gamma^\mu\partial_\mu\psi_{\rm q} - M\psi_{\rm q} + (M-m)\Theta_{\rm V}\psi_{\rm q} = 0, \tag{13.14}$$

where $\Theta_{\rm V} = 1$ inside the quark bag and $\Theta_{\rm V} = 0$ outside. Inside the volume, the dynamics of quarks is governed by the (small) mass m, while outside it is subject to the mass M. Since, despite a great effort, we have not discovered free quarks, M must be very large, and the idea of color confinement indeed requires $M \to \infty$, in order to have quarks confined forever.

However, that limit is not trivial as it turned out. In a series of publications in 1974–75, the MIT-bag model was developed [92, 93, 151] in a way that creates the framework for quark dynamics with confinement, $M \to \infty$. Novel physical concepts were introduced since the confinement condition broke conservation of energy–momentum at the confinement boundary. Namely, in the limit $M \to \infty$, there is no quark quantum wave outside of the hadron volume, and, in order to have a stable physical state, the internal quantum pressure of the confined quarks must be balanced at the confinement boundary by some new external pressure pointing inward.

In a Lorentz-covariant formulation, along with pressure, there is also energy density, which had to be lower outside of the bag than inside, in order to have the inward-pointing pressure. The improvement of the energy–momentum tensor inside the volume region occupied by quarks includes the covariant bag term:

$$T^{\mu\nu} = T^{\mu\nu}_{\rm fields} + g^{\mu\nu}\mathcal{B}. \tag{13.15}$$

On comparing this with, e.g., Eq. (6.6), we indeed see that the bag term increases the energy density, and decreases the pressure within the volume occupied by quantum particles.

On physical grounds, the size of the system is determined by balancing the internal pressure against the external pressure, and thus $\mathcal{B}$ has to enter into the dynamics of quark fields. We follow, in the next few lines,

* In some literature, a four-fold greater value of $\alpha_{\rm s} = 2.2$ is quoted; the difference arises from the factor $\frac{1}{2}$ which relates λ^a to the generator t^a in Eq. (13.56).

the summary of the situation presented in [256]. We will need the surface $\Delta_{\rm S}$-function, which arises from the volume function in a familiar way,

$$\Delta_{\rm S} = -n_\mu \partial^\mu \Theta_{\rm V}, \tag{13.16}$$

where n^μ is the outward space-like normal to the surface of the bag. The static spherical cavity, in spherical coordinates, reads as usual:

$$\delta(R_{\rm h} - r) = \frac{d\Theta(R_{\rm h} - r)}{dr}. \tag{13.17}$$

The action which fully accounts for all physical aspects is

$$\mathcal{S} = \int d^4x \left[\left(\sum_{\rm q} \tfrac{1}{2} \overline{\psi_{\rm q}} i\gamma^\mu \overleftrightarrow{\partial}_\mu \psi_{\rm q} - \mathcal{B} \right) \Theta_{\rm V} - \tfrac{1}{2} \left(\sum_{\rm q} \overline{\psi_{\rm q}} \psi_{\rm q} \right) \Delta_{\rm S} \right], \tag{13.18}$$

where $\overline{\psi_{\rm q}} = \psi_{\rm q}^\dagger \gamma_0$, and γ^μ are Dirac matrices, Eq. (13.26).

To obtain the dynamic equations, we perform variation of the action seeking its stationary point:

$$\psi_{\rm q} \to \psi_{\rm q} + \delta\psi_{\rm q}, \quad \overline{\psi_{\rm q}} \to \overline{\psi_{\rm q}} + \delta\overline{\psi_{\rm q}}. \tag{13.19}$$

Furthermore, the geometry-defining volume $\Theta_{\rm V}$ and surface $\Delta_{\rm S}$ functions change under variation,

$$\Theta_{\rm V} \to \Theta_{\rm V} + \epsilon\Delta_{\rm S}, \quad \Delta_{\rm S} \to \Delta_{\rm S} - \epsilon n_\mu \partial^\mu \Delta_{\rm S}. \tag{13.20}$$

