FREE BANDS AND FREE *-BANDS

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1. Introduction and summary. The word problem for free bands (idempotent semigroups) was solved by Green and Rees [4] in an early paper. They also established certain properties of the free band. This was followed by McLean [6] who provided a general structure theory for bands with some indication as to the structure of the free band. Since then a great many papers have appeared dealing with various aspects of the topic of bands and their varieties. A different solution of the word problem for free bands was recently given by Siekmann and Szabó [9]. For a discussion of bands, see the books [5] and [8].

A *-semigroup is a semigroup provided with a unary operation satisfying the identities $(xy)^* = y^*x^*$, $x^{**} = x$, $x = xx^*x$. The subject of *-semigroups is of much more recent date; it was inaugurated by the paper of Nordahl and Scheiblich [7]. Varieties of *-bands were completely determined by Adair [1], whereas finitely generated free *-bands were studied by Yamada [10]. Free involutorial completely simple semigroups, including free completely simple *-semigroups, were constructed by the authors in [3].

In view of the considerable existing knowledge about bands in general and free bands in particular, it may seem surprising that something new could be said about the structure of the latter. It should come as a lesser surprise that when the structure of a free band has been better understood, its treatment and presentation have become more transparent and considerably simpler. In Section 2 we introduce a new invariant, which could be called a "band reduced" form of a word, which makes it possible to find a copy of the free band analogous to the form that a free group is usually presented. In Section 3 we represent the free band in terms of the structure theorem for arbitrary bands.

The treatment of the free band is also amenable to the case of free *-bands. There are some nontrivial modifications, but the general methods are essentially the same. Hence, in Section 4, we solve the word problem for free *-bands and construct a model for them. We provide, in Section 5, a structure theorem for arbitrary *-bands which is then specialized to give a representation of a free *-band.

The methods and notation we use are closely related to those we employed in the solution of the word problem for orthogroups in [2]. In general, we follow the notation and terminology of the books [5] and [8]. In particular $|A|$ is the cardinality of the set A and $A \setminus B$ is the set theoretic difference of A and B. For a set X, $\mathcal{T}(X)[\mathcal{T}'(X)]$ is the set of functions from X into X written on the right [left] and composed as such.

2. The word problem and a model of a free band. We give here a new description of the free band on a set. Our method has several advantages over the treatment which is already in existence. It produces a canonical form for each word and makes it possible to

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represent the free band in the form of the usual structure theorem for arbitrary bands. This construction provides a convenient frame for a discussion of the free *-band in Section 4.

Let X be a fixed set. The word problem for the free band is based on the free semigroup $F = F(X)$ on X. As usual the elements of F are referred to as *words* and the elements of X are called *variables*. A word $w \in F$ is thought of as a finite product of variables. We sometimes use the free monoid on X obtained by adjoining the empty word $\varnothing$ to F.

NOTATION 2.1. For $w \in F$ let $c(w)$, the *content* of w, be the set of variables in w. Write $w = uxp$ where $c(w) = c(ux)$, $c(w) \neq c(u)$ and $x \in X$. Let $s(w) = u$ and $\sigma(w) = x$. Note that $s(w) = \varnothing$ if $c(w) = \{x\}$. Thus $s(w)\sigma(w)$ is the shortest left cut of w containing all variables of w. Dually write $w = qyv$ where $c(w) = c(yv)$, $c(w) \neq c(u)$ and $y \in X$. Let $\varepsilon(w) = y$ and $e(w) = v$. Thus $\varepsilon(e)e(w)$ is the shortest right cut of w containing all variables of w.

For convenience we treat c, s, σ etc. as operators and omit parentheses. For example $c\sigma s(w)$ means $c(\sigma(s(w)))$. The operators s, σ, ε, e are combined to form $b: F \to F$ which is defined by

$$b(w) = bs(w)\sigma(w)\varepsilon(w)be(w) \qquad (= b(s(w))\sigma(w)\varepsilon(w)b(e(w))).$$

This is an inductive definition on $|c(w)|$. In particular $b(x) = xx$, since $s(x) = e(x) = \varnothing$. Note that $cb(w) = c(w)$.

LEMMA 2.2. *The above mappings have the following properties.*

(i) $bs = sb$, $be = eb$.

(ii) $\sigma b = \sigma$, $\varepsilon b = \varepsilon$.

(iii) $b^2 = b$.

Proof. Let $u \in F$.

(i) If $u = x^k$ for some $k \geq 1$, then

$$bs(x^k) = \varnothing = s(xx) = sb(x^k).$$

If $|c(u)| > 1$, then

$$sb(u) = s(bs(u)\sigma(u)\varepsilon(u)be(u)) = bs(u).$$

This proves that $bs = sb$; the equality $be = eb$ follows dually.

(ii) If $u = x^k$ for some $k \geq 1$, then

$$\sigma b(x^k) = \sigma(xx) = x = \sigma(x^k).$$

If $|c(u)| > 1$, then

$$\sigma b(u) = \sigma(bs(u)\sigma(u)\varepsilon(u)be(u)) = \sigma(u).$$

This shows that $\sigma b = \sigma$; the equality $\varepsilon b = \varepsilon$ follows dually.

(iii) The argument is by induction on $|c(u)|$. If $u = x^k$ for some $k \geq 1$, then

$$b(x^k) = xx = b(xx) = bb(x^k).$$

If $|c(u)| > 1$, then

$$\begin{aligned} bb(u) &= bsb(u)\sigma b(u)\varepsilon b(u)beb(u) \\ &= b^2 s(u)\sigma(u)\varepsilon(u)b^2 e(u) && \text{by parts (i) and (ii)} \\ &= bs(u)\sigma(u)\varepsilon(u)be(u) && \text{by the induction hypothesis} \\ &= b(u) \end{aligned}$$

Therefore $b^2 = b$, as required.

Notation 2.3. We extend the definition of s, σ, ε, e as follows:

$$\begin{aligned} &s^0(w) = w, && e^0(w) = w. \\ &s^{k+1}(w) = s(s^k(w)), && e^{k+1}(w) = e(e^k(w)) \quad \text{for all } k \geq 0. \\ &\sigma^{k+1}(w) = \sigma(s^k(w)), && \varepsilon^{k+1}(w) = \varepsilon(e^k(w)) \quad \text{for all } k \geq 0. \end{aligned}$$

(For large enough k, $s^k(w)$, $\sigma^k(w)$, $\varepsilon^k(w)$, $e^k(w)$ are equal to $\varnothing$.)

Define $s^A(w)$ and $\sigma^A(w)$ so that $\sigma^A(w) \in X$ and $s^A(w)\sigma^A(w)$ is the shortest left cut of w containing all variables of $c(w)\setminus A$. Define dually $e^A(w)$ and $\varepsilon^A(w)$ so that $\varepsilon^A(w) \in X$ and $\varepsilon^A(w)e^A(w)$ is the shortest right cut of w containing all variables of $c(w)\setminus A$. (Again these may be $\varnothing$.)

Lemma 2.4. *Let $u \in F$ and A be a nonempty subset of X. Then*

$$s^A(u) = s^k(u), \qquad \sigma^A(u) = \sigma^k(u).$$

where k is the least integer such that $\sigma^k(u) \notin A$.

