ANZIAM J. 50(2008), 111–127
doi:10.1017/S1446181108000266

NEW GRONWALL–OU-IANG TYPE INTEGRAL INEQUALITIES AND THEIR APPLICATIONS

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(Received 17 July, 2007; revised 11 December, 2008)

Abstract

Some new Gronwall–Ou-Iang type integral inequalities in two independent variables are established. We also present some of its application to the study of certain classes of integral and differential equations.

2000 *Mathematics subject classification*: primary 26D10; secondary 26D15, 34K10, 35B35.

Keywords and phrases: integral inequalities, retarded integral inequality, retarded differential equations, partial delay differential equations.

1. Introduction

One of the most useful methods available for studying a linear and nonlinear system of ordinary differential equations is the use of linear and nonlinear integral inequalities which provide explicit bounds on the unknown functions. Over the last scores of years several new linear and nonlinear integral inequalities have been developed in order to study the behaviour of solutions of such systems. See, for example, [1–13].

In the study of the boundedness of solutions to linear second-order differential equations, Ou-Iang [9] established and applied the following useful nonlinear integral inequality. *If* u, f *are nonnegative continuous functions on* $R_+ = [0, \infty)$, $u_0 \geq 0$ *is a constant and*

$$u^2(t) \leq u_0^2 + 2 \int_0^t f(s)u(s)\, ds \tag{1.1}$$

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for all $t \in R_+$, *then*

$$u(t) \le u_0 + \int_0^t f(s)\,ds, \quad t \in R_+.$$

Like the Gronwall type inequalities, (1.1) is also used to obtain the boundedness of solutions to the unknown function. Furthermore, this result has been extended and generalized by many authors (see [1–3, 5–7, 11–13]). Therefore, the integral inequalities of this type are usually known as the Gronwall–Ou-Iang type inequalities.

Recently, Pachpatte in [13] obtained a useful upper bound involving functions in two independent variables on the inequality

$$u^p(x, y) \le c + p \sum_{i=1}^{n} \left[\int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(x_0)}^{\beta_i(y)} [a_i(s, t)u^p(s, t) + b_i(s, t)u(s, t)]\,dt\,ds \right] \tag{1.2}$$

and its variants, under some suitable conditions on the functions involved in (1.2) and including the constant $p > 1$. These inequalities are applied to study the boundedness of the solutions of the retarded partial differential equation (1.3) with the initial boundary conditions (1.4)

$$\frac{\partial}{\partial y}\left[z^{p-1}(x, y) \frac{\partial}{\partial x} z(x, y) \right] = F[x, y, z(x - h_1(x), y - g_1(y)), \ldots, z(x - h_n(x), y - g_n(y))], \tag{1.3}$$

$$z(x, y_0) = e_1(x), \quad z(x_0, y) = e_2(y), \quad e_1(x_0) = e_2(y_0) = 0. \tag{1.4}$$

The authors Cheung [2], Cheung–Ma [3], Dragomir–Kim [5, 6] and Pachpatte [13] established additional new Gronwall–Ou-Iang type integral inequalities involving functions of two independent variables. Our main aim here, motivated by the works of Cheung, Cheung–Ma, Dragomir–Kim and Pachpatte, is to establish some new and more general Gronwall–Ou-Iang type integral inequalities with two independent variables which are useful in the analysis of certain classes of partial differential equations.

2. Main results

We shall introduce some notation. Let R denote the set of real numbers and $R_+ = [0, \infty)$, $I = [t_0, T)$ be the given subsets of R. Let $\triangle = I_1 \times I_2$, where $I_1 = [x_0, X)$ and $I_2 = [y_0, Y)$ are the given subsets of real numbers R. Denote by $C^i(M, N)$ the class of all i-times continuously differentiable functions defined on set M to the set N. The first-order partial derivatives of a function $z(x, y)$ defined for $x, y \in R$ with respect to x and y are denoted by $D_1 z(x, y)$ and $D_2 z(x, y)$, respectively.

LEMMA 2.1. *Let* $u, a_i \in C(\triangle, R_+)$, $\alpha_i \in C^1(I_1, I_1)$, $\beta_i \in C^1(I_2, I_2)$ *be nondecreasing with* $\alpha_i(x) \le x$ *on* I_1, $\beta_i(y) \le y$ *on* I_2 *for* $i = 1, 2, \ldots, n$. *Let* $\varphi \in C(R_+, R_+)$ *be an increasing function with* $\varphi(\infty) = \infty$ *and* c *be a nonnegative constant. Moreover,*

https://doi.org/10.1017/S1446181108000266 Published online by Cambridge University Press

let $w_1 \in C(R_+, R_+)$ be a nondecreasing function with $w_1 > 0$ on $(0, \infty)$. If

$$\varphi(u(x, y)) \leq c + \sum_{i=1}^{n} \left[\int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(x_0)}^{\beta_i(y)} a_i(s, t) w_1(u(s, t))\, dt\, ds \right] \tag{2.1}$$

for all $x \in I_1$, $y \in I_2$, then, for $x_0 \leq x \leq x_1$, $y_0 \leq y \leq y_1$ with $x_1 \in I_1$, $y_1 \in I_2$,

$$u(x, y) \leq \varphi^{-1}\left\{ G^{-1}\left[G(c) + \sum_{i=1}^{n} \left(\int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(x_0)}^{\beta_i(y)} a_i(s, t)\, dt\, ds \right) \right] \right\}, \tag{2.2}$$

$$\text{where} \quad G(r) = \int_{r_0}^{z} \frac{ds}{w_1[\varphi^{-1}(s)]}, \qquad r \geq r_0 > 0,$$