For a spherical cavity with $n^\mu = (0, \hat{r})$, the following three equations of motion give the stationary point of the action:
the Dirac equation,

$$i\gamma^\mu \partial_\mu \psi_{\rm q}(x) = 0, \qquad x \in {\rm V}; \tag{13.21}$$

the linear boundary condition,

$$i\gamma^\mu n_\mu \psi_{\rm q}(x) = \psi_{\rm q}(x), \qquad x \in {\rm S}; \tag{13.22}$$

the quadratic boundary condition,

$$\tfrac{1}{2} n^\mu \partial_\mu \left(\sum_q \overline{\psi_{\rm q}(x)} \psi_{\rm q}(x) \right) - \mathcal{B} = 0, \qquad x \in {\rm S}. \tag{13.23}$$

A solution satisfying the boundary condition, Eq. (13.22), satisfies the requirement that the normal flow of quark current through the surface of the bag vanishes. To see this, we write this condition, along with the adjoint form:

$$i\gamma^\mu n_\mu \psi_{\rm q} |_{\rm S} = \psi |_{\rm S}, \qquad -i\overline{\psi_{\rm q}} \gamma^\mu n_\mu |_{\rm S} = \overline{\psi} |_{\rm S}\, . \tag{13.24}$$

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To obtain the adjoint form, we used $\gamma^{\mu\,\dagger} = \gamma^0\gamma^\mu\gamma^0$. We construct the outward current at the surface:

$$n_\mu j^\mu\,|_{\mathrm{S}} = n_\mu(\overline{\psi}\gamma^\mu\psi)\,|_{\mathrm{S}} = \pm\overline{\psi}\psi\,|_{\mathrm{S}} = 0. \tag{13.25}$$

Since the right-hand side of Eq. (13.25) is both positive and negative of the same value, as can be seen by using one of the two forms of Eq. (13.24), it must be zero.

Since there is no flow of probability through the surface, the linear boundary condition guarantees confinement of quarks. Moreover, this boundary condition allows us to determine the eigenenergies of quarks in a cavity of a given size R_{h}, as we shall discuss. On the other hand, the quadratic boundary condition Eq. (13.23) allows us to find the size of the system, establishing a balance of forces at the surface; see Eq. (13.8).

We now proceed to obtain examples of solutions of the bag-model dynamic Eq. (13.21), in the static-cavity approximation. γ^μ are the usual covariant Dirac matrices and we use the Bjørken–Drell conventions [74]:

$$\gamma^0 \equiv \beta \equiv \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix},\ \gamma^i \equiv \beta\alpha^i \equiv \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}. \tag{13.26}$$

Here, I_2 is a unit 2×2 matrix, and $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are Pauli's spin matrices:

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{13.27}$$

We are interested in the lowest-energy solution of the Dirac equation, Eq. (13.14), for a stationary spherical cavity. We consider the wave function

$$\psi_{\mathrm{q}}(\vec{r},t) = q_n(\vec{r})e^{-i\omega_n t}\tau_{\mathrm{q}}. \tag{13.28}$$

τ_{q} is the flavor (e.g., isospin or $SU_{\mathrm{f}}(3)$) part of the independent particle wave function, and ω_n is the nth-state eigenenergy. The stationary quark 4-spinor wave function $q(\vec{r})$ satisfies the equation (suppressing the quantum number(s) subscript 'n')

$$(\vec{\alpha}\cdot\vec{p} + \beta m - \omega)q = 0,$$
$$\begin{pmatrix} (m-\omega)I_2 & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & (m+\omega)I_2 \end{pmatrix}\begin{pmatrix} q^{\mathrm{u}} \\ q^{\mathrm{d}} \end{pmatrix} = 0, \tag{13.29}$$

where q^{u} are the upper and q^{d} the lower quark 2-spinor components.