Proof. Note that if $c(u) \subseteq A$, then $\sigma^A(u) = \varnothing$ and if $|(c(u)\setminus A| \leq 1$, then $s^A(u) = \varnothing$. If $t = |c(u)|$, then

$$(\sigma^t(u), \ldots, \sigma(u))$$

is the sequence of varibles of u in order of first occurrence. By definition of k, $s^k(u)\sigma^k(u)$ is the shortest left cut of u containing all variables of u not in A. Therefore $s^k(u)\sigma^k(u) = s^A(u)\sigma^A(u)$ and the lemma follows.

Lemma 2.5. *For all $k \geq 1$, $\sigma^k b = \sigma^k$.*

Proof. The argument is by induction on k. For $k = 1$, the statement is just Lemma 2.2(ii). For $k \geq 1$ and $u \in F$,

$$\begin{aligned} \sigma^{k+1}b(u) &= \sigma s^k b(u) \\ &= \sigma b s^k(u) && \text{by Lemma 2.2(i)} \\ &= \sigma s^k(u) && \text{by case } k = 1 \\ &= \sigma^{k+1}(u). \end{aligned}$$

COROLLARY 2.6. *For any nonempty set A of X, we have*

$$s^A b = bs^A, \qquad \sigma^A b = \sigma^A.$$

Proof. Lemmas 2.4 and 2.5 imply that $s^A(bu) = s^k(bu)$ and $s^A(u) = s^k(u)$ for the same k. The first formula therefore follows from Lemma 2.2(i). A similar argument yields the second formula using Lemma 2.2(ii).

LEMMA 2.7. *Let τ be any congruence on F such that F/τ is a band. Then for any $u \in F$,*

$$u \quad \tau \quad s(u)\sigma(u)\varepsilon(u)e(u).$$

Proof. We show first that

$$c(u) = c(v) \Rightarrow u\tau \quad \mathcal{D} \quad v\tau.$$

Let $c(u) = c(v) = \{x_1, \ldots, x_n\}$. Then $u\tau \quad \mathcal{D} \quad (x_1\tau)\ldots(x_n\tau) \quad \mathcal{D} \quad v\tau$, since $(F/\tau)/\mathcal{D}$ is a semilattice. It then follows that

$$c(u) \subseteq c(v) \Rightarrow D_{v\tau} \leq D_{u\tau}$$

since $c(y) \subseteq c(v)$ implies $c(uv) = c(v)$ so that $D_{v\tau} \leq D_{(uv)\tau} \leq D_{u\tau}$.

To prove the lemma note that since F/τ is a band,

$$u \quad \tau \quad uu = s(u)\sigma(u)w\varepsilon(u)e(u),$$

for some $w \in F \cup \{\varnothing\}$ with $c(w) \subseteq c(u) = c(s(u)\sigma(u)) = c(\varepsilon(u)e(u))$.

NOTATION 2.8. Let $B = b(F)$ with the multiplication

$$u \,.\, v = b(uv).$$

Note that in view of Lemma 2.2(iii), we have

$$B = \{w \in F \mid b(w) = w\}.$$

We now arrive at the first representation of the free band.

THEOREM 2.9. *The mapping b is a homomorphism of F onto B and B is a free band on X.*

Proof. According to the definition of multiplication in B, we must prove that for any $u, v \in F$,

$$b(u)\,.\,b(v)(=b(b(u)b(v))) = b(uv). \tag{1}$$

Applying the definition of b, we thus have to show that

$$[bs(b(u)b(v))][\sigma(b(u)b(v))][\varepsilon(b(u)b(v))][be(b(u)b(v))]$$
$$= [bs(uv)][\sigma(uv)][\varepsilon(uv)][be(uv)]. \tag{2}$$

The argument is by induction on $|c(u) \cup c(v)|$. For the first step, we have $u = x^m$ and

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$v = x^n$ for some $x \in X$ and $m, n \geq 1$. We now compute

$$b(b(x^m)b(x^n)) = b(xxxx) = xx = b(x^m x^n).$$

For the inductive step, we first prove that

$$bs(b(u)b(v)) = bs(uv) \tag{3}$$

On the one hand, we have

$$bs(b(u)b(v)) = \begin{cases} bsb(u) & \text{if } c(v) \subseteq c(u), \\ b(b(u)s^{c(u)}b(v)) & \text{otherwise,} \end{cases}$$

and on the other hand,

$$bs(uv) = \begin{cases} bs(u) & \text{if } c(v) \subseteq c(u), \\ b(us^{c(u)}(v)) & \text{otherwise.} \end{cases}$$

The desired equality in the case $c(v) \subseteq c(u)$ follows directly from Lemma 2.2(i)(iii). For the case $c(v) \nsubseteq c(u)$, $s^{c(u)}b(v) \neq s^0(b(v))$ so that $|c(b(u)s^{c(u)}b(v))| < |c(u) \cup c(v)|$, and analogously $|c(us^{c(u)}v)| < |c(u) \cup c(v)|$, and we may use the induction hypothesis. Indeed,

$$\begin{aligned} b(b(u)s^{c(u)}b(v)) &= b^2(u)bs^{c(u)}b(v) && \text{by the induction hypothesis} \\ &= b(u)bs^{c(u)}(v) && \text{by Lemma 2.2(iii) and Corollary 2.6} \\ &= b(us^{c(u)}(v)) && \text{by the induction hypothesis.} \end{aligned}$$

This proves (3).

Next we show that

$$\sigma(b(u)b(v)) = \sigma(uv). \tag{4}$$

Indeed,

$$\begin{aligned} \sigma(b(u)b(v))) &= \begin{cases} \sigma b(u) & \text{if } c(v) \subseteq c(u) \\ \sigma^{c(u)}b(v) & \text{otherwise} \end{cases} \\ &= \begin{cases} \sigma(u) & \text{if } c(v) \subseteq c(u) & \text{by Lemma 2.2(ii)} \\ \sigma^{c(u)}(v) & \text{otherwise} & \text{by Corollary 2.6} \end{cases} \\ &= \sigma(uv) \end{aligned}$$

which proves (4).

Relation (3) and (4) imply the equality of the first and second brackets on the left and on the right in (2). The equality of the remaining two pairs of brackets follows by duality. This proves (2) and thus also (1), as required.

To prove that B is a free band, we show that if $\bar{b}$ is the congruence induced on F by b, then $\bar{b} \subseteq \tau$ for any congruence τ on F such that F/τ is a band.

Assume $b(u) = b(v)$. By Lemma 2.2(i)(ii) we have

$$bs(u) = sb(u) = sb(v) = bs(v),$$
$$\sigma(u) = \sigma b(u) = \sigma b(v) = \sigma(v).$$

The proof that $u \quad \tau \quad v$ is by induction on $|c(u) \cup c(u)|$. Note that $x^n \quad \tau \quad x^m$ for $x \in X$ since F/τ is a band. By induction we have $s(u) \quad \tau \quad s(v)$ since $bs(u) = bs(v)$ as we just proved and $c(s(u)) \nsubseteq c(u)$. Using this, the fact that $\sigma(u) = \sigma(v)$ (as just shown) and the dual results gives

$$s(u)\sigma(u)\varepsilon(u)e(u) \quad \tau \quad s(v)\sigma(v)\varepsilon(v)e(v).$$

Finally, an application of Lemma 2.7 shows that $u \quad \tau \quad v$.