φ^{-1}, G^{-1} are, respectively, the inverse of φ, G and $x_1 \in I_1$, $y_1 \in I_2$ are chosen so that

$$G(c) + \sum_{i=1}^{n} \left(\int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(x_0)}^{\beta_i(y)} a_i(s, t)\, dt\, ds \right) \in \mathrm{Dom}(G^{-1}),$$

$$G^{-1}\left[G(c) + \sum_{i=1}^{n} \left(\int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(x_0)}^{\beta_i(y)} a_i(s, t)\, dt\, ds \right) \right] \in \mathrm{Dom}(\varphi^{-1})$$

for all $x \in [x_0, x_1]$ and $y \in [y_0, y_1]$.

PROOF. Define a positive function $z(x, y)$ by

$$z(x, y) = c + \epsilon + \sum_{i=1}^{n} \left[\int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} a_i(s, t) w_1(u(s, t))\, dt\, ds \right],$$

where ϵ is an arbitrary small positive number. Then (2.1) can be restated as

$$u(x, y) \leq \varphi^{-1}[z(x, y)]. \tag{2.3}$$

It is easy to observe that $z(x, y)$ is a continuous nondecreasing function for all $x \in I_1$, $y \in I_2$ and

$$\begin{aligned} D_1 z(x, y) &= \sum_{i=1}^{n} \left[\int_{\beta_i(y_0)}^{\beta_i(y)} a_i(\alpha_i(x), t) w_1(u(\alpha_i(x), t))\, dt \right] \alpha_i'(x) \\ &\leq \sum_{i=1}^{n} \left[\int_{\beta_i(y_0)}^{\beta_i(y)} a_i(\alpha_i(x), t) w_1(\varphi^{-1}[z(\alpha_i(x), t)])\, dt \right] \alpha_i'(x) \\ &\leq w_1(\varphi^{-1}[z(x, y)]) \sum_{i=1}^{n} \left[\int_{\beta_i(y_0)}^{\beta_i(y)} a_i(\alpha_i(x), t)\, dt \right] \alpha_i'(x). \end{aligned} \tag{2.4}$$

Using the monotonicity of φ^{-1} and w_1, we deduce

$$w_1(\varphi^{-1}[z(x, y)]) \geq w_1(\varphi^{-1}[z(x_0, y_0)]) = w_1(\varphi^{-1}[c + \epsilon]) > 0. \tag{2.5}$$

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From the definition of G, the inequalities (2.4) and (2.5) give

$$D_1 G(z(x, y)) = \frac{D_1 z(x, y)}{w_1(\varphi^{-1}[z(x, y)])} \leq \sum_{i=1}^{n} \bigg[\int_{\beta_i(y_0)}^{\beta_i(y)} a_i(\alpha_i(x), t)\, dt\bigg] \alpha_i'(t). \tag{2.6}$$

Keeping y fixed in (2.6), setting $x = \sigma$ and integrating it with respect to σ from x_0 to x, $x \in I_1$ and making the change of variable, we obtain

$$G(z(x, y)) \leq G(z(x_0, y)) + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} a_i(s, t)\, dt\, ds.$$

Since $G^{-1}(z)$ is increasing, letting $\epsilon \to 0$ yields

$$z(x, y) \leq G^{-1}\bigg[G(c) + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} a_i(s, t)\, dt\, ds\bigg] \tag{2.7}$$

for

$$G(c) + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} a_i(s, t)\, dt\, ds \in \mathrm{Dom}(G^{-1}).$$

Using (2.7) in (2.3), we get the required inequality in (2.2). This completes the proof of the lemma. □

In what follows, for any functions $f_i, g_i \in C(R_+, R_+)$, define

$$I_{\alpha_i,\beta_i}[f_i(s, t) + g_i(s, t)] \equiv \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} [f_i(s, t) + g_i(s, t)]\, dt\, ds.$$

THEOREM 2.2. *Let u, $f_i \in C(\triangle, R_+)$, $\alpha_i \in C^1(I_1, I_1)$, $\beta_i \in C^1(I_2, I_2)$ be nondecreasing with $\alpha_i(x) \leq x$ on I_1, $\beta_i(y) \leq y$ on I_2 for $i = 1, 2, \ldots, n$. Let c be a nonnegative constant. Moreover, assume that $\varphi \in C(R_+, R_+)$ and $w_1 \in C(R_+, R_+)$ are defined as in Lemma 2.1. If*

$$\varphi(u(x, y)) \leq c + \sum_{i=1}^{n} I_{\alpha_i,\beta_i}[f_i(s, t) u(s, t) w_1(u(s, t))] \tag{2.8}$$

for all $x \in I_1$, $y \in I_2$, then, for $x_0 \leq x \leq x_2$, $y_0 \leq y \leq y_2$ with $x_2 \in I_1$, $y_2 \in I_2$,

$$u(x, y) \leq \varphi^{-1}\bigg\{\Omega^{-1}\bigg[G^{-1}\bigg(G[\Omega(c)] + \sum_{i=1}^{n} I_{\alpha_i,\beta_i}[f_i(s, t)]\bigg)\bigg]\bigg\},$$

where

$$\Omega(r) = \int_{r_0}^{z} \frac{ds}{\varphi^{-1}(s)}, \quad r \geq r_0 > 0, \quad G(z) = \int_{z_0}^{z} \frac{ds}{w_1[\varphi^{-1}(\Omega^{-1}(s))]}, \quad z \geq z_0 > 0,$$