When there is no potential inside the bag, each of the four components 'k' of the spinor q has to satisfy the energy–momentum condition obtained by 'squaring' the Dirac equation, i.e., the Klein–Gordon equation:

$$[\omega^2 - m_{\rm q}^2 - (i\,\vec{\nabla})^2]q^k = 0. \tag{13.30}$$

We recall the spherical decompositions

$$\vec{\nabla} = \hat{r}\,\frac{\partial}{\partial r} - i\frac{\hat{r}}{r} \times \vec{L}, \qquad \vec{L} = \vec{r} \times i\vec{\nabla}, \tag{13.31}$$

$$\vec{\nabla}^{\,2} = \frac{d^2}{dr^2} + \frac{2}{r}\,\frac{d}{dr} - \frac{\vec{L}^2}{r^2}, \tag{13.32}$$

and recognize that the components of the Dirac cavity solution have the form

$$q^k = N j_l(x\,r/R_{\rm h}) \sum_{\mu=-l}^{l} c^{l\mu}_{jj_z}(k) Y_{l\mu}(\Omega), \tag{13.33}$$

where x is obtained from an eigenvalue condition, and $Y_{l\mu}(\Omega)$ are the usual spherical functions of fixed angular momentum l, μ. The Clebsch–Gordan coefficients $c^{l\mu}_{jj_z}$ are fixed by construction of a spinor spherical wave of good total angular momentum j and its z-axis projection, $j \le j_z \le -j$. Equation (13.30) now implies that

$$\omega \equiv \frac{\varepsilon}{R_{\rm h}} = \sqrt{\frac{x^2}{R_{\rm h}^2} + m_{\rm q}^2}. \tag{13.34}$$

The no-node, lowest-energy quark-cavity solution is

$$q_0^{\rm u} = N_0 j_0(x\,r/R_{\rm h})\chi_s, \quad j_0(z) = \frac{\sin z}{z}. \tag{13.35}$$

We use the 2-spinor $\chi_s,\ s = \pm\frac{1}{2}$ for spin-up and -down quarks:

$$\chi_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{13.36}$$

To obtain the corresponding lower components of the Dirac spinor, we use Eq. (13.29):

$$q^{\rm d} = \frac{1}{\omega + m}\vec{\sigma} \cdot i\,\vec{\nabla} q^{\rm u}. \tag{13.37}$$

The spherical decomposition Eq. (13.31), along with $\vec{L}q_0^{\rm u} = 0$, and the spherical-Bessel-function property,

$$j_1 = -\frac{d}{dz}j_0 = \frac{\sin z}{z^2} - \frac{\cos z}{z}, \tag{13.38}$$

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yields the lowest angular ($j = \frac{1}{2}$) quark wavefunction Eq. (13.28):

$$\psi_\mathrm{q}(\vec{r}, t) = N \begin{pmatrix} j_0(xr/R_\mathrm{h})\chi_s \\ \dfrac{x}{R_\mathrm{h}(\omega + m_\mathrm{q})} j_1(xr/R_\mathrm{h}) i\sigma_r \chi_s \end{pmatrix} e^{-i\omega t} \tau_\mathrm{q}. \tag{13.39}$$

The radial spin matrix has been introduced:

$$\sigma_r = \hat{r} \cdot \vec{\sigma}, \qquad \sigma_r^2 = I. \tag{13.40}$$

Equation (13.39) can easily be cast into the form Eq. (13.33), with the lower components q^d having $l = 1$.

The linear boundary condition, Eq. (13.22), at the surface of the bag, that is at a given radius R_h of the bag, takes the form

$$-i(\vec{\gamma} \cdot \vec{n})\psi\,|_\mathrm{s} = \psi\,|_\mathrm{s}, \tag{13.41}$$

which eigenvalue condition will now be used to fix the value of x. The surface-normal vector for a spherical bag is $\vec{n} = \hat{r}$. The boundary condition reads

$$-i\sigma_r q^\mathrm{d} \chi_s\,|_{r=R_\mathrm{h}} = q^\mathrm{u} \chi_s\,|_{r=R_\mathrm{h}}, \quad i\sigma_r q^\mathrm{u} \chi_s\,|_{r=R_\mathrm{h}} = q^\mathrm{d} \chi_s\,|_{r=R_\mathrm{h}}\,. \tag{13.42}$$

Using Eq. (13.39), we obtain

$$j_0(x) = \frac{x}{\sqrt{x^2 + (m_\mathrm{q} R_\mathrm{h})^2} + m_\mathrm{q} R_\mathrm{h}} j_1(x), \tag{13.43}$$

which takes the explicit form [93]

$$1 - x \cot x = \sqrt{x^2 + (m_\mathrm{q} R_\mathrm{h})^2} + m_\mathrm{q} R_\mathrm{h}. \tag{13.44}$$