REMARK 2.10. A straightforward inductive argument can be used to show that the congruence induced on F by b is the congruence β defined inductively as follows:

$$u \quad \beta \quad v \Leftrightarrow s(u) \quad \beta \quad s(v), \qquad \sigma(u) = \sigma(v), \qquad \varepsilon(u) = \varepsilon(v), \qquad e(u) \quad \beta \quad e(v),$$

where we formally set $\varnothing \quad \beta \quad \varnothing$.

This β actually coincides with β defined in ([**5**], Lemma IV.4.6) where the additional requirement that $c(u) = c(v)$ is superfluous. This can be easily proved by induction.

We now describe an algorithm for computing $b(w)$.

Step 1: Double the word w to ww.

Step 2: Write $ww = w_0x_0w'x_1w_1$ where $x_0 \in X$ and w_0x_0 is the shortest left cut of ww (or w) which contains all variables of w and $x_1 \in X$ and x_1w_1 is the shortest right cut of ww (or w) which contains all variables of w. This factorization is possible because of the form of ww.

Step 3: Delete the word w' retaining the word $w_0x_0x_1w_1$. Note that this amounts to deleting all letters in ww which occur earlier and later in the word ww.

Step 4: Apply Steps 1, 2, and 3 to w_0 thereby obtaining the words $w_{00}x_{00}$ and $x_{01}w_{01}$. Also apply the same steps to w_1 thereby obtaining the words $w_{10}x_{10}$ and $x_{11}w_{11}$.

Continue this procedure on w_{00}, w_{01}, w_{10} and w_{11} until the end. This procedure must finish since at each step, the content of each $w_{i_1 \ldots i_k}$ is one less than in the preceding step.

We thus arrive at a word of the form

$$x_{00\ldots0} \ldots x_{00}x_{01}x_0x_1x_{10}x_{11} \ldots x_{11\ldots1},$$

where the number of 0's in the first subscript equals the number of 1's in the last subscript which in its turn equals $|c(w)|$. This word is our $b(w)$. The length of $b(w)$ is $2(2^{|c(w)|} - 1)$.

From the above steps one can easily devise a test for a word to have the property that $b(w) = w$.

We illustrate the above procedure by the following example. Let $w = (xyxy^3xz)^2y$. The algorithm consists, briefly, of two steps: doubling and taking certain subwords. In each of the following steps the underlined subword is doubled in the next line.

Step 1: $(xyxy^3xz)^2y(xyxy^3xz)^2y$ doubling w.

Step 2: $\underline{xyxy^3x}\ zx\ \underline{zy}$ deleting

(where $s(w) = xyxy^3x$, $\sigma(w) = z$, $\varepsilon(w) = x$, $e(w) = zy$).

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Step 3: $xyxy^3xxyxy^3xzxzyzy$ doubling.

Step 4: $\underset{\sim}{x}\,yy\,\underset{\sim}{x}\,zx\,\underset{\sim}{z}\,yz\,\underset{\sim}{y}$ deleting.

Step 5: $xxyyxxzxzzyzyy$ doubling.

$b(w) = xxyyxxzxzzyzyy$

A different canonical form for words in the treatment of the free band was devised by Siekmann and Szabó [**9**]. The attraction of their approach is that the canonical word is the unique shortest word representing an element of the free band. They show that a word can be reduced to the shortest form by replacing uu by u and pqr by pr if $c(q) \subseteq c(p) = c(r)$. It may be of some interest that our canonical word can be reduced to theirs by making only the substitutions of the form uu by u.

Our expanded canonical word may be much longer than theirs, that is the word of least length. But our canonical word reflects the intrinsic nature of the usual solution of the word problem. The principal advantage of our treatment is that at every step the procedure is unique and mechanical; there is no search for suitable reductions and so it is easier to apply. In particular, the route which leads from a word to its canonical form is unique.

We also compute the reduced form of w by the methods of [**9**] in two different ways.

Step 1: $xyxy^3xzy$ replace u^2 by $u = xyxy^3xz$

Step 2: $xyxy^2xzy$ replace y^2 by y

Step 3: $xyxyxzy$ replace y^2 by y

Step 4: $xyxzy$ replace $(xy)^2$ by xy.

Step 1: $xyxy^3xzy$ replace u^2 by $u = xyxy^3xz$

Step 2: xyy^3xzy replace $(xy)x(y^3x)$ by xyy^3x since $c(x) \subseteq c(xy) = c(y^3x)$

Step 3: $xyyxzy$ replace $xy(y)^2yx$ by $xyyx$

Step 4: $xyxzy$ replace yy by y.

The word $b(w)$ is transformed into its reduced form by several replacements of u^2 by u as follows.

$$(xx)(yy)(xx)zx(zz)yz(yy)$$
$$xy(xzxz)yzy$$
$$xyx(zyzy)$$
$$xyxzy.$$

3. The structure of a free band. Our aim here is to express the free band on a set in terms of the usual structure theorem for arbitrary bands. To this end, we start with some statements concerning words, then state the structure theorem for bands, and finally

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construct the desired representation of the free band. The final result shows transparently how the representation was put together in terms of words.

NOTATION 3.1. In what follows we need some combinations of the invariants introduced in Notation 2.1. For $w \in F$, let

$$\lambda(w) = bs(w)\sigma(w),$$
$$\rho(w) = \varepsilon(w)be(w).$$

In particular $\lambda(x) = x = \rho(x)$.

LEMMA 3.2. *Let A be a nonempty subset of X.*
(i) $\lambda b = \lambda = \lambda^2, \quad \rho b = \rho = \rho^2.$
(ii) $bs^A = s^A\lambda, \quad be^A = e^A\rho.$
(iii) $\sigma^A = \sigma^A\lambda, \quad \varepsilon^A = \varepsilon^A\rho.$

Proof. These statements are refinements of Lemma 2.2 and Corollary 2.6. Their proofs can be obtained with minor modifications from those already given. For example to prove part of (ii) compare with Corollary 2.6 and compute

$$\begin{aligned} bs^A(u) &= s^A b(u) = s^k b(u) = s^k(\lambda(u)\rho(u)) \\ &= s^k\lambda(u) = s^A\lambda(u). \end{aligned}$$

LEMMA 3.3. *Let $u, v \in F$.*
(i) $\lambda(uv) = \lambda(u\lambda(v)) = \lambda(ub(v)) = \lambda(b(u)v).$
(ii) $\rho(uv) = \rho(\rho(u)v) = \rho(b(u)v) = \rho(ub(v)).$

Proof. We only prove part (i). Statement (ii) is the dual. There are two cases to consider.

Case 1: $c(v) \subseteq c(u)$. In this case v does not influence $\lambda(uv)$ since $s(uv) = s(u)$ and $\sigma(uv) = \sigma(u)$. Therefore

$$\lambda(uv) = \lambda(u) = \lambda(u\lambda(v)) = \lambda(ub(v)).$$

Similarly $\lambda(b(u)v) = \lambda(b(u))$ and thus the result follows by Lemma 3.2(i).

