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THE OPEN MAPPING AND CLOSED GRAPH THEOREM FOR EMBEDDABLE TOPOLOGICAL SEMIGROUPS

BY
DOUGLASS L. GRANT[(1)]

Some extensions of the open mapping and closed graph theorem are proved for certain classes of commutative topological semigroups, namely those embeddable as open subsets of topological groups. Preliminary results of independent interest include investigations of properties which "lift" from embeddable semigroups to the groups in which they are embedded, and from semigroup homomorphisms to homomorphisms of the groups.

1. **Definitions and preliminary results.** This paper extends the open mapping and closed graph theorem to a class of commutative topological semigroups, namely those which are embeddable as open subsets of their groups of quotients. Such semigroups have been characterized by Rothman [9], and further results concerning them appear in [1], [3] and [7].

Throughout the paper, the letters S, T will denote commutative, cancellative topological semigroups which are embeddable as open subsets of the topological groups G, H, respectively. By Theorem 2.2 of [7], there is no loss of generality in assuming that G, H are the respective groups of quotients. If $f:S\to T$ is a homomorphism (not necessarily continuous), its extension to these groups, as defined in [9], will be denoted by $h:G\to H$. The symbol $\mathbf{V}(S, x)$ will denote the filter of neighbourhoods of x in S. If G is a topological group and e its identity, $\mathbf{V}(G, e)$ will be denoted by $\mathbf{V}(G)$. All spaces are assumed Hausdorff. All other notation is as in [5].

Let $\mathbf{C}$ be a class of embeddable topological semigroups. We say that a semigroup S is a $B(\mathbf{C})$ semigroup if every continuous and almost open homomorphism from S onto a semigroup in $\mathbf{C}$ is open. The symbol $\mathbf{S}$ will denote the class of all embeddable topological semigroups. It is easy to see that each locally compact semigroup has the $B(\mathbf{S})$ property, since each continuous and almost open mapping of a locally compact topological space into a Hausdorff space is open.

REMARK. Before we establish another class of semigroups with the $B(\mathbf{S})$ property, we must observe that certain properties of an embeddable semigroup are

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preserved in the group of quotients. Various elementary arguments show that commutativity, local compactness, first countability, and separability are among the properties so preserved. Complete metrizability and the Baire property are also preserved, by p. 314 and p. 256 of [2], respectively.

The next result relates the properties of the homomorphism f of semigroups to those of its extension h.

PROPOSITION 1. *Let $f:S\to T$ be a homomorphism. If f is* (*a*) *continuous*, (*b*) *open*, (*c*) *almost continuous*, (*d*) *almost open*, (*e*) *one-to-one*, (*f*) *onto*, *or* (*g*) *endowed with the closed graph property, then $h:G\to H$ has the same property.*

Proof. The proof of (a) is embodied in that of Theorem 4.1 of [9], while (e) and (f) are obvious.

(b) Let $V\in \mathbf{V}(G)$. For any $s_0\in S$, $f(s_0V\cap S)$ is in $\mathbf{V}(T, f(s_0))$. Then,

$$f(s_0)^{-1}f(s_0V\cap S)\subseteq h(V),$$

and the latter is in $\mathbf{V}(H)$.

(c) Let $x\in G$, $B\in \mathbf{V}(H, h(x))$. Then $h(x)=h(s_1s_2^{-1})=f(s_1)f(s_2)^{-1}$, where $s_1, s_2\in S$. Then there are neighbourhoods C_1, C_2 of $f(s_1)$, $f(s_2)$, respectively, such that $C_1C_2^{-1}\subseteq B$. Then,

$$Cl_Gh^{-1}(B)\supseteq Cl_Sf^{-1}(C_1\cap T)[Cl_Sf^{-1}(C_2\cap T)]^{-1}.$$

Since $Cl_Sf^{-1}(C_i\cap T)$ is a neighbourhood of s_i in S, and so in G, for $i=1,2$, it follows that $Cl_Gh^{-1}(B)$ is a neighbourhood of x, and h is almost continuous.

(d) Let $V\in \mathbf{V}(G)$. Then, for any $a\in S$,

$$Cl_Hh(V) = Cl_Hh(a^{-1}aV)\supseteq h(a^{-1})Cl_Tf(aV\cap S),$$

and the latter is in $\mathbf{V}(H)$. Therefore, h is almost open.

(g) Denote the graphs of f and h by $R(f)$ and $R(h)$, respectively, and let $(a,y)\in Cl_{G\times H}R(h)$. Then, for any $U\times V\in \mathbf{V}(G\times H)$, we have $(aU\times yV)\cap R(h)\neq\varnothing$. Letting $s_0\in S$, we then see that

$$(s_0U\times h(s_0a^{-1})yV)\cap R(h)\neq\varnothing.$$

Since S is open in G, we may let $U=s_0^{-1}B$ and $V=f(s_0)^{-1}C$, where $B\in \mathbf{V}(S, s_0)$ and $C\in \mathbf{V}(T, f(s_0))$. For any such B, C, then, $(B\times h(a^{-1})yC)\cap R(f)$ is non-empty, whence, for each $(B, C)\in \mathbf{V}(S, s_0)\times \mathbf{V}(T, f(s_0))$, there exists $s_{B,C}\in B$ such that $f(s_{B,C})\in h(a)^{-1}yC$. Let $\mathbf{B}=\{s_{B,C}:(B, C)\in \mathbf{V}(S, s_0)\times \mathbf{V}(T, f(s_0))\}$. Since T is Hausdorff, $\bigcap \mathbf{V}(T, f(s_0))=\{f(s_0)\}$, and so $f(\mathbf{B})$ converges to $h(a)^{-1}yf(s_0)$. Clearly, $\mathbf{B}$ converges to s_0, and so $f(s_0)=h(a)^{-1}yf(s_0)$, since $R(f)$ is closed. Then, $y=h(a)$, and so $R(h)$ is closed.