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Ω^{-1}, φ^{-1}, G^{-1} are, respectively, the inverses of Ω, φ, G and $x_2 \in I_1$, $y_2 \in I_2$ are chosen so that

$$G(\Omega(c)) + \sum_{i=1}^{n} I_{\alpha_i,\beta_i}[f_i(s,t)] \in \mathrm{Dom}(G^{-1}),$$

$$G^{-1}\Big[G(\Omega(c)) + \sum_{i=1}^{n} I_{\alpha_i,\beta_i}[f_i(s,t)]\Big] \in \mathrm{Dom}(\Omega^{-1}),$$

$$\Omega^{-1}\Big\{G^{-1}\Big[G(\Omega(c)) + \sum_{i=1}^{n} I_{\alpha_i,\beta_i}[f_i(s,t)]\Big]\Big\} \in \mathrm{Dom}(\varphi^{-1})$$

for all $x \in [x_0, x_2]$ and $y \in [y_0, y_2]$.

PROOF. Let us first assume that $c > 0$. Define a positive function $z(x, y)$ by

$$z(x, y) = c + \sum_{i=1}^{n} I_{\alpha_i,\beta_i}[f_i(s,t)u(s,t)w_1(u(s,t))].$$

Then $z(x, y) > 0$, $z(x_0, y) = z(x, y_0) = c$ and (2.8) can be restated as

$$u(x, y) \le \varphi^{-1}[z(x, y)]. \tag{2.9}$$

It is easy to observe that $z(x, y)$ is a continuous nondecreasing function for all $x \in I_1$, $y \in I_2$ and

$$\begin{aligned} D_1 z(x, y) &= \sum_{i=1}^{n}\Big[\int_{\beta_i(y_0)}^{\beta_i(y)} [f_i(\alpha_i(x),t)u(\alpha_i(x),t)w_1(u(\alpha_i(x),t))]\,dt\Big]\alpha_i'(x) \\ &\le \sum_{i=1}^{n}\Big[\int_{\beta_i(y_0)}^{\beta_i(y)} \Big[f_i(\alpha_i(x),t)\varphi^{-1}[z(\alpha_i(x),t)]w_1(\varphi^{-1}[z(\alpha_i(x),t)])\Big]\,dt\Big]\alpha_i'(x) \\ &\le \varphi^{-1}[z(x,y)]\sum_{i=1}^{n}\Big[\int_{\beta_i(y_0)}^{\beta_i(y)} \Big[f_i(\alpha_i(x),t)w_1(\varphi^{-1}[z(\alpha_i(x),t)])\Big]\,dt\Big]\alpha_i'(x). \end{aligned}$$

Using the monotonicity of φ^{-1} and z, we deduce

$$\varphi^{-1}[z(x, y)] \ge \varphi^{-1}[z(x_0, y_0)] = \varphi^{-1}(c) > 0.$$

From the definition of Ω and the above relation,

$$\begin{aligned} D_1\Omega(z(x, y)) &= \frac{D_1 z(x, y)}{\varphi^{-1}[z(x, y)]} \\ &\le \sum_{i=1}^{n}\Big[\int_{\beta_i(y_0)}^{\beta_i(y)} \Big[f_i(\alpha_i(x),t)w_1(\varphi^{-1}[z(\alpha_i(x),t)])\Big]\,dt\Big]\alpha_i'(x). \end{aligned} \tag{2.10}$$

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Keeping y fixed in (2.10), setting $x = \sigma$ and integrating it with respect to σ from x_0 to x, $x \in I_1$ and making the change of variable, we obtain

$$\Omega(z(x, y)) \le \Omega(c) + \sum_{i=1}^{n} I_{\alpha_i,\beta_i}[f_i(s, t)w_1(\varphi^{-1}[z(s, t)])]. \tag{2.11}$$

Now, an application of Lemma 2.1 to (2.11) gives

$$z(x, y) = \Omega^{-1}\bigg[G^{-1}\bigg(G(\Omega(c)) + \sum_{i=1}^{n} I_{\alpha_i,\beta_i}[f_i(s, t)]\bigg)\bigg]. \tag{2.12}$$

Using (2.12) in (2.9), we get the required inequality.