Case 2: $c(v) \nsubseteq c(u)$. In this case

$$s(uv) = us^{c(u)}(v), \qquad \sigma(uv) = \sigma^{c(u)}(v).$$

Therefore

$$\begin{aligned} \lambda(uv) &= b(us^{c(u)}(v))\sigma^{c(u)}(v) \\ &= b(b(u)bs^{c(u)}(v))\sigma^{c(u)}(v) \\ &= b(b(u)bs^{c(u)}\lambda(v))\sigma^{c(u)}\lambda(v) \quad \text{by Lemmas 2.2(iii) and 3.2(ii).} \\ &= b(us^{c(u)}\lambda(v))\sigma^{c(u)}\lambda(v) \\ &= \lambda(u\lambda(v)). \end{aligned}$$

The proof that $\lambda(uv) = \lambda(ub(v))$ is similar.

To prove that $\lambda(uv)=\lambda(b(u)v)$ we proceed by induction on $|c(v)|$. If $v=x^r$, then $x\notin c(u)$ and

$$\lambda(ux^r)=b(u)x=\lambda(b(u)x^r).$$

If $|c(v)|>1$, then as in the earlier parts of this proof,

$$\begin{aligned}\lambda(uv)&=b(us^{c(u)}(v))\sigma^{c(u)}(v)\\&=\lambda(us^{c(u)}(v))\rho(us^{c(u)}(v))\sigma^{c(u)}(v), \quad \text{by definition of } b,\\&=\lambda(b(u)s^{c(u)}(v))\rho(b(u)s^{c(u)}(v))\sigma^{c(u)}(v)\end{aligned}$$

(the first substitution $u\to b(u)$ is by induction and the second follows from the formula $\rho(uv)=\rho(b(u)v)$ which is dual to the formula $\lambda(uv)=\lambda(ub(v))$ proved above)

$$\begin{aligned}&=b(b(u)s^{c(u)}(v))\sigma^{c(u)}(v)\\&=\lambda(b(u)v).\end{aligned}$$

We will now represent the free band on X in terms of the usual structure theorem for bands, which we quote below.

For any set X and $F\in\mathcal{T}'(X)$, the notation $\langle F\rangle$ means that F is a constant, and $Fx=\langle F\rangle\in X$ for all $x\in X$. Similarly for $\Phi\in\mathcal{T}(X)$, $\langle\Phi\rangle$ means $x\Phi=\langle\Phi\rangle$ for all $x\in X$.

THEOREM 3.4 ([**8**], Theorem II.1.6). *Let Y be a semilattice; for every $\alpha\in Y$ let L_α and R_α be nonempty sets such that $L_\alpha\cap L_\beta=R_\alpha\cap R_\beta=\varnothing$ if $\alpha\neq\beta$ and let $S_\alpha=L_\alpha\times R_\alpha$. For $\alpha\geq\beta$ let the following functions be given*

$$F_{\beta,\alpha}:S_\alpha\to\mathcal{T}'(L_\beta)\quad \textit{with}\quad F_{\beta,\alpha}:(i,\mu)\to F^{(i,\mu)}_{\beta,\alpha},$$
$$\Phi_{\alpha,\beta}:S_\alpha\to\mathcal{T}(R_\beta)\quad \textit{with}\quad \Phi_{\alpha,\beta}:(i,\mu)\to \Phi^{(i,\mu)}_{\alpha,\beta}.$$

Assume that for any $\alpha,\beta\in Y$ and $(i,\mu)\in S_\alpha$, $(j,\nu)\in S_\beta$, the following conditions are satisfied.

(i) $\langle F^{(i,\mu)}_{\alpha,\alpha}\rangle=i$, $\langle\Phi^{(i,\mu)}_{\alpha,\alpha}\rangle=\mu$,

(ii) $\langle F^{(i,\mu)}_{\alpha\beta,\alpha}F^{(j,\nu)}_{\alpha\beta,\beta}\rangle=k$, $\langle\Phi^{(i,\mu)}_{\alpha,\alpha\beta}\Phi^{(j,\nu)}_{\beta,\alpha\beta}\rangle=\xi$ *for some* $(k,\xi)\in S_{\alpha\beta}$.

(iii) *in the notation of* (ii), *for any* $\gamma<\alpha\beta$

$$F^{(k,\xi)}_{\gamma,\alpha\beta}=F^{(i,\mu)}_{\gamma,\alpha}F^{(j,\nu)}_{\gamma,\beta},\qquad \Phi^{(k,\xi)}_{\alpha\beta,\gamma}=\Phi^{(i,\mu)}_{\alpha,\gamma}\Phi^{(j,\nu)}_{\beta,\gamma}.$$

Let $S=\bigcup_{\alpha\in Y}S_\alpha$, and with the same notation, define $\circ$ on S by

$$(i,\mu)\circ(j,\nu)=(k,\xi).$$

Then S is a band, and conversely, every band can be so constructed. As notation let $S=B(Y;L_\alpha,R_\alpha,F_{\beta,\alpha},\Phi_{\alpha,\beta})$.

We are now in a position to construct a band S, using the structure theorem for bands, which will then be shown to be the free band on X.

Let Y be the free semilattice on X thought of as the set of all finite non-empty subsets of X under union. For $A \in Y$ let

$$L_A = \{w \in F \mid \lambda(w) = w,\ c(w) = A\},$$
$$R_A = \{w \in F \mid \rho(w) = w,\ c(w) = A\}.$$

For $C, D \in Y$, $C \subseteq D$, let $F_{D,C} : S_C = L_C \times R_C \to \mathcal{T}'(L_D)$ be defined by

$$F_{D,C}^{(u,v)}(w) = \lambda(uvw) \quad (w \in L_D,\ (u, v) \in S_C),$$

and let $\Phi_{C,D} : S_C \to \mathcal{T}(R_D)$ be defined by

$$(w)\Phi_{C,D}^{(u,v)} = \rho(wuv) \quad (w \in R_D,\ (u, v) \in S_C).$$

LEMMA 3.5. *The maps $F_{D,C}$ and $\Phi_{C,D}$ defined above satisfy the conditions of Theorem* 3.4 *and therefore can be used to define a band $S = B(Y; L_A, R_A, F_{D,C}, \Phi_{C,D})$.*

Proof. Assume $C, D \in Y$ and $(u, v) \in S_C$, $(p, q) \in S_D$.

(i) $F_{C,C}^{(u,v)}(w) = \lambda(uvw) = \lambda(u) = u$

since $c(u) = c(v) = c(w) = C$. Therefore $\langle F_{C,C}^{(u,v)} \rangle = u$. Dually $\langle \Phi_{C,C}^{(u,v)} \rangle = v$.

