

## FINITE-DIMENSIONAL SUBSPACES OF THE $p$ -ADIC SPACE $\ell^\infty$

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ABSTRACT. For finite-dimensional subspaces of  $\ell^\infty$  over a non-archimedean valued base field  $K$  we study orthocomplementation as related to the Hahn-Banach property and strictness. As a corollary we obtain that, if  $K$  is not spherically complete, a closed hyperplane in  $c_0$  having the Hahn-Banach property is orthocomplemented (Theorem 2.1, Remark 2).

**1. Introduction.** Our starting point is the study of orthocomplemented subspaces of  $c_0$  (carried out in full in [1], 5.4). Here the behaviour of subspaces of finite codimension is crucial which justifies special attention. In this note we have taken the (equivalent) dual view point *i.e.* the study of finite-dimensional subspaces of  $\ell^\infty$ . It contains all the essentials of [1].

Throughout  $K$  is a non-archimedean complete valued field whose valuation  $|\cdot|$  is non-trivial. All Banach spaces are over  $K$ . We shall use the notations and conventions of [2]. In particular we recall that for Banach spaces  $E$  and  $F$  the expression  $E \sim F$  indicates that  $E$  and  $F$  are isomorphic *i.e.* that there exists a linear isometrical bijection  $E \rightarrow F$ .

Let  $D$  be a closed subspace of some Banach space  $E$ . We say that  $D$  is *strict (in  $E$ )* if for each  $x \in E$  the function  $d \mapsto \|x - d\|$  ( $d \in D$ ) has a minimum, equivalently if the quotient map  $\pi: E \rightarrow E/D$  is strict in the sense of [2], page 172.  $D$  is said to be *HB (in  $E$ )* if each  $f \in D'$  extends to an  $\tilde{f} \in E'$  such that  $\|\tilde{f}\| = \|f\|$ . Recall that  $D$  is *orthocomplemented (in  $E$ )* if there exists an orthoprojection of  $E$  onto  $D$ . Obviously,

PROPOSITION 1.1. *Orthocomplemented subspaces are strict and HB.*

If  $K$  is spherically complete every finite-dimensional subspace is orthocomplemented ([2], 4.35(i), (iii)) making the program set out in the Abstract trivial. Hence, most following results will be of interest only if  $K$  is not spherically complete.

**2. One-dimensional subspaces of  $\ell^\infty$ .** We can prove the following curious theorem.

THEOREM 2.1. *Suppose  $K$  is not spherically complete. Let  $D = Kx$  ( $x = (x_1, x_2, \dots) \in \ell^\infty$ ) be a one-dimensional subspace of  $\ell^\infty$ . Then the following are equivalent.*

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- ( $\alpha$ )  $D$  is HB.
- ( $\beta$ )  $D$  is strict.
- ( $\gamma$ )  $D$  is orthocomplemented.
- ( $\delta$ )  $\max_n |x_n|$  exists.

PROOF. Clearly,  $(\gamma) \Rightarrow (\alpha)$ ,  $(\gamma) \Rightarrow (\beta)$ . Assume  $(\alpha)$ . Then there exists an  $f \in (\ell^\infty)'$  for which  $\|f\| \|x\| = 1 = |f(x)|$ , and one verifies immediately that  $\text{Ker } f$  is an orthocomplement of  $Kx$ , so we proved  $(\alpha) \Rightarrow (\gamma)$ . If  $|x_m| = \max_n |x_n|$  then one verifies easily that  $\{(y_1, y_2, \dots) \in \ell^\infty : y_m = 0\}$  is an orthocomplement of  $Kx$  which proves  $(\delta) \Rightarrow (\gamma)$ . To arrive at the remaining implication  $(\beta) \Rightarrow (\delta)$ , suppose  $|x_n| < \|x\|$  for all  $n \in \mathbb{N}$ ; we shall prove that  $D$  is not strict. Let  $B_1 \supset B_2 \supset \dots$  be bounded discs in  $K$  whose intersection is empty. Let  $r_n$  be the diameter of  $B_n$ ; we may suppose that  $r_1 > r_2 > \dots$ . Set  $B_0 := K$  and  $r_0 := \infty$ . Define a function  $\varphi: K \rightarrow [0, \infty)$  by

$$\varphi(\lambda) = \lim_{n \rightarrow \infty} \inf_{\mu \in B_n} |\lambda - \mu|.$$