If $c = 0$, we carry out the above procedure with $\varepsilon > 0$ instead of c and subsequently let $\varepsilon \to 0$. This completes the proof. □

THEOREM 2.3. *Let $u, f_i, g_i \in C(\triangle, R_+)$, $\alpha_i \in C^1(I_1, I_1)$, $\beta_i \in C^1(I_2, I_2)$ be nondecreasing with $\alpha_i(x) \le x$ on I_1, $\beta_i(y) \le y$ on I_2 for $i = 1, 2, \ldots, n$. Let c be a nonnegative constant. Moreover, assume that $\varphi \in C(R_+, R_+)$ and $w_1 \in C(R_+, R_+)$ are defined as in Theorem 2.2. If*

$$\varphi(u(x, y)) \le c + \sum_{i=1}^{n} I_{\alpha_i,\beta_i}[f_i(s, t)u(s, t)w_1(u(s, t)) + g_i(s, t)u(s, t)] \tag{2.13}$$

for all $x \in I_1$, $y \in I_2$, then, for $x_0 \le x \le x_2$, $y_0 \le y \le y_2$ with $x_2 \in I_1$, $y_2 \in I_2$,

$$u(x, y) \le \varphi^{-1}\bigg\{\Omega^{-1}\bigg[G^{-1}\bigg(G[p(x, y)] + \sum_{i=1}^{n} I_{\alpha_i,\beta_i} f_i(s, t)\bigg)\bigg]\bigg\},$$

where

$$p(x, y) = \Omega(c) + \sum_{i=1}^{n} I_{\alpha_i,\beta_i}[g(s, t)],$$

$$\Omega(r) = \int_{r_0}^{r} \frac{ds}{\varphi^{-1}(s)}, \quad r \ge r_0 > 0, \quad G(z) = \int_{z_0}^{z} \frac{ds}{w_1[\varphi^{-1}(\Omega^{-1}(s))]}, \quad z \ge z_0 > 0,$$

Ω^{-1}, φ^{-1}, G^{-1} are, respectively, the inverses of Ω, φ, G and $x_2 \in I_1$, $y_2 \in I_2$ are chosen so that

$$G_i(\Omega(c)) + \sum_{i=1}^{n} I_{\alpha_i,\beta_i}[f_i(s, t) + g(s, t)] \in \mathrm{Dom}(G_i^{-1}),$$

$$G_i^{-1}\bigg[G_i(\Omega(c)) + \sum_{i=1}^{n} I_{\alpha_i,\beta_i}[f_i(s, t) + g(s, t)]\bigg] \in \mathrm{Dom}(\Omega^{-1}),$$

$$\Omega^{-1}\bigg\{G_i^{-1}\bigg[G_i(\Omega(c)) + \sum_{i=1}^{n} I_{\alpha_i,\beta_i}[f_i(s, t) + g(s, t)]\bigg]\bigg\} \in \mathrm{Dom}(\varphi^{-1})$$

for all $x \in [x_0, x_2]$ and $y \in [y_0, y_2]$.

PROOF. Let us first assume that $c > 0$. Define a positive function $z(x, y)$ by

$$z(x, y) = c + \sum_{i=1}^{n} I_{\alpha_i,\beta_i}[f_i(s, t)u(s, t)w_1(u(s, t)) + g_i(s, t)u(s, t)]. \tag{2.14}$$

Then $z(x, y) > 0$, $z(x_0, y) = z(x, y_0) = c$ and (2.13) can be restated as

$$u(x, y) \le \varphi^{-1}[z(x, y)]. \tag{2.15}$$

It is easy to observe that $z(x, y)$ is a continuous nondecreasing function for all $x \in I_1$, $y \in I_2$ and

$$\begin{aligned} D_1 z(x, y) &= \sum_{i=1}^{n}\bigg[\int_{\beta_i(y_0)}^{\beta_i(y)} \Big[f_i(\alpha_i(x), t)u(\alpha_i(x), t)w_1(u(\alpha_i(x), t)) \\ &\quad + g_i(\alpha_i(x), t)u(\alpha_i(x), t)\Big]\, dt\bigg]\alpha_i'(x) \\ &\le \sum_{i=1}^{n}\bigg[\int_{\beta_i(y_0)}^{\beta_i(y)} \Big[f_i(\alpha_i(x), t)\varphi^{-1}[z(\alpha_i(x), t)]w_1(\varphi^{-1}[z(\alpha_i(x), t)]) \\ &\quad + g_i(\alpha_i(x), t)\varphi^{-1}[z(\alpha_i(x), t)]\, dt\bigg]\alpha_i'(x)\bigg] \\ &\le \varphi^{-1}[z(x, y)]\sum_{i=1}^{n}\bigg[\int_{\beta_i(y_0)}^{\beta_i(y)} \Big[f_i(\alpha_i(x), t)w_1(\varphi^{-1}[z(\alpha_i(x), t)]) \\ &\quad + g_i(\alpha_i(x), t)\Big]\, dt\bigg]\alpha_i'(x). \end{aligned} \tag{2.16}$$

Using the monotonicity of φ^{-1} and z, we deduce

$$\varphi^{-1}[z(x, y)] \ge \varphi^{-1}[z(x_0, y_0)] = \varphi^{-1}(c) > 0.$$

From the definition of Ω and the above relation,

$$\begin{aligned} D_1\Omega(z(x, y)) &= \frac{D_1 z(x, y)}{\varphi^{-1}[z(x, y)]} \\ &\le \sum_{i=1}^{n}\bigg[\int_{\beta_i(y_0)}^{\beta_i(y)} \Big[f_i(\alpha_i(x), t)w_1(\varphi^{-1}[z(\alpha_i(x), t)]) \\ &\quad + g_i(\alpha_i(x), t)\Big]\, dt\bigg]\alpha_i'(x). \end{aligned}$$