(ii) $F_{C\cup D,C}^{(u,v)} F_{C\cup D,D}^{(p,q)}(w) = F_{C\cup D,C}^{(u,v)}(\lambda(pqw))$

$$= \lambda(uv\lambda(pqw)) = \lambda(uvpqw) \quad \text{by Lemma 3.3(i)}$$
$$= \lambda(uvp) \text{ since } c(w) = c(uv) \cup c(pq) = c(u) \cup c(p).$$

Therefore

$$\langle F_{C\cup D,C}^{(u,v)} F_{C\cup D,D}^{(p,q)} \rangle = \lambda(uvp)$$

and dually

$$\langle \Phi_{C,C\cup D}^{(u,v)} \Phi_{D,C\cup D}^{(p,q)} \rangle = \rho(vpq).$$

(iii) For $A \supseteq C \cup D$, we obtain

$$F_{A,C\cup D}^{(\lambda(uvp),\rho(vpq))}(w) = \lambda(\lambda(uvp)\rho(vpq)w)$$
$$= \lambda(b(uvpq)w) = \lambda(uvpqw) \quad \text{by Lemma 3.3(i).}$$
$$F_{A,C}^{(u,v)} F_{A,D}^{(p,q)}(w) = \lambda(uv\lambda(pqw))$$
$$= \lambda(uvpqw) \quad \text{by Lemma 3.3(i).}$$

The other condition is the dual.

We are now ready for the second representation of the free band.

THEOREM 3.6. *The mapping χ defined by*

$$\chi : w \to (\lambda(w), \rho(w)) \quad (w \in F)$$

is a homomorphism of F onto S, as given in Lemma 3.5, *which induces the same congruence as b. Thus S is a free band on X.*

Proof. Let $u, v \in F$ and denote $C = c(u)$, $D = c(v)$. Then

$$\langle F_{C\cup D,C}^{(\lambda(u),\rho(u))} F_{C\cup D,D}^{(\lambda(v),\rho(v))} \rangle = \lambda(\lambda(u)\rho(u)\lambda(v)), \tag{1}$$
$$\langle \Phi_{C,C\cup D}^{(\lambda(u),\rho(u))} \Phi_{D,C\cup D}^{(\lambda(v),\rho(v))} \rangle = \rho(\rho(u)\lambda(v)\rho(v)). \tag{2}$$

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In addition, by Lemma 3.3(i),

$$\lambda(\lambda(u)\rho(u)\lambda(v))=\lambda(b(u)\lambda(v))=\lambda(uv) \tag{3}$$

and analogously

$$\rho(\rho(u)\lambda(v)\rho(v))=\rho(uv). \tag{4}$$

We compute

$$\begin{aligned}\chi(u)\chi(v)&=(\lambda(u),\rho(u))(\lambda(v),\rho(v))\\&=(\lambda(\lambda(u)\rho(u)\lambda(v)),\rho(\rho(u)\lambda(v)\rho(v))) &&\text{by (1), (2)}\\&=(\lambda(uv),\rho(uv)) &&\text{by (3), (4)}\\&=\chi(uv)\end{aligned}$$

and χ is indeed a homomorphism.

Next let $(u, v)\in S$. Then $c(u)=c(v)$ which implies that $\lambda(uv)=\lambda(u)=u$ and $\rho(uv)=\rho(v)=v$. It follows that $\chi(uv)=(u, v)$ and χ maps F onto S.

That χ and b induce the same congruence on F means

$$\chi(u)=\chi(v)\Leftrightarrow b(u)=b(v)\quad (u, v\in F). \tag{5}$$

In view of (5) and the fact that $b(w)=\lambda(w)\rho(w)$, it remains to prove that

$$\lambda(u)=\lambda(v),\qquad \rho(u)=\rho(v)\Leftrightarrow\lambda(u)\rho(u)=\lambda(v)\rho(v)\quad (u, v\in F). \tag{6}$$

The direct implication is trivial. It was established in the proof of Theorem 2.9 that $b(u)=b(v)$ implies $bs(u)=bs(v)$ and $\sigma(u)=\sigma(v)$. This together with the duals of these statements proves the converse implication in relation (6).

The final assertion of the theorem now follows from Theorem 2.9.

4. A construction of a free *-band. An *involutorial semigroup* is a pair $(S, {}^*)$ where S is a semigroup and * is an involution on S, that is S satisfies the identities $(xy)^*=y^*x^*$ and $x^{**}=x$. If, in addition, S satifises the identities $x=x^2$ and $x=xx^*x$, then $(S, {}^*)$, or simply S, is a **-band*.

Our discussion of the word problem for the free *-band on a fixed set X is based on the free involutorial semigroup $F^*=F^*(X)$. Let $I=X\cup X^*$ where X^* is a set in one-to-one correspondence with X *via* $x\to x^*$. The free involutorial semigroup F^* is defined as $(F(I), {}^*)$ where * is given on $F(I)$, the free semigroup on I, as follows. For $i\in I$,

$$i^*=\begin{cases}x^* & \text{if}\quad i=x\in X,\\ x & \text{if}\quad i=x^*\in X^*.\end{cases}$$

If $w=i_1\ldots i_n\in F(I)$, then

$$w^*=i_n^*\ldots i_1^*.$$

The methods we use in the present section to solve the word problem for the free

*-band follow closely those for bands we developed in Section 3. In particular, we introduce a mapping $b^*: F^* \to F^*$ which plays the role of b in Section 3 and allows us to describe the free *-band as $b^*(F^*)$.

The invariants s and σ given in Section 2 have here their analogues s_x and σ_x, respectively. We will not need the analogues of ε and e (which could be defined similarly) because of the operation *.

NOTATION 4.1. For $w \in F^*$, let

$$c_X(w) = \{x \in X \mid x \text{ or } x^* \text{ occurs in } w\}.$$

Write $w = uip$ for $i \in I$ with $c_X(w) = c_X(ui)$ and $c_X(w) \neq c_X(u)$. Let

$$s_X(w) = u, \qquad \sigma_X(w) = i.$$

As usual, $s_X(w) = \varnothing$ if $|c_X(w)| = 1$. Therefore $s_X(w)\sigma_X(w)$ is the shortest left cut of w such that $c_X(s_X(w)\sigma_X(w)) = c_X(w)$.

The maps s_X and σ_X are combined to form $b^*: F^* \to F^*$ defined by

$$b^*(w) = b^* s_X(w)\sigma_X(w)[b^* s_X(w^*)\sigma_X(w^*)]^*.$$

In particular $b^*(i) = ii$ for all $i \in I$.

We also require the usual extension of s_X. Thus

$$s_X^0(w) = w,$$
$$s_X^{k+1}(w) = s_X(s_X^k(w)) \quad \text{for all} \quad k \geq 0.$$

For $\varnothing \neq A \subseteq X$ define $s_X^A(w)\sigma_X^A(w)$ to be the shortest left cut of w such that $\sigma_X^A(w) \in I$ and $c_X(s_X^A(w)\sigma_X^A(w)) = c_X(w) \setminus A$.

LEMMA 4.2. *Let $u, v \in F^*$ and $\varnothing \neq A \subseteq X$.*

(i) $$s_X^A(w) = s_X^k(w), \qquad \sigma_X^A(w) = \sigma_X^k(w)$$

where k is the least integer such that $c_X(s_X^k(w)\sigma_X^k(w)) = c_X(w) \setminus A$.

(ii) $$s_X(uv) = \begin{cases} s_X(u) & \text{if} \quad c_X(v) \subseteq c_X(u), \\ s_X(u)s_X^{c_X(u)}(v) & \text{otherwise.} \end{cases}$$

Proof. See the proofs of the corresponding results in Section 2.