Then  $d := \lim_{n \rightarrow \infty} r_n = \inf \varphi$ , but  $\min \varphi$  does not exist. Furthermore  $d$  is strictly positive.

We shall construct  $c_1, c_2, \dots \in K$  such that  $y := (c_1x_1, c_2x_2, \dots) \in \ell^\infty$  and

$$(*) \quad \|y - \lambda x\| = \varphi(\lambda) \|x\| \quad (\lambda \in K).$$

(Then  $\min_\lambda \|y - \lambda x\|$  does not exist so  $D$  is not strict in  $\ell^\infty$ .) In fact, let  $n \in \mathbb{N}$ . If  $x_n = 0$  we set  $c_n := 0$ . Otherwise, take a  $k(n) \in \mathbb{N}$  for which  $r_{k(n)}|x_n| \leq d\|x\|$  and choose any  $c_n \in B_{k(n)}$ . Then  $c_1, c_2, \dots$  is bounded so  $y \in \ell^\infty$ . To prove  $(*)$  let  $\lambda \in K$ . There is a unique  $m \in \{0, 1, \dots\}$  such that  $\lambda \in B_m \setminus B_{m+1}$ .

(i) We first show  $\|y - \lambda x\| \leq \varphi(\lambda) \|x\|$  i.e.  $|c_n - \lambda| |x_n| \leq \varphi(\lambda) \|x\|$  for each  $n$ . That is obvious if  $x_n = 0$  so assume  $x_n \neq 0$ . If (a)  $m \geq k(n)$  then  $\lambda \in B_m \subset B_{k(n)}$ . Since also  $c_n \in B_{k(n)}$  we find  $|c_n - \lambda| |x_n| \leq r_{k(n)} |x_n| \leq d\|x\| \leq \varphi(\lambda) \|x\|$ ; if (b)  $m < k(n)$  then  $c_n \in B_{k(n)} \subset B_{m+1}$  while  $\lambda \notin B_{m+1}$  so that  $|c_n - \lambda| = \varphi(\lambda)$  and  $|c_n - \lambda| |x_n| = \varphi(\lambda) |x_n| \leq \varphi(\lambda) \|x\|$ .

(ii) To prove  $\|y - \lambda x\| \geq \varphi(\lambda) \|x\|$  let  $\varepsilon > 0$ . We may assume  $\varepsilon + d < r_m$ . There is an  $n \in \mathbb{N}$  such that  $d\|x\| < (d + \varepsilon) |x_n|$ . Then  $r_{k(n)} \leq \frac{d\|x\|}{|x_n|} < d + \varepsilon < r_m$  so that  $k(n) > m$  and like in (b) above,  $\|y - \lambda x\| \geq |c_n - \lambda| |x_n| = \varphi(\lambda) |x_n| > d/(d + \varepsilon) \varphi(\lambda) \|x\|$ .

REMARK 1. The crucial implication  $(\beta) \Rightarrow (\delta)$  is false if  $K$  is spherically complete ( $Kx$  satisfies  $(\alpha)$ ,  $(\beta)$ ,  $(\gamma)$  but not always  $(\delta)$  if the valuation is dense).