Keeping y fixed in (2.15), setting $x = \sigma$ and integrating it with respect to σ from x_0 to x, $x \in I_1$ and making the change of variable, we obtain

$$\Omega(z(x, y)) \le \Omega(c) + \sum_{i=1}^{n} I_{\alpha_i,\beta_i}\Big[f_i(s, t)w_1(\varphi^{-1}[z(s, t)]) + g_i(s, t)\Big].$$

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Let $x \leq X$, $y \leq Y$ be arbitrary numbers and denote

$$p(x, y) = \Omega(c) + \sum_{i=1}^{n} I_{\alpha_i,\beta_i}[g_i(s, t)].$$

From the above relation, we deduce

$$\Omega(z(x, y)) \leq p(X, Y) + \sum_{i=1}^{n} I_{\alpha_i,\beta_i}\Big[f_i(s, t)w_1(\varphi^{-1}(z(s, t)))\Big].$$

Now, an application of Lemma 2.1 gives

$$z(x, y) \leq \Omega^{-1}\bigg[G^{-1}\bigg(G(p(X, Y)) + \sum_{i=1}^{n} I_{\alpha_i,\beta_i}[f_i(s, t)]\bigg)\bigg].$$

Using the inequality (2.16) in the inequality (2.14), we get

$$u(x, y) \leq \varphi^{-1}\bigg\{\Omega^{-1}\bigg[G^{-1}\bigg(G(p(X, Y)) + \sum_{i=1}^{n} I_{\alpha_i,\beta_i}[f_i(s, t)]\bigg)\bigg]\bigg\}.$$

Taking $x = X$, $y = Y$ in the above inequality, since X and Y are arbitrary, we can prove the desired inequality.

If $c = 0$, we carry out the above procedure with $\varepsilon > 0$ instead of c and subsequently let $\varepsilon \to 0$. This completes the proof. □

COROLLARY 2.4. *Let $u, f_i, g_i \in C(\triangle, R_+)$, $\alpha_i \in C^1(I_1, I_1)$, $\beta_i \in C^1(I_2, I_2)$ be nondecreasing with $\alpha_i(x) \leq x$ on I_1, $\beta_i(y) \leq y$ on I_2 for $i = 1, 2, \ldots, n$. Let $p > 1$ and $c \geq 0$ be constants. Moreover, assume that $w_1 \in C(R_+, R_+)$ is defined as in Theorem 2.2. If*

$$u^p(x, y) \leq c + p\sum_{i=1}^{n} I_{\alpha_i,\beta_i}[f_i(s, t)u(s, t)w_1(u(s, t)) + g_i(s, t)u(s, t)]$$

for all $x \in I_1$, $y \in I_2$, then, for $x_0 \leq x \leq x_2$, $y_0 \leq y \leq y_2$ with $x_2 \in I_1$, $y_2 \in I_2$,

$$u(x, y) \leq \bigg[G^{-1}\bigg(G[B(x, y)] + (p-1)\sum_{i=1}^{n} I_{\alpha_i,\beta_i}[f_i(s, t)]\bigg)\bigg]^{1/(p-1)},$$

where $B(x, y) = c^{(p-1)/p} + (p-1)\sum_{i=1}^{n} I_{\alpha_i,\beta_i}[g_i(s, t)]$, G^{-1} is the inverse function of

$$G(z) = \int_{z_0}^{z} \frac{ds}{w_1[s^{1/(p-1)}]}, \qquad z \geq z_0 > 0,$$

and $x_2 \in I_1$, $y_2 \in I_2$ are chosen so that

$$G[B(x, y)] + (p-1)\sum_{i=1}^{n} I_{\alpha_i,\beta_i}[f_i(s, t)] \in \mathrm{Dom}(G_i^{-1})$$

for all $x \in [x_0, x_2]$ and $y \in [y_0, y_2]$.

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THEOREM 2.5. *Let $u, f_i, g_i \in C(\triangle, R_+)$, $\alpha_i \in C^1(I_1, I_1)$, $\beta_i \in C^1(I_2, I_2)$ be nondecreasing with $\alpha_i(x) \le x$ on I_1, $\beta_i(y) \le y$ on I_2 for $i = 1, 2, \ldots, n$. Let $\varphi \in C(R_+, R_+)$ be an increasing function with $\varphi(\infty) = \infty$ and c be a nonnegative constant. Moreover, let $w_1, w_2 \in C(R_+, R_+)$ be nondecreasing functions with $w_1, w_2 > 0$ on $(0, \infty)$. If*

$$\varphi(u(x, y)) \le c + \sum_{i=1}^{n} I_{\alpha_i,\beta_i}[f_i(s, t)w_1(u(s, t)) + g_i(s, t)w_2(u(s, t))] \tag{2.17}$$

for all $x \in I_1$, $y \in I_2$, then, for $x_0 \le x \le x_2$, $y_0 \le y \le y_2$ with $x_2 \in I_1$, $y_2 \in I_2$, we have the following property.