When $m_\mathrm{q} R_\mathrm{h} \to 0$, the lowest eigenvalue is $x_0 = 2.04$. The (kinetic) energy of a massless quark in the confining radius R_h is

$$\omega_\mathrm{bag}(m_\mathrm{q} = 0) = \frac{2.04}{R_\mathrm{h}}. \tag{13.45}$$

The first radial excitation found, on solving Eq. (13.44) for the second-lowest eigenvalue, is relatively high, with $x_1 = 5.40$, more than doubling the kinetic energy.

For massive quarks we have

$$\omega_\mathrm{bag} = \frac{\sqrt{x_0^2 + (m_\mathrm{q} R_\mathrm{h})^2}}{R_\mathrm{h}}, \qquad 2.04 < x_0 < \pi. \tag{13.46}$$

As indicated in Eq. (13.46), with increasing $m_\mathrm{q} R_\mathrm{h}$, x_0 increases, never reaching the singularity of $\cot x$ at $x_0(m_\mathrm{q} R_\mathrm{h} \to \infty) = \pi$.

The quark-wave-function normalization N, in Eq. (13.39), is easily obtained:

$$\int d^3r\,\overline{\psi_{\rm q}}\gamma_0\psi_{\rm q} = 4\pi N^2 R_{\rm h}^3\int_0^1 dz\,z^2\left(j_0^2(xz)+\frac{x}{1-x\cot x}j_1^2(xz)\right).$$

We have used the eigenvalue condition Eq. (13.44) and substituted $z = r/R_{\rm h}$. The quadratic boundary condition Eq. (13.23) is

$$\mathcal{B} = \frac{1}{2}\,\frac{d}{dr}\left(\sum_q \overline{\psi_{\rm q}}\psi_{\rm q})\right)\Bigg|_{R_{\rm h}}. \tag{13.47}$$

We leave it to the reader as an exercise to show that Eq. (13.47) is indeed equivalent to the intuitive requirement that the total energy contained in the bag volume be at a minimum with respect to the radius of the bag, Eq. (13.7).

13.3 Nonabelian gauge invariance

The color hyperfine magnetic interaction, Eq. (13.4), is the residual force between quarks in the perturbative vacuum. It defines the hadron spectrum. To understand this force properly, we need to understand quantum chromodynamics and, in particular, the quark–quark interaction better. Akin to spin, the color charge of quarks is an internal quantum number, but it resembles in its properties more the electrical charge, so much so that one also speaks of 'color charge': like electrical charge, color charge is thought to be the source of a force field. Since we have so far not been able to build an apparatus to distinguish among the three fundamental colors, akin to the way electro-magnetic fields differentiate the spin and charge states, all colors must appear on an exactly equal footing.

Therefore, the theory of color forces, i.e., quantum chromodynamics (QCD), is based on the principle of gauge invariance extended to include invariance under arbitrary rotations (redefinition of color-principal 'axis') in the three-component color space. The resulting theory of strong interactions based on color forces has been called quantum chromodynamics in order to underline its formal similarity to quantum electrodynamics (QED). The form of the QCD Lagrangian is the generalization of QED required when one is considering invariance under local gauge color transformations.

Before introducing QCD, let us recall how, within QED, the principle of gauge invariance operates. It allows for changes in the electromagnetic potential, leaving the electro-magnetic fields $F^{\mu\nu}$, i.e., $\vec{E}$ and $\vec{B}$, unchanged. The electro-magnetic potential $A^\mu = (A_0, \vec{A})$ is thus not

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defined uniquely. We further recall that a measurement, in general, does not allow one to observe the phase factor of a quantum wave.