REMARK 2. Dualizing  $(\beta) \Rightarrow (\gamma)$  simply leads to the following ([1], (4.3)). *If  $K$  is not spherically complete then any closed HB hyperplane in  $c_0$  is orthocomplemented.*

**3. Finite-dimensional subspaces of  $\ell^\infty$ .** The situation becomes more complicated when we start considering subspaces with dimension greater than 1. Yet, we can save part of Theorem 2.1. To this end we have the following lemma.

LEMMA 3.1. *If  $D$  is a finite-dimensional subspace of  $\ell^\infty$  then  $\ell^\infty/D \sim \ell^\infty$ .*

PROOF. Identifying  $\ell^\infty$  and  $c'_0$  in the natural way we have  $D = H^\perp := \{f \in c'_0 : f = 0 \text{ on } H\}$  for some closed finite-codimensional subspace  $H$  of  $c_0$ . The restriction  $f \mapsto f|_H$  ( $f \in c'_0$ ) induces a map  $c'_0/H^\perp \rightarrow H'$  which is an isomorphism by [2], 3.16(vi). Gruson's Theorem [2], 5.9 tells us that  $H$  has an orthonormal base so  $H \sim c_0$  whence  $H' \sim \ell^\infty$ . We find  $\ell^\infty/D = c'_0/H^\perp \sim H' \sim \ell^\infty$ .

THEOREM 3.2. *Suppose that  $K$  is not spherically complete. Let  $D$  be a finite-dimensional subspace of  $\ell^\infty$ .*

- (a) *If  $D$  is strict then  $D$  is HB.*
- (b) *The following are equivalent.*
  - ( $\alpha$ )  *$D$  is HB and has an orthogonal (orthonormal) base.*
  - ( $\beta$ )  *$D$  is strict and has an orthogonal (orthonormal) base.*
  - ( $\gamma$ )  *$D$  is orthocomplemented.*
  - ( $\delta$ ) *For each  $x \in D$ ,  $\max_n |x_n|$  exists.*

PROOF. (a) Let  $f \in D', f \neq 0$  and set  $S := \text{Ker } f$ . Let  $\pi: \ell^\infty \rightarrow \ell^\infty/S$  be the quotient map and  $\pi_D: D \rightarrow D/S$  be its restriction. The formula  $f = g \circ \pi_D$  defines a  $g \in (D/S)'$ . From the strictness of  $D$  in  $\ell^\infty$  it follows directly that  $D/S$  is strict in  $\ell^\infty/S$ . Now  $D/S$  is one-dimensional and  $\ell^\infty/S \sim \ell^\infty$  (Lemma 3.1) so by Theorem 2.1  $g$  extends to some  $\tilde{g} \in (\ell^\infty/S)'$  for which  $\|\tilde{g}\| = \|g\|$ . Then  $\tilde{g} \circ \pi$  extends  $f$  and its norm equals  $\|f\|$ . We see that  $D$  is HB in  $\ell^\infty$ .

(b) ( $\gamma$ )  $\Rightarrow$  ( $\beta$ ). Clearly  $D$  is strict. Since  $D$  is orthocomplemented in  $\ell^\infty$  it is a quotient of  $\ell^\infty$  so  $D'$  is isomorphic to a closed subspace of  $(\ell^\infty)' \sim c_0$  and so  $D' \sim K^n$  for some  $n \in \mathbb{N}$ . Then  $D \sim D'' \sim K^n$  and  $D$  has an orthonormal base.

( $\beta$ )  $\Rightarrow$  ( $\alpha$ ). Follows directly from (a).

( $\alpha$ )  $\Rightarrow$  ( $\delta$ ). Let  $x \in D$ . Then, since  $D$  has an orthogonal base,  $Kx$  is orthocomplemented in  $D$  hence  $Kx$  is HB in  $D$ . Now also  $D$  is HB in  $\ell^\infty$  so  $Kx$  is HB in  $\ell^\infty$  and from Theorem 2.1 ( $\alpha$ )  $\rightarrow$  ( $\delta$ ) it follows that  $\max_n |x_n|$  exists.