LEMMA 4.3. *Let $u \in F^*$ and let $\hat{u}$ be obtained from u by replacing every occurrence of j by jj^*j for all $j \in I$. Denote $\sigma(\hat{u}) = i^*$, $i \in I$. Then $\sigma_X(u) = i$. Moreover, $s_X(u)i$ is obtained from $s(\hat{u})$ by replacing jj^*j by j whenever jj^*j was introduced in the transition from u to $\hat{u}$.*

Proof. The variable $i^* = \sigma(\hat{u})$ is the first occurrence of the last variable (element of I) in $\hat{u}$ to occur from the left. Since i precedes i^* in $\hat{u}$ and i always occurs next to i^* in $\hat{u}$, it follows that $\hat{u} = vii^*iw$ where i and i^* do not occur in v, but all other variables of $\hat{u}$ do occur in v. Also $s(\hat{u}) = vi$. It follows that $\sigma_X(u) = i$, and if the occurrences of jj^*j in v which were introduced by the transition from u to $\hat{u}$ are replaced by j, the result will be $s_X(u)$.

LEMMA 4.4. *Let τ be any congruence on F^* such that F^*/τ is a *-band. Then, for any $u \in F^*$,*

$$u \quad \tau \quad s_X(u)\sigma_X(u)[s_X(u^*)\sigma_X(u^*)]^*.$$

Proof. Let $\hat{u}$ be as defined in Lemma 4.3. Then $u \quad \tau \quad \hat{u}$. Also since F^*/τ is in particular a band, by Lemma 2.7 we have

$$\hat{u} \quad \tau \quad s(\hat{u})\sigma(\hat{u})\varepsilon(\hat{u})e(\hat{u}).$$

Let $\sigma(\hat{u}) = x^*$ and $\varepsilon(\hat{u}) = y^*$ for some $x^*, y^* \in I$. Since

$$x, y \in c(s(\hat{u})\sigma(\hat{u})) = c(\varepsilon(\hat{u})e(\hat{u}))$$

and F^*/τ is a band, we get

$$\hat{u} \quad \tau \quad s(\hat{u})\sigma(\hat{u})\varepsilon(\hat{u})e(\hat{u}) \quad \tau \quad s(\hat{u})\sigma(\hat{u})xy\varepsilon(\hat{u})e(\hat{u}).$$

Lemma 4.3 shows that

$$s(\hat{u})\sigma(\hat{u})xy\varepsilon(\hat{u})e(\hat{u}) \quad \tau \quad s_X(u)xx^*xyy^*y(s_X(u^*))^*.$$

This last word is τ related to

$$s_X(u)\sigma_X(u)[s_X(u^*)\sigma_X(u^*)]^*,$$

since $xx^*x \quad \tau \quad x = \sigma_X(u)$ and $yy^*y \quad \tau \quad y = (\sigma_X(u^*))^*$.

We are finally ready for a characterization of the free *-band.

THEOREM 4.5. *The mapping b^* is a homomorphism of F^* onto B^*. Moreover, B^* is a free *-band on X.*

Proof. The proof that b^* is a multiplicative homomorphism is similar to the proof that b is a homomorphism in the proof of Theorem 2.9. To check that b^* preserves * compute

$$\begin{aligned}(b^*(w))^* &= [b^*(s_X(w)\sigma_X(w)(b^*s_X(w^*)\sigma_X(w^*))^*]^* \\ &= b^*s_X(w^*)\sigma_X(w^*)[b^*s_X(w)\sigma_X(w)]^* \\ &= b^*(w^*).\end{aligned}$$

Let β^* be the congruence on F^* induced by b^*. We next show that F^*/β^* is a *-band. It is necessary to show that for all $u \in F^*$,

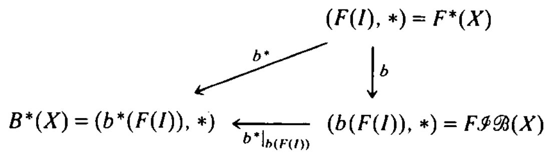
$$u \quad \beta^* \quad uu, \qquad u \quad \beta^* \quad uu^*u.$$

Both of these are obvious since $c_X(u) = c_X(uu) = c_X(uu^*)$ and therefore $b^*(u) = b^*(uu) = b^*(uu^*u)$.

To prove that B^* is a free *-band we must show that $\beta^* \subseteq \tau$ for any congruence τ on F^* such that F^*/τ is a *-band. The proof is similar to the proof of the corresponding result for bands. Lemma 4.4 is used.

In the next result we establish a connection between b and b^*. A description of the free *-band on X can be given by the methods developed in ([**3**], Section 3). The free involutorial semigroup on X is defined as $(F(I), *)$. The free involutorial band on X, $F\mathcal{IB}(X)$ is the quotient of $F(I)$ by the congruence generated by (u, uu) for $u \in F(I)$. The results of Section 3 show that $F\mathcal{IB}(X) \cong (b(F(I)), *)$. The free *-band on X is the quotient of $F\mathcal{IB}(X)$ (and therefore of $b(F(I))$) by the congruence ρ generated by (u, uu^*u) for $u \in b(F(I))$.

THEOREM 4.6. *The diagram*



*commutes, where b preserves the *-operation and $b^*|_{b(F(I))}$ is a homomorphism of $F\mathcal{IB}(X)$ onto $B^*(X)$ which induces the congruence*

$$\rho = \{(u, uu^*u) \mid u \in b(F(I))\}^*.$$

Proof. We first prove that b preserves *. If $x_1 \dots x_n$, where $x_i \in X$, is any word in F^*, then

$$\begin{aligned} b((x_1 \dots x_n)^*) &= b(x_n^* \dots x_1^*) = b(x_n^*) \dots b(x_1^*) \\ &= x_n^* x_n^* \dots x_1^* x_1^* = (x_1 x_1 \cdots x_n x_n)^* \\ &= (b(x_1) \dots b(x_n))^* = (b(x_1 \dots x_n))^*. \end{aligned}$$

To show that the diagram commutes we establish that $b^*b(u) = b^*(u)$. The proof is by induction on $|c(u)|$. To start note that $b^*b(x^n) = b^*(xx) = b^*(x^n)$. For the inductive step compute

$$\begin{aligned} b^*b(u) &= b^*s_Xb(u)\sigma_Xb(u)[b^*s_X(b(u))^*\sigma_X(b(u))^*]^*, \\ b^*(u) &= b^*s_X(u)\sigma_X(u)[b^*s_X(u^*)\sigma_X(u^*)]^*. \end{aligned}$$

By duality and the fact that b preserves *, it is enough to show that

$$\sigma_x = \sigma_X b, \tag{1}$$

$$b^*s_X = b^*s_X b. \tag{2}$$

The proof of (1) is also by induction on $|c(u)|$. To start note that for $n \geq 1$,

$$\sigma_X(x^n) = x = \sigma_X(xx) = \sigma_X b(x^n).$$

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For the inductive step compute

$$\sigma_X(u)=\begin{cases}\sigma(u) & \text{if } \sigma(u)\notin c_Xs(u),\\ \sigma_X(su) & \text{otherwise},\end{cases}$$

$$\sigma_Xb(u)=\sigma_X(sb(u)\sigma(u))=\sigma_X(bs(u)\sigma(u)) \quad \text{by Lemma 2.2(i)}$$

$$=\begin{cases}\sigma(u) & \text{if } \sigma(u)\notin c_Xbs(u)=c_Xs(u),\\ \sigma_Xbs(u) & \text{otherwise.}\end{cases}$$

Therefore $\sigma_X(u)=\sigma_Xb(u)$ if $\sigma(u)\notin c_Xbs(u)$. Otherwise equality follows by induction since $|cs(u)|<|c(u)|$.

