SOME FIXED POINT THEOREMS IN METRIC AND BANACH SPACES

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1. Introduction. The purpose of this paper is two-fold. Sections 2 and 3 are motivated by an observation that certain theorems concerning "diminishing orbital diameters" (introduced in [1]) are true under weaker assumptions. Specifically, we investigate the relationship between that concept and alternate conditions such as "asymptotic regularity", and in the process we sharpen some metric space results established in [1;5]. Mention is made in these sections of examples which show that certain additional weakenings of our hypotheses cannot be made, but we include in detail only the one which seemed to us most intricate.

In the fourth section, some new fixed point theorems are obtained in Banach spaces. These theorems are for nonlinear mappings T for which $V = I - T$ is "convex" (that is, $\|V(\frac{x+y}{2})\| \leq 1/2[\|V(x)\| + \|V(y)\|]$ for $x, y \in \text{dom } T$). Although the assumption of convexity of $I - T$ is stringent, a significant feature of this section is the inclusion of examples showing that even in the presence of this assumption, other weakenings of our hypotheses are not possible.

2. Diminishing orbital diameters. Let f be a mapping of a metric space M into itself. For $x \in M$, let

$$O(x) = \{x, f(x), f^2(x), \dots\},$$

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and for $A \subset M$, let the diameter of A be:

$$\delta A = \sup\{ d(x,y) : x, y \in A\}$$

Definition [1]. If for each $x \in M$ it is the case that $\lim_n \delta O(f^n(x)) < \delta O(x)$ whenever $\delta O(x) > 0$, then f is said to have diminishing orbital diameters on M.

A number of results have been established for mappings of this type, perhaps the principal one being that every non-expansive mapping (i.e., a mapping f for which $d(f(x), f(y)) \leq d(x,y)$, $x, y \in \text{dom } f$) with diminishing orbital diameters which maps a weakly compact subset of a Banach space into itself always has a fixed point (see [1;6]; also see [7] for applications of this theorem).

One of the first observations made concerning mappings of this type was that the condition of diminishing orbital diameters has strong implications in compact settings.

THEOREM 2.1. *Let* $f: M \to M$ *be nonexpansive and have diminishing orbital diameters. Suppose for some* $x \in M$, $\lim_k f^{n_k}(x) = z$. *Then* $\lim_n f^n(x) = z$ *and* $z = f(z)$.

This theorem was proved in [1] (a slightly more general version appears in [5]) using a result of Edelstein [3] which states that for nonexpansive f, if $\lim_k f^{n_k}(x) = z$ then z generates an isometric sequence. (This means that $d(f^m(z), f^n(z)) = d(f^{m+k}(z), f^{n+k}(z))$, $k = 0, 1, 2, \ldots$.) More perceptive application of Edelstein's result makes it clear that the condition of diminishing orbital diameters is really not the crucial factor in the theorem, but rather that f be non-isometric on $O(x)$ whenever $\delta O(x) > 0$. Thus, one can prove:

THEOREM 2.2. *Let* $f: M \to M$ *be nonexpansive; for each* $x \in M$, *assume* f *is not an isometry on* $O(x)$ *if* $\delta O(x) > 0$. *If for some* $x \in M$, $\lim_k f^{n_k}(x) = z$, *then* $\lim_n f^n(x) = z$ *and* $f(z) = z$.

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Proof. If follows immediately from Edelstein's result that $\delta O((z)) = 0$, and thus that $f(z) = z$.

That Theorem 2.2. implies Theorem 2.1 is easily seen. If $f: M \to M$ has diminishing orbital diameters at x then $\delta O(f^n(x)) < \delta O(x)$ for some n if $\delta O(x) > 0$, and one may infer from this that f is not an isometry on $O(x)$.

3. Asymptotically regular mappings. In this section we shall be interested in making some observations about the relationship between mappings with the "non-isometric on orbits" condition of Theorem 2.2 and asymptotically regular mappings. As before, M denotes a metric space.

Definition [2]. A mapping $f: M \to M$ is asymptotically regular on M if $\lim_n d(f^n(x), f^{n+1}(x)) = 0$ for each $x \in M$.

Interest in asymptotically regular maps stems from the fact that strong conclusions may be drawn concerning convergence of sequences of iterates of such mappings to fixed points; see Browder-Petryshyn [2]. In compact settings, the connection with concepts studied here is firm.

REMARK 3.1. If M is compact and $f: M \to M$ nonexpansive, then the following are equivalent.