( $\delta$ )  $\Rightarrow$  ( $\gamma$ ). From Theorem 2.1 ( $\delta$ )  $\Rightarrow$  ( $\gamma$ ) it follows that every one-dimensional subspace of  $D$  is orthocomplemented. But then  $D$  is orthocomplemented ([2], 4.35(iii)).

4. **A counterexample and a problem.** For a finite-dimensional subspace  $D$  of  $\ell^\infty$  consider the following two questions.

QUESTION 1. If  $D$  is HB, does it follow that  $D$  is strict?

QUESTION 2. If  $D$  is strict, does it follow that  $D$  is orthocomplemented?

The answers to both of them are affirmative if  $D$  is one-dimensional (Theorem 2.1), but not settled by Theorem 3.2 if  $D$  has dimension  $> 1$ .

We regret to have to leave Question 2 as an open problem. An equivalent formulation is the following (see [1], Section 4 Problem I).

PROBLEM. Let  $K$  be not spherically complete. Let  $H$  be a closed subspace of  $c_0$  of finite codimension. Suppose every  $f \in H'$  extends to an  $\tilde{f} \in c'_0$  such that  $\|\tilde{f}\| = \|f\|$ . Does it follow that  $H$  is orthocomplemented? (Compare Theorem 2.1, Remark 2).

However we shall answer Question 1 in the negative by constructing in Example 4.3 a two-dimensional subspace  $D$  of  $\ell^\infty$  that is HB but not strict by taking the adjoint of some suitable strict quotient map  $c_0 \rightarrow D'$ . For this reason we first describe all strict quotients of  $c_0$  (compare the fact that every Banach space of countable type over a densely valued field is a quotient of  $c_0$ , see [3], 3.1).

THEOREM 4.1 ([4], 2.3). For a Banach space  $F \neq \{0\}$  the following are equivalent.

- ( $\alpha$ ) There is a strict quotient map  $c_0 \rightarrow F$ .
- ( $\beta$ )  $F$  is of countable type and  $\|F\| = |K|$ .

PROOF. We only need to prove ( $\beta$ )  $\Rightarrow$  ( $\alpha$ ). We may assume that the valuation is dense. Choose  $\mu_1, \mu_2, \dots \in K, 0 < |\mu_1| < |\mu_2| < \dots, \lim_{n \rightarrow \infty} |\mu_n| = 1$ . For each  $n$ , let  $X_n$  be a  $|\mu_n|$ -orthogonal base of  $F$  such that  $|\mu_n| \leq \|z\| < 1$  for each  $z \in X_n$ . Let  $Y$  be a maximal orthogonal system in  $\{x \in F : \|x\| = 1\}$ . Then  $X := Y \cup X_1 \cup X_2 \cup \dots$  is countable, say  $X = \{x_1, x_2, \dots\}$ . It is not hard to see that every  $x \in F$  with  $\|x\| \leq 1$  admits a (not necessarily unique) representation  $x = \sum_{i=1}^\infty \lambda_i x_i$  where  $\lambda_i \in K, |\lambda_i| \leq 1, \lambda_i \rightarrow 0$ . But this implies that the map  $\pi: c_0 \rightarrow F$  given by

$$\pi((\lambda_1, \lambda_2, \dots)) = \sum_{i=1}^\infty \lambda_i x_i$$

is a quotient map and sends the closed unit ball of  $c_0$  onto the closed unit ball of  $F$ . Now, since  $\|F\| = |K|$ , the same is true for arbitrary closed balls about 0 rather than the unit ball. But this means that  $\pi$  is a strict quotient map.

EXAMPLE 4.2. Let  $K$  be separable with a dense valuation (e.g., let  $K$  be the completion of the algebraic closure of  $\mathbb{Q}_p$ ). Then there exists a closed subspace  $H$  of  $c_0$ , with codimension 2, that is strict but not HB.