We now show how a change in the choice of the quantum phase is related to the change in gauge of the potential A^μ. The effect of a local change in the phase of the wave function,

$$\psi \to \psi' = e^{-i\alpha}\psi, \tag{13.48}$$

$$\partial_\mu \psi \to \partial_\mu (e^{-i\alpha}\psi) = e^{-i\alpha}[\partial_\mu \psi - (i\partial_\mu \alpha)\psi], \tag{13.49}$$

can be compensated for by the simultaneous gauge transformation of the electro-magnetic potential A_μ:

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \frac{1}{e}(\partial_\mu \alpha). \tag{13.50}$$

This occurs if the quantum fields and potential are 'minimally coupled':

$$[(\partial_\mu + ieA_\mu)\psi]' = (\partial_\mu + ieA'_\mu)\psi' = e^{-i\alpha}(\partial_\mu + ieA_\mu)\psi. \tag{13.51}$$

We see that the generalized covariant derivative,

$$\partial_\nu \to D_\nu = \partial_\nu + ieA_\nu, \tag{13.52}$$

remains gauge invariant, up to an overall phase factor $e^{-i\alpha}$.

We will now generalize the principle of gauge invariance to the case of QCD. The additional difficulty is the non-commutative, i.e., *nonabelian*, aspect of the transformation, which is associated with the fact that there is not just one but several charges, i.e., colors. which a particle can carry: the usual 4-spinor wave function of a spin-$\frac{1}{2}$ particle, ψ, becomes in our case a component of a 12-spinor in color space:

$$\Psi = \begin{pmatrix} \psi_\mathrm{r} \\ \psi_\mathrm{g} \\ \psi_\mathrm{b} \end{pmatrix}. \tag{13.53}$$

As long as the RGB (red, green, blue) 'color' is not observable, the color-gauge transformation generalizing Eq. (13.48) can, apart from introducing a phase, also rotate (mix) the color components of the wave function. The arbitrariness of the quantum wave is now expressed by the transformation

$$\Psi \to \Psi' = V\Psi, \qquad V^\dagger = V^{-1}, \qquad \det(V) = 1. \tag{13.54}$$

Since the complex rotations of a three-dimensional spinor-vector are described by unitary 3×3 matrices $V = (v_{ik})$ of unit determinant, the symmetry group of the gauge transformations is $SU_\mathrm{c}(3)$, where the subscript reminds us of color, and will be omitted where confusion with flavor symmetry $SU_\mathrm{f}(3)$ is unlikely.

The flavor $SU_{\rm f}(3)$ symmetry arising from the near degeneracy in energy of the three 'light' u, d, and s quarks is quite different in its nature. The color symmetry permits us to rename *locally* the color of quarks. The flavor symmetry is a global symmetry: once a definition of flavor has been chosen at CERN it applies at the BNL, as long as both laboratories belong to a region of the Universe occupying the same vacuum state. However, even this situation has an exception, namely when, after chiral symmetry has locally been restored in a high-energy heavy-ion collision, the chiral-symmetry-breaking vacuum is reformed, and the dynamic processes that are occurring could lead to a physical vacuum state in which the definition of flavor is different from that already established in the remainder of the world. We will not further discuss in this book these disoriented chiral states, which are a current topic of research.

Any unitary matrix V with $\det(V) = 1$, Eq. (13.54), can be written as the imaginary exponential of a hermitian traceless matrix L:

$$V = \exp(iL), \qquad L^\dagger = L, \qquad \mathrm{tr}(L) = 0. \tag{13.55}$$

All traceless hermitian 3×3 matrices can be expressed as linear combinations of the eight generators t_a of the Lie group $SU(3)$, using eight real variables θ_a:

$$L = \frac{1}{2}\sum_{a=1}^{8} \theta_a \lambda_a, \qquad t_a \equiv \frac{1}{2}\lambda_a. \tag{13.56}$$

The 'fundamental' (Gell-Mann) 3×3-matrix representation of the $SU(3)$ algebra is well known. There are $n_{\rm c}^2 - 1 = 8$, with $n_{\rm c} = 3$ for $SU_{\rm c}(3)$, $\lambda_a, a = 1, \ldots, 8$, matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{13.57}$$

The λ_a have been constructed in close analogy to the Pauli matrices σ_i, Eq. (13.27), and the t_a are analogous to the spin matrices $s_i = \sigma_i/2$. The first three λ_a correspond (up to the added third trivial row and column) to the Pauli matrices. While λ_8 is the second traceless diagonal 3×3

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matrix we can construct, the pairs λ_4, λ_5 and λ_6, λ_7 are generalizations of the $SU(2)$ σ_1 and σ_2 matrices, which are similar to λ_1 and λ_2.