1) f has diminishing orbital diameters on M.

2) f is asymptotically regular on M.

3) f is not an isometry on $O(x)$ if $\delta O(x) > 0$, $x \in M$.

The equivalence of these conditions follows from the fact that (2) implies (3) trivially, and existing fixed point theorems imply every sequence of iterates converges to a fixed point of f under either assumption (1) or (3). It is also easy to see that in any case, if $f: M \to M$ is asymptotically regular then f satisfies (3). For if $x \neq f(x)$ and f is an isometry on $O(x)$ we have

$$0 \neq d(x, f(x)) = d(f(x), f^2(x)) = \cdots = d(f^n(x), f^{n+1}(x)) \to 0$$

which is absurd. Thus nonexpansiveness of f is not needed.

The following theorem is proved in [5].

THEOREM 3.1. Suppose M is compact and let $f: M \to M$ be continuous and have diminishing orbital diameters on M. Then every sequence $\{f^n(x)\}$ contains a subsequence which converges to a fixed point of f.

A similar result holds for asymptotically regular maps. We state this theorem without proof since it is a consequence of Theorem 3.3 and Remark 3.3.

THEOREM 3.2. Theorem 3.1 remains true if the assumption of diminishing orbital diameters is replaced by asymptotic regularity.

In general, a continuous mapping f may be asymptotically regular and yet not have diminishing orbital diameters, even in a compact set. Thus the hypothesis of Theorem 3.2 does not imply the hypothesis of Theorem 3.1. It is also the case that a mapping may have diminishing orbital diameters on a set, yet not be asymptotically regular. However, we note the following:

REMARK 3.2. If M is compact, $f: M \to M$ continuous with diminishing orbital diameters, then for each $x \in M$, if $\lim_n d(f^n(x), f^{n+1}(x))$ exists, it must be zero.

Proof. Let $x \in M$. By Theorem 3.1 some subsequence of $\{f^n(x)\}$ converges to a fixed point of f. Say $f^{n_k}(x) \to z = f(z)$. Thus,

$$\lim_k f^{n_k+1}(x) = z,$$

so

$$\lim_k d(f^{n_k}(x), f^{n_k+1}(x)) = 0.$$

Therefore, if the indicated limit exists, it must be zero.

That this limit need not always exist, however, is seen by the following example. In this example M is compact, $f: M \to M$ is continuous and has diminishing orbital diameters, but f is not asymptotically regular.

EXAMPLE 3.1. Consider the following points in the plane,

$$a_n = (1/n, 0), \ n = 1, 2, \dots ;$$

$$x_k = (0, 1/n), \ k = 2^n, \ n = 0, 1, 2, \dots ;$$

and define the remaining points x_n as indicated in the figure (Fig. 1). Then $M = \cup\{a_n, x_n, (0,0)\}$ is a compact subset of the plane. Define $f\colon M \to M$ as follows:

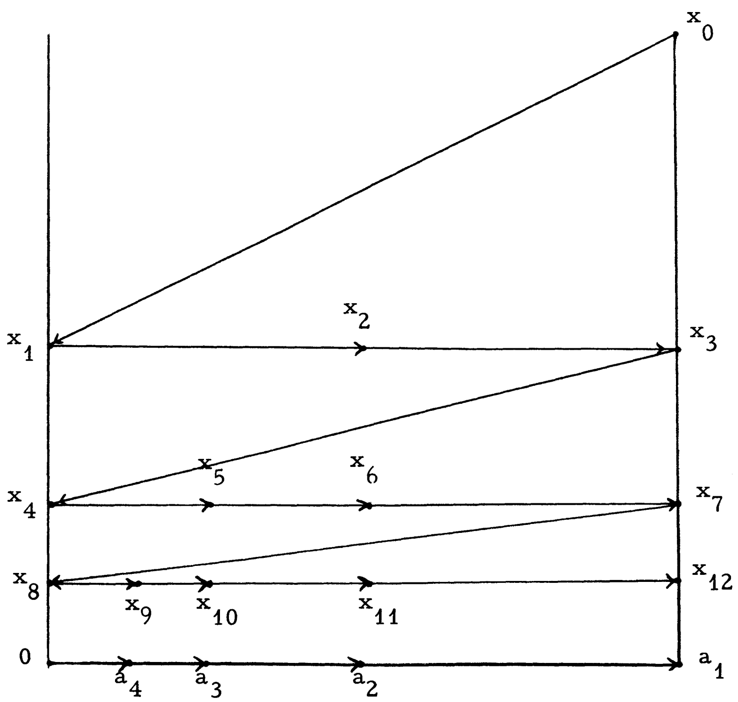

Figure 1

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$$f(x_n) = x_{n+1}, \quad n = 0, 1, 2, \ldots ;$$