(1) *For the case $w_2(u) \le w_1(u)$,*

$$u(x, y) \le \varphi^{-1}\bigg\{G_1^{-1}\bigg(G_1(c) + \sum_{i=1}^{n} I_{\alpha_i,\beta_i}[f_i(s, t) + g(s, t)]\bigg)\bigg\}.$$

(2) *For the case $w_1(u) \le w_2(u)$,*

$$u(x, y) \le \varphi^{-1}\bigg\{G_2^{-1}\bigg(G_2(c) + \sum_{i=1}^{n} I_{\alpha_i,\beta_i}[f_i(s, t) + g(s, t)]\bigg)\bigg\},$$

where

$$G_i(r) = \int_{r_0}^{z} \frac{ds}{w_i[\varphi^{-1}(s)]}, \quad r \ge r_0 > 0 \quad (i = 1, 2),$$

and φ^{-1}, G_i^{-1} are, respectively, the inverses of φ, G_i and $x_2 \in I_1$, $y_2 \in I_2$ are chosen so that

$$G_i(c) + \sum_{i=1}^{n} I_{\alpha_i,\beta_i}[f_i(s, t) + g_i(c, t)] \in \mathrm{Dom}(G_i^{-1})$$

for all $x \in [x_0, x_2]$ and $y \in [y_0, y_2]$.

PROOF. Let us first assume that $c > 0$. Define a positive function $z(x, y)$ by

$$z(x, y) = c + \sum_{i=1}^{n} I_{\alpha_i,\beta_i}[f_i(s, t)w_1(u(s, t)) + g_i(s, t)w_2(u(s, t))].$$

Then $z(x, y) > 0$, $z(x_0, y) = z(x, y_0) = c$ and (2.17) can be restated as

$$u(x, y) \le \varphi^{-1}[z(x, y)]. \tag{2.18}$$

It is easy to observe that $z(x, y)$ is a continuous nondecreasing function for all $x \in I_1$, $y \in I_2$ and

$$D_1 z(x, y) \le \sum_{i=1}^{n}\bigg[\int_{\beta_i(y_0)}^{\beta_i(y)}\Big[f_i(\alpha_i(x), t)w_1(\varphi^{-1}[z(\alpha_i(x), t)]) + g_i(\alpha_i(x), t)w_2(\varphi^{-1}[z(\alpha_i(x), t)])\Big]\,dt\bigg]\alpha_i'(x).$$

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(1) When $w_2(u) \le w_1(u)$, using the monotonicity of φ^{-1} and z, we deduce

$$D_1 z(x, y) \le w_1(\varphi^{-1}[z(x, y)]) \sum_{i=1}^{n} \left[\int_{\beta_i(y_0)}^{\beta_i(y)} [f_i(\alpha_i(x), t) + g_i(\alpha_i(x), t)]\, dt \right] \alpha_i'(x).$$

From the definition of G_1 and the above relation,

$$D_1 G_1(z(x, y)) \le \sum_{i=1}^{n} \left[\int_{\beta_i(y_0)}^{\beta_i(y)} [f_i(\alpha_i(x), t) + g_i(\alpha_i(x), t)]\, dt \right] \alpha_i'(x). \tag{2.19}$$

Keeping y fixed in (2.19), setting $x = \sigma$ and integrating it with respect to σ from x_0 to x, $x \in I_1$ and making the change of variable, we obtain

$$z(x, y) \le G_1^{-1}\left(G(c) + \sum_{i=1}^{n} I_{\alpha_i, \beta_i}[f_i(s, t) + g_i(s, t)] \right). \tag{2.20}$$

Using (2.18) and (2.20) in (2.17), we get the required inequality.

If $c = 0$, we carry aut the above procedure with $\varepsilon > 0$ instead of c and subsequently let $\varepsilon \to 0$.

(2) When $w_1(u) \le w_2(u)$, the proof can be completed similarly. This completes the proof. □

THEOREM 2.6. *Let $u, f_i, g_i \in C(\triangle, R_+)$, $\alpha_i \in C^1(I_1, I_1)$, $\beta_i \in C^1(I_2, I_2)$ be nondecreasing with $\alpha_i(x) \le x$ on I_1, $\beta_i(y) \le y$ on I_2 for $i = 1, 2, \ldots, n$. Let $\varphi \in C(R_+, R_+)$ be an increasing function with $\varphi(\infty) = \infty$ and c be a nonnegative constant. Moreover, let $w_1, w_2 \in C(R_+, R_+)$ be a nondecreasing function with $w_1, w_2 > 0$ on $(0, \infty)$. If*

$$\varphi(u(x, y)) \le c + \sum_{i=1}^{n} I_{\alpha_i, \beta_i}[f_i(s, t)u(s, t)w_1(u(s, t)) + g_i(s, t)u(s, t)w_2(u(s, t))] \tag{2.21}$$

for all $x \in I_1$, $y \in I_2$, then, for $x_0 \le x \le x_2$, $y_0 \le y \le y_2$ with $x_2 \in I_1$, $y_2 \in I_2$, we have the following property.