The following commutation and anticommutation relations can be used to define the algebra of the $SU(3)$ group:

$$[t_a, t_b] = i \sum_c f_{abc} t_c, \tag{13.58}$$

$$\{t_a, t_b\} = (1/3)\delta_{ab} \mathrm{I}_3 + \sum_c d_{abc} t_c, \tag{13.59}$$

where I_3 is the 3×3 unit matrix. f_{abc} and d_{abc} are the antisymmetric and symmetric 'structure constants', respectively, of the Lie group $SU(3)$, which can be determined in a straightforward fashion from Eqs. (13.58) and (13.59). One of their frequently used properties is

$$\sum_{k,l=1}^{n_\mathrm{c}^2-1} f_{ikl} f_{jkl} = n_\mathrm{c} \delta_{ij}. \tag{13.60}$$

Another important relation, which is related to the definition of the color charge, is found by considering the trace of Eq. (13.59):

$$\mathrm{tr}(t_a t_b) = \tfrac{1}{2}\delta_{ab}. \tag{13.61}$$

A second frequently needed representation of the $SU(3)$ algebra in terms of the 8×8 matrices is called 'adjoint' representation. This representation plays with regard to the eight-component glue field a similar role to that which the fundamental representation plays with regard to the quark field. Pushing the analogy to the spin, we are looking for the equivalent of spin-1 representation. The matrix representation of generators in the adjoint representation is

$$(T_a)_{bc} = -i f_{abc}, \ a, b, c = 1, \ldots, 8. \tag{13.62}$$

T_a satisfy the same algebra as the generators t_a of the fundamental representation, Eqs. (13.58) and (13.59). The trace of the product of two T_a, an analogous result to Eq. (13.61), follows from Eq. (13.60) in view of Eq. (13.62):

$$\mathrm{tr}(T_a T_b) = n_\mathrm{c} \delta_{ab}. \tag{13.63}$$

We now make the color-rotation matrix V space-dependent, $V \to V(x)$, allowing that the eight real parameters ϑ_a, in Eq. (13.56), depend on x,

$$V(x) = e^{-i\vartheta_a(x) t_a}, \tag{13.64}$$

with summation over a repeated group index such as a being implicitly understood henceforth. In analogy to Eqs. (13.48) and (13.49), the local nonabelian gauge transformation of matter fields is

$$\Psi \to \Psi' = V\Psi, \tag{13.65}$$

$$\partial_\mu \Psi \to \partial_\mu \Psi' = V[\partial_\mu \Psi + V^\dagger(\partial_\mu V)\Psi], \tag{13.66}$$

where we have used Eq. (13.54). Since $V^\dagger V = 1$, we have

$$(\partial_\mu V^\dagger)V = -V^\dagger(\partial_\mu V). \tag{13.67}$$

Instead of the term $(i\partial_\mu\alpha)$, in Eq. (13.49), we have now a matrix term $V^\dagger(\partial_\mu V)$; the entire expression is multiplied by a matrix V, Eq. (13.64), rather than a phase factor $e^{i\alpha}$.

13.4 Gluons

We introduce now the dynamic color-potential field A_μ. To proceed in analogy to QED, we have to couple the color potential to a product of quark–antiquark spinors. Thus, A_μ must be a 3×3 matrix, given the color structure of the spinor Ψ. The three-component quark wave function forms a triplet (fundamental) representation of the color group $SU(3)$, while the wave function of an antiquark forms an antitriplet. From the product of a triplet and an antitriplet, intuitively, we can understand that one can form an $SU(3)$ singlet; what remains is an octet of states. In analogy to spin, for which the product of two spin-$\frac{1}{2}$ particles can be a singlet ($S = 0$) or a triplet ($S = 1$) state, we write

$$3_\mathrm{c} \times \bar{3}_\mathrm{c} = 8_\mathrm{c} \oplus 1_\mathrm{c}. \tag{13.68}$$