PROOF. According to [3], 1.14 there is a three-dimensional Banach space  $F$  over the (non-spherically complete) field  $K$  such that

- (i) every two-dimensional subspace of  $F$  has an orthonormal base,
- (ii)  $F$  has no orthogonal base.

From (i) we obtain  $\|F\| = |K|$  so by the previous theorem there exists a strict quotient map  $\pi: c_0 \rightarrow F$ . Choose  $e \in F, \|e\| = 1$ ; there exists an  $a \in c_0$  with  $\pi(a) = e$  and  $\|a\| = 1$ . Then  $H := \pi^{-1}(Ke) = D + Ka$  (where  $D := \text{Ker } \pi$ ) has codimension 2.

To prove strictness of  $H$ , let  $x \in c_0 \setminus H$ ; we show that  $\|x - h_0\| \leq \|x - h\|$  ( $h \in H$ ) for some  $h_0 \in H$ . First, by (i) the space  $[\pi(x), \pi(a)]$  has an orthogonal base so we can find a  $\lambda_0 \in K$  such that

$$(1) \quad \|\pi(x - \lambda_0 a)\| \leq \|\pi(x - \lambda a)\| \quad (\lambda \in K).$$

Secondly, by strictness of  $\pi$  there is a  $v \in c_0$  such that

$$(2) \quad \pi(v) = \pi(x - \lambda_0 a) \quad \text{and} \quad \|v\| = \|\pi(x - \lambda_0 a)\|.$$

There exists a  $d_0 \in D$  such that  $v = x - \lambda_0 a - d_0$ . Now set  $h_0 := \lambda_0 a + d_0 \in H$ . For any  $h = \lambda a + d$  ( $\lambda \in K, d \in D$ ) we have, using (1) and (2),

$$\|x - h_0\| = \|v\| = \|\pi(x - \lambda_0 a)\| \leq \|\pi(x - \lambda a)\| \leq \|x - \lambda a - d\| = \|x - h\|.$$

Next, suppose  $H$  is HB in  $c_0$ ; we derive a contradiction.

In fact, let  $f \in H'$  given by

$$f(\lambda a + d) = \lambda \quad (\lambda \in K, d \in D).$$

Then  $\|f\| = 1$  (the choice of  $a$  entails that  $Ka$  and  $D$  are orthogonal) and it extends to an  $\tilde{f} \in c'_0$  for which  $\|\tilde{f}\| = 1$ . Then, for each  $x \in \text{Ker } \tilde{f}$

$$\|a - x\| = \|\tilde{f}\| \|a - x\| \geq |\tilde{f}(a - x)| = |\tilde{f}(a)| = 1 = \|a\|$$

so that  $\text{Ker } \tilde{f} \perp Ka$ . With  $\pi$  as above,  $\pi(\text{Ker } \tilde{f})$  is a two-dimensional space and from  $\text{Ker } \tilde{f} \perp Ka$  it follows that  $\pi(\text{Ker } \tilde{f}) \perp K\pi(a) = Ke$ . Then, together with (i), this would imply that  $F = Ke + \pi(\text{Ker } \tilde{f})$  has an orthogonal base, conflicting (ii).

From this we easily obtain

EXAMPLE 4.3. Let  $K$  be as in Example 4.2. Then there exists a two-dimensional subspace  $D$  of  $\ell^\infty$  that is HB but not strict.

PROOF. Let  $H \subset c_0$  be as in the previous Example. By reflexivity the ‘vertical’ maps in the commutative diagram (where the indicated maps are the ‘natural’ ones)

$$\begin{array}{ccccc} H & \xrightarrow{i} & c_0 & \xrightarrow{\pi} & c_0/H \\ \downarrow & & \downarrow & & \downarrow \\ H'' & \xrightarrow{i''} & c''_0 & \xrightarrow{\pi''} & (c_0/H)'' \end{array}$$

are isomorphisms. The adjoint  $\pi': (c_0/H)' \rightarrow c'_0 = \ell^\infty$  is easily seen to be an isometry. Strictness of  $\pi''$  means precisely that  $D := \text{Im } \pi'$  is HB in  $c'_0$ . Also,  $H$  is not HB in  $c_0$  so  $i': c'_0 \rightarrow H'$  is not a strict quotient map which means that its kernel  $D$  is not strict in  $c'_0$ .