(1) *For the case $w_2(u) \le w_1(u)$,*

$$u(x, y) \le \varphi^{-1}\left\{ \Omega^{-1}\left[H_1^{-1}\left(H_1(\Omega(c)) + \sum_{i=1}^{n} I_{\alpha_i, \beta_i}[f_i(s, t) + g_i(s, t)] \right) \right] \right\}.$$

(2) *For the case $w_1(u) \le w_2(u)$,*

$$u(x, y) \le \varphi^{-1}\left\{ \Omega^{-1}\left[H_2^{-1}\left(H_2(\Omega(c)) + \sum_{i=1}^{n} I_{\alpha_i, \beta_i}[f_i(s, t) + g_i(s, t)] \right) \right] \right\},$$

where

$$\Omega(r) = \int_{r_0}^{r} \frac{ds}{\varphi^{-1}(z(s))}, \quad r \geq r_0 > 0,$$

$$H_i(z) = \int_{z_0}^{z} \frac{ds}{w_i[\varphi^{-1}(\Omega^{-1}(s))]}, \quad z \geq z_0 > 0 \quad (i = 1, 2),$$

φ^{-1}, Ω^{-1}, H_i^{-1} *are, respectively, the inverses of* φ, Ω, H_i *for* $i = 1, 2$ *and* $x_2 \in I_1$, $y_2 \in I_2$ *are chosen so that*

$$H_i(\Omega(c)) + \sum_{i=1}^{n} I_{\alpha_i,\beta_i}[f_i(s, t) + g_i(s, t)] \in \mathrm{Dom}(H_i^{-1}),$$

$$H_i^{-1}\bigg(H_i(\Omega(c)) + \sum_{i=1}^{n} I_{\alpha_i,\beta_i}[f_i(s, t) + g_i(s, t)]\bigg) \in \mathrm{Dom}(\Omega^{-1})$$

for all $x \in [x_0, x_2]$ *and* $y \in [y_0, y_2]$.

PROOF. Let us first assume that $c > 0$. Define a positive function $z(x, y)$ by

$$z(x, y) = c + \sum_{i=1}^{n} I_{\alpha_i,\beta_i}[f_i(s, t)u(s, t)w_1(u(s, t)) + g_i(s, t)u(s, t)w_2(u(s, t))].$$

Then $z(x, y) > 0$, $z(x_0, y) = z(x, y_0) = c$ and (2.21) can be restated as

$$u(x, y) \leq \varphi^{-1}[z(x, y)].$$

It is easy to observe that $z(x, y)$ is a continuous nondecreasing function for all $x \in I_1$, $y \in I_2$ and

$$D_1 z(x, y) \leq \varphi^{-1}[z(x, y)] \sum_{i=1}^{n} \bigg[\int_{\beta_i(y_0)}^{\beta_i(y)} \Big[f_i(\alpha_i(x), t)w_1(\varphi^{-1}[z(\alpha_i(x), t)]) + g_i(\alpha_i(x), t)w_2(\varphi^{-1}[z(\alpha_i(x), t)])\Big]\,dt\bigg]\alpha_i'(x).$$

Using the monotonicity of φ^{-1} and z, we deduce

$$\varphi^{-1}[z(x, y)] \geq \varphi^{-1}[z(x_0, y_0)] = \varphi^{-1}(c) > 0.$$

From the definition of Ω and the above relation,

$$D_1 \Omega(z(x, y)) \leq \sum_{i=1}^{n} \bigg[\int_{\beta_i(y_0)}^{\beta_i(y)} \Big[f_i(\alpha_i(x), t)w_1(\varphi^{-1}[z(\alpha_i(x), t)]) + g_i(\alpha_i(x), t)w_2(\varphi^{-1}[z(\alpha_i(x), t)])\Big]\,dt\bigg]\alpha_i'(x). \tag{2.22}$$

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Keeping y fixed in (2.21), setting $x=\sigma$ and integrating it with respect to σ from x_0 to x, $x \in I_1$ and making the change of variable, we obtain

$$\Omega(z(x,y)) \le \Omega(c) + \sum_{i=1}^{n} I_{\alpha_i,\beta_i}[f_i(s,t)w_1(\varphi^{-1}[z(s,t)]) + g_i(s,t)w_2(\varphi^{-1}[z(s,t)])]. \tag{2.23}$$

Now, an application of Theorem 2.5 to (2.23), we can prove the desired inequalities.

If $c=0$, we carry out the above procedure with $\varepsilon > 0$ instead of c and subsequently let $\varepsilon \to 0$. This completes the proof. □

3. Applications

In this section, we will show that our results are useful in proving the global existence of solutions to certain differential equations with time delay. First consider the partial differential equation involving several retarded arguments with the initial boundary conditions

$$D_2\Big(z^{p-1}(x,y)D_1z(x,y)\Big) = F[x,y,z(x-h_1(x),y-k_1(y)),\dots,z(x-h_n(x),y-k_n(y))], \tag{3.1}$$

$$z(x,y_0)=e_1(x), \quad z(x_0,y)=e_2(y), \quad e_1(x_0)=e_2(y_0)=0, \tag{3.2}$$

where $p>1$ is a constant, $F \in C(\triangle \times R^n, R)$, $e_1 \in C^1(I_1,R_+)$, $e_2 \in C^1(I_2,R_+)$ and $h_i \in C^1(I_1,R_+)$, $k_i \in C^1(I_2,R_+)$ are nonincreasing and such that $x-h_i(x) \ge 0$, $x-h_i(x) \in C^1(I_1,I_1)$, $y-k_i(y) \ge 0$, $y-k_i(y) \in C^1(I_2,I_2)$, $h_i'(x)<1$, $k_i'(y)<1$ and $h_i(x_0)=k_i(y_0)=0$ for $i=1,\dots,n$ and $x \in I_1$, $y \in I_2$.