PROPOSITION 2. *Every complete metrizable semigroup has the $B(\mathbf{S})$ property.*

Proof. Let S be such a semigroup. Then G is complete metrizable, by the Remark above, and so G is a $B(\mathbf{A})$ group, by Theorem 31.3 of [5]. If $f:S\to T$ is continuous, almost open and onto, its extension h also has these properties, by Proposition 1. Hence, h is open, and so is f.

Let $\mathbf{C}_1$ be the class of first countable semigroups.

PROPOSITION 3. *Every locally countably compact, and hence every countably compact, semigroup is a* $B(\mathbf{C}_1)$ *semigroup.*

This follows at once since a countably compact subset of a first countable space is closed [2, p. 230]. This result holds without our underlying assumption of embeddability.

2. **Closed graph theorems.** We now extend Theorem 1 of [6] to embeddable topological semigroups.

THEOREM 4. *Let P be a property which is transmitted from an embeddable semigroup to the associated group,* $\mathbf{C}$ *be the category of embeddable semigroups with property P,* $\mathbf{C}^*$ *the class of topological groups with the same property. Further assume that* $\mathbf{C}^*$ *is right fitting with respect to continuous, almost open homomorphisms. Let S, T be semigroups with $S \in \mathbf{C}$. If H is a $B_r(\mathbf{C}^*)$ group and if $f:S\to T$ is almost continuous, almost open, and has a closed graph, then f is continuous.*

Proof. By Proposition 1, the extension $h:G\to H$ of f is almost continuous and almost open, and has the closed graph property. Since $G \in \mathbf{C}^*$, it follows from Theorem 1 of [6] that h is continuous. Hence, its restriction f is also continuous.

In a like manner, the next result follows from Theorem 2 of [6] and Theorem 2.6 of [4].

THEOREM 5. *Let S, T be Hausdorff semigroups, $f:S\to T$ a homomorphism with closed graph. If H is a $B_r(\mathbf{A})$ group and f is almost continuous, then f is continuous. If G is a $B(\mathbf{A})$ group and f is almost open and onto, then f is open.*

We now generalize another result of Husain [5, Proposition 32.11] to the class of embeddable semigroups.

PROPOSITION 6. *If S is a (i) separable, or (ii) Lindelöf semigroup and T is a Baire semigroup, then any homomorphism f from S onto T is almost open, and any homomorphism g from T into S is almost continuous.*

Proof. (i) By the Remark above, G is separable and H is a Baire group. By Proposition 32.11 of [5], h is almost open, and it then follows that f is almost open. The other statement follows by a dual argument.

(ii) Let $B \in \mathbf{V}(S, x_0)$, $V \in \mathbf{V}(G)$ such that $V^{-1}V \subseteq x_0^{-1}B$. Then there exists a countable subset A of S such that $\{aV \cap S : a \in A\}$ covers S. Hence,

$$\{h(aV) \cap T : a \in A\}$$

covers T. Since T has the Baire property, for some $a' \in A$, $Cl_T[h(a'V) \cap T]$ has non-void interior. Let $p \in U \subseteq Cl_T[h(a'V) \cap T]$, where U is open in T. Then

$T \cap [h(x_0)p^{-1}U]$ is an open neighbourhood of $h(x_0)$, and furthermore,

$$\begin{aligned} T \cap [h(x_0)p^{-1}U] &\subseteq \{h(x_0)p^{-1}Cl_T[h(a'V) \cap T]\} \cap T \\ &\subseteq Cl_T[h(x_0)(h(a'V) \cap T)^{-1}(h(a'V) \cap T)] \\ &\subseteq Cl_T[h(x_0)h(V^{-1}V)] \subseteq Cl_T h(B) = Cl_T f(B). \end{aligned}$$

Hence, f is almost open.

Now, let $k:H \rightarrow G$ extend g, $a \in T$, $B \in \mathbf{V}(S, g(a))$, and select $V \in \mathbf{V}(G)$ such that $V^{-1}V \subseteq g(a)^{-1}B$. By an argument dual to that above, we then obtain an open $U \in \mathbf{V}(H)$ such that $U \subseteq Cl_G k^{-1}[g(a)^{-1}B]$. Then $aU \cap T$ is an open neighbourhood of a in T such that $aU \cap T \subseteq Cl_T[g^{-1}(B)]$, and so g is almost continuous.

With these results in hand, we can now prove a quite general version of the open mapping and closed graph theorem.

THEOREM 7. *Let S be any separable (or Lindelöf) semigroup, T any Baire semigroup. If S is a $B(\mathbf{S})$ semigroup, then any continuous homomorphism f from S onto T is open. If H is a $B_r(\mathbf{A})$ group, any homomorphism g from T into S with closed graph is continuous.*

Proof. By Proposition 6, f is almost open, and so it is open, since S is a $B(\mathbf{S})$ semigroup. The other statement follows directly from Proposition 6 and Theorem 5.

Since both the $B(\mathbf{S})$ property and the Baire property are implied by either local compactness or complete metrizability, and since either of the latter properties suffices to make the group of quotients a $B_r(\mathbf{A})$ group, we may freely impose either on S or T in Theorem 7 to obtain special cases of the theorem. Nothing is gained by proceeding to the compact case, however, for a compact, cancellative topological semigroup is a topological group [8].

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COLLEGE OF CAPE BRETON,
SYDNEY, NOVA SCOTIA.