$$f(a_n) = a_{n-1}, \quad n = 2, 3, \ldots ;$$

$$f(a_1) = (0, 0) = f((0, 0)).$$

Then $\delta O(f^n(x_k)) > 1$ while $\lim_n \delta O(f^n(x_k)) = 1$. Clearly, $\{d(f^{n+1}(x_o), f^n(x_o))\}$ does not converge.

A very trivial example shows that Theorem 3.2 is not true with "asymptotically regular" replaced by the condition of Theorem 2.2, namely "f is not isometric on O(x) whenever $\delta O(x) > 0$", and Theorem 3.1 and 3.2 cannot be jointly generalized using this condition. We can achieve a unifying of 3.1 and 3.2 as follows:

THEOREM 3.3. *Let M be compact and suppose $f: M \to M$ is continuous and has the property that for each $x \in M$ with $\delta O(x) > 0$ there exists an integer n such that $\overline{O(f^n(x))} \nsupseteq \overline{O(x)}$. Then every sequence $\{f^n(x)\}$ contains a subsequence which converges to a fixed point of f.*

Because its proof is a routine modification of the proof of Theorem 3.1 [5, Theorem II], we only show how Theorem 3.3 implies Theorem 3.1 and 3.2. In fact we can take a "mixed" hypothesis.

REMARK 3.3. Let $x \in M$ and suppose $f: M \to M$ is continuous. If either f has diminishing orbital diameters at x or if f is asymptotically regular at x, and if $\delta O(x) > 0$, then for some n, $\overline{O(f^n(x))} \nsupseteq \overline{O(x)}$.

Proof. Choose $x \in M$ with $\delta O(x) > 0$. If f has diminishing orbital diameters at x, then for some n, $\delta O(f^n(x)) < \delta O(x)$; hence $\overline{O(f^n(x))} \nsupseteq \overline{O(x)}$. Suppose, then, that f is asymptotically regular at x. $\delta O(x) > 0$ implies $x \neq f(x)$. Suppose $x \in \overline{O(f(x))}$. Then $x = \lim_k f^{n_k}(x)$ for some subsequence $\{f^{n_k}(x)\}$ of $\{f^n(x)\}$. Therefore $f(x) = \lim_k f^{n_k+1}(x)$. Hence

$$\lim_k d(f^{n_k}(x), f^{n_k+1}(x)) = d(x, f(x)).$$

Asymptotic regularity of f at x implies $x = f(x)$ contradicting $\delta O(x) > 0$; thus $x \notin \overline{O(f(x))}$ so $\overline{O(x)} \neq \overline{O(f(x))}$.

By the comment following the example we see that our "mixed" hypothesis in the above remark cannot contain as an alternative the "non-isometry on $O(x)$" condition. That is, being non-isometric on $O(x)$ is not sufficient to imply $\overline{O(x)} \neq \overline{O(f^n(x))}$ for some n. It can be shown, however, that the latter is true if f is nonexpansive.

4. Mappings in Banach spaces. Let K be a convex subset of a Banach space X. A mapping $V:K \to X$ is called *convex* if

$$\left\|V\left(\frac{x+y}{2}\right)\right\| \leq 1/2\left[\|V(x)\| + \|V(y)\|\right]$$

for all $x, y \in K$.

THEOREM 4.1. *Let K be a nonempty, weakly compact, convex subset of a Banach space X. Suppose $T:K \to K$ is continuous and suppose $I - T$ is convex on K. If $\inf_{x \in K} \|x - Tx\| = 0$, then T has a fixed point in K.*

Proof. For each $r \geq 0$, let

$$H_r = \{z \in K: \|z - Tz\| \leq r\}.$$

Because $\inf_{x\in K} \|x - Tx\| = 0$, $H_r \neq \emptyset$ if $r > 0$. Convexity of $I - T$ easily implies H_r is convex and continuity of T implies H_r is closed. As closed convex subsets of the weakly compact set K, the set H_r are also weakly compact and it follows that

$$\bigcap_{r>0} H_r \neq \emptyset$$

yielding at least one fixed point for T.