Since the color-gauge-field quantum must have the same quantum numbers as a quark–antiquark pair, if it is to be able to be the product of their annihilation, it must contain at least eight non-matrix fields; the ninth singlet field corresponds to the colorless case. In analogy to the real field A_μ in QED, we now choose a hermitian matrix A_μ to represent the massless gluons, in the form of a linear combination of eight Gell-Mann matrices with eight real Yang–Mills fields $A^a_\mu(x)$:

$$\mathrm{A}_\mu(x) = \tfrac{1}{2}A^a_\mu(x)\lambda_a = A^a_\mu(x)t_a. \tag{13.69}$$

We proceed in analogy to Eq. (13.50): if the potential A_μ changes under a local color-gauge transformation according to

$$\mathrm{A}_\mu \to \mathrm{A}' = V\mathrm{A}_\mu V^\dagger + \frac{1}{g}V(\partial_\mu V^\dagger), \tag{13.70}$$

the minimally coupled derivative, $\partial_\mu + ig\mathrm{A}_\mu$, remains invariant in form under the gauge transformation, up to the nonabelian phase factor

$$(\partial_\mu + ig\mathrm{A}'_\mu)\Psi' = V(\partial_\mu + ig\mathrm{A}_\mu)\Psi. \tag{13.71}$$

To show this, we have to use Eq. (13.66) and remember Eq. (13.67).

It is often helpful to check the form of equations, and in particular relative signs, by remembering that the transition from QED to QCD equations can be effected with the introduction of a gauge-covariant derivative, both for quark and glue fields, defined by

$$\mathrm{D}_\nu = \partial_\nu + igt_a A^a_\nu = \partial_\nu + ig\mathrm{A}_\nu. \tag{13.72}$$

In view of Eqs. (13.66) and (13.71), we have shown that this covariant derivative transforms under nonabelian gauge transformations as

$$\mathrm{D}_\nu \to \mathrm{D}'_\nu = V\mathrm{D}_\nu V^\dagger. \tag{13.73}$$

Similarly, the eight-component field-strength tensor,

$$\mathrm{F}_{\mu\nu} = t_a F^a_{\mu\nu}, \tag{13.74}$$

defined as,

$$\mathrm{F}_{\mu\nu} = \partial_\mu \mathrm{A}_\nu - \partial_\nu \mathrm{A}_\mu + ig\left[\mathrm{A}_\mu, \mathrm{A}_\nu\right], \tag{13.75}$$

or equivalently,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf_{abc}A^b_\mu A^c_\nu, \tag{13.76}$$

remains up to a phase form-invariant under a local color-gauge transformation, i.e.,

$$\mathrm{F}_{\mu\nu} \to \mathrm{F}'_{\mu\nu} = V\mathrm{F}_{\mu\nu}V^\dagger. \tag{13.77}$$

This transformation property is most easily proved once we realize that we can define the field-strength tensor using the commutator of the covariant derivative, Eq. (13.72),

$$[\mathrm{D}_\mu, \mathrm{D}_\nu] \equiv ig\mathrm{F}_{\mu\nu}, \tag{13.78}$$

which verifies Eq. (13.77) in view of Eq. (13.73), remembering properties of V, Eq. (13.54).

13.5 The Lagrangian of quarks and gluons

The complete gauge-invariant Lagrangian of quantum chromodynamics is then

$$\mathcal{L}_{\rm QCD} = \sum_{\rm f} \overline{\Psi}_{\rm f}\gamma^\mu(i\partial_\mu - g{\rm A}_\mu)\Psi_{\rm f} - m_{\rm f}\overline{\Psi}_{\rm f}\Psi_{\rm f} - \tfrac{1}{2}\,{\rm tr}({\rm F}_{\mu\nu}{\rm F}^{\mu\nu}). \quad (13.79)$$

Here, the summation over the different quark flavors f has been made explicit. The form similarity to the Lagrangian of QED,

$$\mathcal{L}_{\rm QED} = \overline{\psi}\gamma^\mu(i\partial_\mu - eA_\mu)\psi - m\overline{\psi}\psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}\ , \quad (13.80)$$

is evident.