To prove (2) we again proceed by induction on $|c(u)|$. At one point we use the original inductive hypothesis and at another the inductive hypothesis on (2) itself. The induction is started by noting that for $n\geq 1$,

$$s_X(x^n)=\varnothing=s_X(xx)=s_Xb(x^n).$$

For the inductive step compute

$$b^*s_X(u)=\begin{cases}b^*s(u) & \text{if } \sigma(u)\notin c_Xs(u),\\ b^*s_Xs(u) & \text{otherwise},\end{cases}$$

$$b^*s_Xb(u)=b^*s_X(sb(u)\sigma b(u))=b^*s_X(bs(u)\sigma(u)) \quad \text{by Lemma 2.2(i)(ii)}$$

$$=\begin{cases}b^*bs(u) & \text{if } \sigma(u)\notin c_Xbs(u)=c_Xs(u),\\ b^*s_Xbs(u) & \text{otherwise.}\end{cases}$$

If $\sigma(u)\notin c_X(s(u))$, then $b^*s(u)=b^*bs(u)$ by the original inductive hypothesis. Otherwise equality follows by induction since $|cs(u)|<|c(u)|$.

The final statement to prove is that $b^*|_{b(F(I))}$ induces ρ. Let $b^*|_{b(F(I))}$ induce θ on $b(F(I))$. It was shown in the proof of Theorem 4.5 that $b^*(u)=b^*(uu^*u)$. Consequently $\rho\subseteq\theta$. It follows that the map $b(F(I))/\rho\to b(F(I))$ defined by $u\rho\to u\theta$ is an epimorphism. By ([3], Theorem 3.1), $b(F(I))/\rho$ is a free *-band and therefore this map is the unique homomorphism which extends $(ii)\rho\to(ii)\theta$ (all $i\in I$). On the other hand $b(F(I))/\theta=B^*$ is a free *-band, and therefore this map is the unique isomorphism extending $(ii)\rho\to(ii)\theta$. The map is in particular one-to-one and therefore $\theta\subseteq\rho$.

5. The structure of *-bands and free *-bands. The purpose of this section is establishing a structure theorem for *-bands of the general form of Theorem 3.4 for bands. We can say intuitively that for *-bands we need a half of the ingredients for a band, namely only the functions $F_{\beta,\alpha}$; the other half is determined by the symmetry caused by the *-operation. In the proof of the structure theorem we do not specialize Theorem 3.4 to this case but prove it directly using the set of projections. This gives additional insight into the structure of *-bands. We conclude with a presentation of the free *-band on a set in terms of the structure theorem for general *-bands.

CONSTRUCTION 5.1. Let Y be a semilattice. For each $\alpha\in Y$, let L_α be a non-empty set such that $L_\alpha \cap L_\beta = \varnothing$ if $\alpha \neq \beta$ and let $B_\alpha = L_\alpha \times L_\alpha$. For any $\alpha, \beta \in Y$ such that $\alpha \geq \beta$, let $F_{\beta,\alpha}: B_\alpha \to \mathcal{T}'(L_\beta)$ be a function, in notation $(i, j) \to F_{\beta,\alpha}^{(i,j)}$, satisfying the following conditions: for $(i, j) \in B_\alpha$ and $(k, l) \in B_\beta$,

(i) $\langle F_{\alpha,\alpha}^{(i,j)} \rangle = i$,

(ii) $\langle F_{\alpha\beta,\alpha}^{(i,j)} F_{\alpha\beta,\beta}^{(k,l)} \rangle = p$, $\langle F_{\alpha\beta,\beta}^{(l,k)} F_{\alpha\beta,\alpha}^{(j,i)} \rangle = q$,

(iii) for $\gamma < \alpha\beta$ and the notation as in (ii),

$$F_{\gamma,\alpha}^{(i,j)} F_{\gamma,\beta}^{(k,l)} = F_{\gamma,\alpha\beta}^{(p,q)}.$$

Let $B = \bigcup_{\alpha \in Y} B_\alpha$ and with the above notation, define the operations

(iv) $(i, j)(k, l) = (p, q)$,

(v) $(i, j)^*(j, i)$.

Denote the algebraic system so constructed by $B^*(Y; L_\alpha, F_{\beta,\alpha})$.

Note that condition (ii) is needed in order to state condition (iii) including the needed notation. It actually requires that the products of the form $F_{\alpha\beta,\alpha}^{(i,j)} F_{\alpha\beta,\beta}^{(k,l)}$ be constant functions.

We prove next that the above construction gives a *-band.

LEMMA 5.2. *$B = B^*(Y; L_\alpha, F_{\beta,\alpha})$ is a *-band.*

Proof. In view of conditions (i) and (ii), it follows that (iii) is also valid for the case when $\gamma = \alpha\beta$. Using this general version of condition (iii), the associative law requires only a routine argument. Condition (i) implies that all elements are idempotents. Therefore B is a band.

With the notation as in the above construction, we obtain

$$((i, j)(k, l))^* = (p, q)^* = (q, p),$$
$$(k, l)^*(i, j)^* = (l, k)(j, i) = (u, v)$$

where

$$\langle F_{\alpha\beta,\beta}^{(l,k)} F_{\alpha\beta,\alpha}^{(j,i)} \rangle = u, \qquad \langle F_{\alpha\beta,\alpha}^{(i,j)} F_{\alpha\beta,\beta}^{(k,l)} \rangle = v.$$

Comparing this with part (ii) in the construction, we get that $u = q$ and $v = p$, as required. It follows at once that for any $x \in B$, $x^{**} = x$ and $xx^*x = x$. Therefore B is a *-band.

Conversely, to each *-band B we will now construct an isomorphic copy of B of the above form. For any *-band S, let $P(S)$ be the set of projections of S. (Recall $w \in P(S)$ if and only if $w = w^*$.)

LEMMA 5.3. *Let $S = (Y; S_\alpha)$ be a *-band. For each $\alpha \in Y$, let $L_\alpha = S_\alpha \cap P(S)$ and $B_\alpha = L_\alpha \times L_\alpha$. For any $\alpha, \beta \in Y$ such that $\alpha \geq \beta$, and for any $(p, q) \in B_\alpha$, define a function $F_{\beta,\alpha}^{(p,q)}$ by*

$$F_{\beta,\alpha}^{(p,q)}(r) = pqrqp \quad (r \in L_\beta).$$

*Then $(p, q) \to F_{\beta,\alpha}^{(p,q)}$ is a function $F_{\beta,\alpha}: B_\alpha \to \mathcal{T}'(L_\beta)$. These parameters satisfy the conditions of Construction 5.1, so by Lemma 5.2, $B = B^*(Y; B_\alpha, F_{\beta,\alpha})$ is a *-band. The* *mapping*

$$\chi: a \to (aa^*, a^*a) \quad (a \in S)$$

is an isomorphism of S onto B.