THEOREM 4.2. Let K be as in Theorem 4.1. Suppose $T: K \to K$ is nonexpansive and suppose $I - T$ is convex on K. Then T has a fixed point in K.

Proof. By a result of Göhde [8] we have $\inf_{x \in K} \|x - Tx\| = 0$; hence 4.2 follows from 4.1.

THEOREM 4.3. Let K and X be as in Theorem 4.1. Suppose $T: K \to K$ is nonexpansive and suppose $I - T$ is convex on K. If T is asymptotically regular on K, then every sequence $\{T^n x\}$ contains a subsequence which converges weakly to a fixed point of T, and moreover, every such weakly convergent subsequence has a fixed point of T as limit.

Proof. Let $x \in K$. Because K is weakly compact, some subsequence $\{T^{n_k}x\}$ of $\{T^n x\}$ converges weakly, say to w. Thus,

$$w \in \bigcap_{k=1}^{\infty} \overline{\operatorname{conv}}\{T^{n_k}x, T^{n_{k+1}}x, \dots\}.$$

The fact that $I - T$ is convex is readily seen to imply that, for each k,

$$\|w - Tw\| \leq \sup_{i \geq k} \|T^{n_i}x - T(T^{n_i}x)\|.$$

Therefore, using asymptotic regularity of T,

$$\|w - Tw\| \leq \lim_k \|T^{n_k}x - T(T^{n_k}x)\| = 0$$

and w is fixed under T.

EXAMPLE 4.1. Let B be the unit ball in any infinite dimensional Banach space X and let

$$B_{1/2} = \{x \in X: \|x\| \leq 1/2\}.$$

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Let φ be any continuous function mapping $B \to B_{1/2}$ which has no fixed point. (Because $B_{1/2}$ is not compact, such a mapping will exist.) Let $[x, \varphi(x)[$ be the half-line emanating from x which contains $\varphi(x)$. Then the length of $[x, \varphi(x)[\cap B$ is at least $1/2$. Define Tx to be the point of this ray with distance $1/2$ from x. Then $T:B \to B$ and $\|x - Tx\| \equiv 1/2$, $x \in B$. Thus $I - T$ is convex. T is continuous because φ is. Since X may be chosen so that B satisfies the hypotheses on K in Theorem 4.1 (e.g., assume X is reflexive), this example shows that $\inf \|x - Tx\| = 0$ cannot be removed in Theorem 4.1, nor can nonexpansiveness of T in Theorem 4.2 be replaced by continuity of T.

EXAMPLE 4.2. Now consider the space $\mathcal{C}[0,1]$ of continuous functions. Let

$$K = \{ f \in \mathcal{C}[0,1] : f(0) = 0,\ f(1) = 1,\ 0 \leq f(x) \leq 1 \}.$$

Define $\varphi : K \to K$ as follows:

$$\varphi(f)(x) = xf(x), \quad f \in K.$$

As seen in [4], φ is nonexpansive on K and has no fixed point. Let $f, g \in K$.

$$\left\|\frac{f+g}{2} - \varphi\left(\frac{f+g}{2}\right)\right\| = \sup\left\{ \left|\frac{f(x)+g(x)}{2} - x\left(\frac{f(x)+g(x)}{2}\right)\right| : x \in [0,1]\right\}$$

$$\leq \sup\left\{\frac{1}{2}(1 - x)f(x) : x \in [0,1]\right\}$$

$$+ \sup\left\{\frac{1}{2}(1 - x)g(x) : x \in [0,1]\right\}$$

$$= 1/2\{ \|f - \varphi(f)\| + \|g - \varphi(g)\| \}.$$

Thus $I - \varphi$ is convex on K. Also,

$$\|\varphi^n(f) - \varphi^{n+1}(f)\| = \sup\{ |x^n f(x) - x^{n+1} f(x)| : x \in [0,1]\}$$

$$= \sup\{ (x^n - x^{n+1}) f(x) : x \in [0,1]\}$$

$$\leq \sup\{ (x^n - x^{n+1}) : x \in [0,1]\} \leq \frac{1}{n+1},$$

so φ is asymptotically regular on K. Therefore the hypothesis of weak compactness is essential in each of the Theorems 4.1 - 4.3.

It might be noted that the procedure of Example 4.1 could be used to obtain a continuous map $T:B \to B$ such that $\|x - Tx\| \equiv k$, $x \in B$, for any fixed k, $0 < k < 1$. An interesting question arises as to whether such T exists for which $\|x - Tx\| \equiv 1$, $x \in B$.

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