In view of Eq. (13.61), we have an exact correspondence between the gluon and photon terms:

$$\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}(B^2 - E^2)\bigg|_{\rm QED},$$
$$\frac{1}{2}\,{\rm tr}({\rm F}^{\mu\nu}{\rm F}_{\mu\nu}) = \frac{1}{4}\sum_a F_a^{\mu\nu}F^a_{\mu\nu} = \frac{1}{2}\sum_a (B_aB^a - E_aE^a)\bigg|_{\rm QCD}. \quad (13.81)$$

The key difference is in the nonlinear term entering into the definition Eq. (13.76) of $F^a_{\mu\nu}$, which is quadratic in the color potentials A^a_μ.

The analog of the Maxwell equations may be derived for the color field from the QCD Lagrangian, Eq. (13.79):

$$[{\rm D}_\nu, {\rm F}^{\mu\nu}] = g{\rm j}^\mu. \quad (13.82)$$

The color indices of the matter fields Ψ define the matrix current ${\rm j}^\mu$ obtained from $\mathcal{L}_{\rm QCD}$,

$$j^a_\mu = \overline{\Psi}\gamma^\mu t_a\Psi. \quad (13.83)$$

We can write Eq. (13.82) in the more conventional form

$$\partial_\nu F_a^{\mu\nu} = gj_a^\mu + gJ_a^\mu, \quad (13.84)$$

where we encounter the gluon current J_a^μ:

$$J_a^\mu = f_{abc}A_\nu^bF_c^{\mu\nu}, \quad {\rm J}^\mu = -i[{\rm A}_\nu, {\rm F}^{\mu\nu}]. \quad (13.85)$$

The nonlinear term, Eq. (13.85), placed on the right-hand side of Eq. (13.84), shows that the color field acts also as its own source. In other words, the quanta of the color field, *gluons*, carry color charge themselves. This is the source of the substantial physical difference between QCD and QED. From Eq. (13.84), one can further see that j^a_μ does not obey a continuity equation, which means that the color charge of quarks alone is not conserved. This is not surprising since quarks can emit or absorb gluons, which carry color. Only if we add the color charge of the gluon field, represented by the second term on the right-hand side of Eq. (13.84), is a conserved color current obtained.

In view of the similarity in form of $\mathcal{L}_{\mathrm{QCD}}$ to $\mathcal{L}_{\mathrm{QED}}$, many of the well-known other formal properties carry over. We refrain from systematically developing this here, even though we will call upon these similarities as needed in further developments.

14 Perturbative QCD

14.1 Feynman rules

The nonabelian gauge theory of quarks and gluons, proposed in section 13.5 and called QCD, has widely been accepted as the fundamental theory of strong interactions, with both quarks and gluons being the carriers of the strong-interaction charge [123]. The evidence for the validity of QCD as a dynamic theory governing hadronic reactions is overwhelming, and this is not the place where this matter should be argued. Rather, we will show how QGP-related practical results can be derived from the complex theoretical framework. There are many books dealing with more applications of QCD and the interested reader should consult these for further developments [110, 194, 280].

Akin to QED, QCD is a 'good' renormalizable theory. QCD is known to be also an *asymptotically free* theory, viz., the running coupling constant α_{s}, see Eq. (14.12), is a diminishing function as the energy scale increases. Therefore, the high-energy, or, equivalently, the short-distance behavior is amenable to a perturbative expansion. On the other hand, perturbative QCD has 'fatal' defects at large distances, which are signaled by the growth and the ultimate divergence of α_{s} as the scale of energy diminishes (infrared 'slavery'). Consequently, at any reasonable physical distance of relevance to the 'macroscopic' QGP, we have to deal with an intrinsically strongly coupled, nonperturbative physical system. A perturbative treatment ignores this, and, in principle, must be unreliable in problems in which the confinement scale becomes relevant. The question of when exactly this occurs will be one of the important issues we will aim to resolve, using as criterion $\alpha_{\mathrm{s}} \leq 1$.

The perturbative approach, which applies to short-distance phenomena, has been tested extensively in high-energy processes. When the 'short distance' grows and approaches 0.5 fm, the perturbative expansion of QCD may still apply insofar as its results are restricted to the physics occurring in the deconfined, viz., QGP, phase. The rules of perturbative QCD follow the well-known Feynman rules of QED, allowing for the glue–glue