Proof. For $\alpha \geq \beta$, $(p, q) \in B_\alpha$ and $r \in L_\alpha$, $pqrqp = (pqr)(pqr)^*$ so that $pqrqp$ is a projection. In addition, $\alpha \geq \beta$ implies that $pqrqp \in L_\beta$ and thus $F^{(p,q)}_{\alpha,\beta} \in \mathcal{T}'(L_\beta)$. We now verify that the functions $F_{\beta,\alpha}$ satisfy conditions (i), (ii) and (iii) of Construction 5.1.

(i) $F^{(p,q)}_{\alpha,\alpha}(r) = pqrqp = p$ for any $r \in L_\alpha$;

(ii) $F^{(p,q)}_{\alpha\beta,\alpha}F^{(r,s)}_{\alpha\beta,\beta}(t) = pqrstsrqp = pqrqp$ is independent of t; we thus may put $\langle F^{(p,q)}_{\alpha\beta,\alpha}F^{(r,s)}_{\alpha\beta,\beta}\rangle = u$, $\langle F^{(s,r)}_{\alpha\beta,\beta}F^{(q,p)}_{\alpha\beta,\alpha}\rangle = v$;

(iii) for $\gamma < \alpha\beta$, with the above notation, on the one hand,

$$F^{(p,q)}_{\gamma,\alpha}F^{(r,s)}_{\gamma,\beta}(t) = pqrstsrqp,$$

and on the other hand, from above, $u = pqrqp$ and analogously, $v = srqrs$ so that

$$\begin{aligned} F^{(u,v)}_{\gamma,\alpha\beta}(t) = uvtvu &= (pqr)qpsr(qrs)t(srq)rspq(rqp) \\ &= (pqr)qrstsrq(rqp) = pqrstsrqp \end{aligned}$$

which verifies condition (iii) as well. Thus we may form the *-band $B = B^*(Y; B_\alpha, F_{\beta,\alpha})$.

It is easy to see that χ is a bijection of S onto B. For $a \in S_\alpha$ and $b \in S_\beta$, we obtain

$$(a\chi)(b\chi) = (aa^*, a^*a)(bb^*, b^*b) = (p, q)$$

where

$$\begin{aligned} p = F^{(aa^*,a^*a)}_{\alpha\beta,\alpha}F^{(bb^*,b^*b)}_{\alpha\beta,\beta}(t) &= (aa^*)(a^*a)(bb^*)(b^*b)t(b^*b)(bb^*)(a^*a)(aa^*) \\ &= abtb^*a^* = (ab)(ab)^* \end{aligned}$$

and similarly $q = (ab)^*(ab)$. It follows that $(a\chi)(b\chi) = (ab)\chi$. Also

and χ is an isomorphism.

It is convenient to introduce the following concept.

Definition 5.4. For a given *-band S, we call $B = B^*(Y; B_\alpha, F_{\beta,\alpha})$ constructed in Lemma 5.3 the *standard representation* of S.

Note that in the above lemma, the multiplication takes on the form

$$(p,q)(r,s) = (pqrqp, srqrs) = ((pqr)(pqr)^*, (qrs)^*(qrs)).$$

We may summarize the principal results of this section as follows.

Theorem 5.5. *The semigroup $B^*(Y; L_\alpha, F_{\beta,\alpha})$ in Construction 5.1 is a *-band. Conversely, every *-band admits a standard representation.*

We now give a model for the free *-band $F\mathcal{B}^*$ on X in terms of the structure

Theorem 5.5. We first mimic the procedure in Section 3. Let

$$\lambda^*(w) = b^* s_X(w)\sigma_X(w) \quad (w \in F^*).$$

Let Y be the free semilattice on X thought of as the set of all finite non-empty subsets of X under union. For $A \in Y$, let

$$L_A = \{w \in F^* \mid \lambda^*(w) = w, c_X(w) = A\}.$$

For $C, D \in Y$, $C \subseteq D$, let $F_{D,C}: B_C = L_C \times L_C \to \mathcal{T}'(L_D)$ be a function defined by

$$F_{D,C}^{(u,v^*)}(w) = \lambda^*(uvw) \quad (u, v^* \in L_C, w \in L_D).$$

Lemma 5.6. *The mappings $F_{D,C}$ satisfy the conditions of Construction* 5.1 *and therefore can be used to define a *-band $S^* = B^*(Y; L_A, F_{D,C})$.*

Proof. The argument is essentially the same as in the proof of Lemma 3.5. Note that the latter proof required several preliminaries on λ and b. Preliminaries of the same type can be established for λ^* and b^* in essentially the same way.

The final result gives the structure of free *-bands.

Theorem 5.7. *The mapping χ^* given by*

$$\chi^*: w \to (\lambda^*(w), \lambda^*(w^*)) \quad (w \in F^*)$$

*is a homormorphism of F^* onto S^* which induces the same congruence as b^*. Consequently S^* is a free *-band on X.*

Proof. See the proof of Theorem 3.6.

Alternatively, we may follow the recipe devised in Lemma 5.3. Indeed, let Y be as above and for each $A \in Y$, let

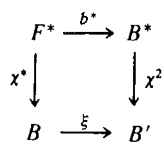
$$L'_A = \{w \in b^*(F^*) \mid w = w^*, c_X(w) = A\}.$$

For $C, D \in Y$, $C \subseteq D$, let $F'_{D,C}: B_C \to \mathcal{T}'(L_D)$ be a function defined by

$$F'^{(u,v)}_{D,C}(w) = uvwvu \quad (u, v \in L_C, w \in L_D).$$

Then Lemma 5.3 yields the isomorphism χ of B^* onto $B' = B^*(Y; L'_A, F'_{D,C})$.

The following diagram illustrates the homomorphisms we have discussed:



where ξ is the isomorphism of B onto B^* defined by

$$\xi: (u, v) \to (uu^*, vv^*) \quad ((u, v^*) \in B).$$

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REFERENCES

1. C. L. Adair, Bands with involution, *J. Algebra* **75** (1982), 297–314.

2. J. A. Gerhard and M. Petrich, The word problem for orthogroups, *Canad. J. Math.* **33** (1981), 893–900.

3. J. A. Gerhard and M. Petrich, Free involutorial completely simple semigroups, *Canad. J. Math.* **37** (1985), 271–295.

4. J. A. Green and D. Rees, On semi-groups in which $x^r = x$, *Proc. Cambridge Phil. Soc.* **48** (1952), 35–40.

5. J. M. Howie, *An introduction to semigroup theory*, (Academic Press, 1976).

6. D. McLean, Idempotent semigroups, *Amer. Math. Monthly* **61** (1954), 110–113.

7. T. E. Nordahl and H. E. Scheiblich, Regular *-semigroups, *Semigroup Forum* **16** (1978), 369–377.

8. M. Petrich, *Lectures in semigroups,* (Akademie-Verlag, 1977).

9. J. Siekmann and P. Szabó, A noetherian and confluent rewrite system for idempotent semigroups, *Semigroup Forum* **25** (1982), 83–110.

10. M. Yamada, Finitely generated free *-bands, *Semigroup Forum* **29** (1984), 13–16.

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