5. **Embeddings into  $\ell^\infty$ .** Which finite-dimensional Banach spaces are isomorphic to a (strict, HB, orthocomplemented) subspace of  $\ell^\infty$ ?

THEOREM 5.1. Let  $E$  be a finite-dimensional Banach space,  $E \neq \{0\}$ .

- (i) If  $|K|$  is dense  $E$  is isomorphic to a subspace of  $\ell^\infty$ .
- (ii) If  $|K|$  is discrete  $E$  is isomorphic to a subspace of  $\ell^\infty$  if and only if  $\|E\| = |K|$ .

PROOF. (i) For each  $t \in (0, 1)$ ,  $E$  has a  $t$ -orthogonal base ([2], 3.15) so one easily constructs for each  $n \in \mathbb{N}$  a linear injection  $T_n: E \rightarrow c_0 \hookrightarrow \ell^\infty$  such that  $(1 - \frac{1}{n})\|x\| \leq \|T_n x\| \leq \|x\|$  for all  $x \in E$ . The formula  $x \mapsto (T_1 x, T_2 x, \dots)$  defines a linear isometry of  $E$  into  $\times_{\mathbb{N}} \ell^\infty \sim \ell^\infty(\mathbb{N} \times \mathbb{N}) \sim \ell^\infty$ .

(ii) If  $E$  is embeddable then clearly  $\|E\| = \|\ell^\infty\| = |K|$ . Conversely, if  $\|E\| = |K|$  then, since  $K$  is spherically complete  $E$  has an orthonormal base, hence  $E \sim K^n \hookrightarrow \ell^\infty$  for some  $n \in \{1, 2, \dots\}$ .

REMARK 1. A similar proof works for a Banach space  $E$  of countable type.

REMARK 2. If  $K$  is spherically complete and  $E$  is a finite-dimensional Banach space isomorphic to a subspace of  $\ell^\infty$  then  $E$  is automatically strict, HB, orthocomplemented and  $E$  has an orthogonal base. For non-spherically complete  $K$  we have the following.

THEOREM 5.2. *Let  $K$  be not spherically complete, let  $E$  be a finite-dimensional Banach space,  $E \neq \{0\}$ .*

- (i)  *$E$  is isomorphic to some HB subspace of  $\ell^\infty \Leftrightarrow \|E'\| = |K|$ .*
- (ii)  *$E$  is isomorphic to some orthocomplemented subspace of  $\ell^\infty \Leftrightarrow E$  has an orthonormal base.*

PROOF. (i) If  $E$  is isomorphic to some HB subspace of  $\ell^\infty$  then  $\|E'\| \subset \|(\ell^\infty)'\| = \|c_0\| = |K|$ . Conversely, if  $\|E'\| = |K|$  then there exists, by Theorem 4.1, a strict quotient map  $c_0 \rightarrow E'$  whose adjoint  $E'' \rightarrow c'_0$  leads to an inclusion map  $E \rightarrow E'' \rightarrow c'_0 = \ell^\infty$  where  $E \sim E''$  and  $E''$  is HB.

(ii) If  $E$  is orthocomplemented then by Theorem 3.2 it has an orthonormal base. The converse is clear.

REMARK 1. We do not have a similar characterization of ' $E$  is isomorphic to a strict subspace of  $\ell^\infty$ '.

REMARK 2. The condition  $\|E'\| = |K|$  in (i) above is not equivalent to  $\|E\| = |K|$  ([3], 1.15).

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