The following theorem deals with a boundedness on the solution of (3.1).

THEOREM 3.1. *Assume that $F : \triangle \times R^n \to R$ is a continuous function for which there exists continuous nonnegative functions $f_i(x,y)$, $g_i(x,y)$ for $i=1,\dots,n$ and $x \in I_1$, $y \in I_2$ such that*

$$\begin{cases} |F(x,y,u_1,\dots,u_n)| \le \sum_{i=1}^{n}\{f_i(x,y)|u_i|w_1(|u_i|) + g_i(x,y)|u_i|\}, \\ |e_1(x)+e_2(y)| \le c, \end{cases} \tag{3.3}$$

where c is a constant. Let

$$M_i = \max_{x\in I_1}\frac{1}{1-h_i'(x)}, \quad N_i = \max_{y \in I_2}\frac{1}{1-k_i'(y)}, \quad i=1,\dots,n. \tag{3.4}$$

If $z(x,y)$ is any solution of the problem (3.1) with the condition (3.2), then

$$\|z(x,y)\| \le \bigg[G^{-1}\bigg(G\Big[\overline{B}(x,y)\Big] + (p-1)\sum_{i=1}^{n} I_{\phi_i,\psi_i}\Big[\overline{f_i}(s,t)\Big]\bigg)\bigg]^{1/(p-1)},$$

https://doi.org/10.1017/S1446181108000266 Published online by Cambridge University Press

where $\overline{B}(x, y) = c^{(p-1)/p} + (p-1)\sum_{i=1}^{n} I_{\alpha_i,\beta_i}[\overline{g_i}(s, t)]$, G^{-1} *is the inverse function of*

$$G(z) = \int_{z_0}^{z} \frac{ds}{w_1\big[s^{1/(p-1)}\big]}, \quad z \geq z_0 > 0,$$

for all $(x, y) \in \triangle_1$, *where* $\phi(x) = x - h_i(x)$, $\psi(y) = y - k_i(y)$ *and*

$$\begin{array}{l}\overline{f_i}(\sigma, \tau) = M_i N_i f_i(\sigma + h_i(s), \tau + k_i(t)),\\ \overline{g_i}(\sigma, \tau) = M_i N_i g_i(\sigma + h_i(s), \tau + k_i(t)),\end{array} \quad \sigma, s \in I_1, \ \tau, t \in I_2.$$

PROOF. It is easy to see that the solution $z(x, y)$ of the problem (3.1) satisfies the equivalent integral equation

$$z^p(x, y) = \int_{x_0}^{x}\int_{y_0}^{y} F\big(s, t, z(s - h_1(s), t - k_1(t)), \ldots, z(s - h_n(s), t - k_n(t))\big)\,dt\,ds + e_1(x) + e_2(y).$$

From (3.3) and making the change of variables, we have

$$\begin{aligned}|z(x, y)|^p &\leq c + pI_{x,y}\sum_{i=1}^{n}\Big\{f_i(x, y)|z(s - h_i(s), t - k_i(t))|w_1(|z(s - h_i(s), t - k_i(t))|)\\ &\quad + g_i(x, y)|z(s - h_i(s), t - k_i(t))|\Big\}\\ &\leq c + p\sum_{i=1}^{n} I_{\phi_i,\psi_i}\Big\{\overline{f_i}(\sigma, \tau)|z(\sigma, \tau)|w_1(|z(\sigma, \tau)|) + \overline{g_i}(x, y)|z(\sigma, \tau)|\Big\}. \quad (3.5)\end{aligned}$$

Now, a suitable application of the inequality given in Corollary 2.4 to (3.5) yields the desired result. This completes the proof. □

REMARK 1. Consider the partial differential equation (3.1) with the initial boundary condition (3.2). Assume that $F : \triangle \times R^n \to R$ is a continuous function for which there exists continuous nonnegative functions $g_i(x, y)$ such that

$$|F(x, y, u_1, \ldots, u_n)| \leq \sum_{i=1}^{n} g_i(x, y)|u_i|. \quad (3.6)$$

Let M_i and N_i be functions defined by (3.4). If $z(x, y)$ is any solution of the problem (3.1) with the condition (3.2), then the solution $z(x, y)$ can be written as

$$z^p(x, y) = \int_{x_0}^{x}\int_{y_0}^{y} F\big(s, t, z(s - h_1(s), t - k_1(t)), z(s - h_n(s), t - k_n(t))\big)\,dt\,ds + e_1(x) + e_2(y). \quad (3.7)$$

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From (3.6) and (3.7), making the change of variables, we have

$$\begin{aligned}|z(x,y)|^p &\le c + p\int_{x_0}^{x}\int_{y_0}^{y}\sum_{i=1}^{n} g_i(x,y)|z(s-h_i(s), t-k_i(t))|\,dt\,ds\\ &\le c + p\sum_{i=1}^{n} I_{\phi_i,\psi_i}[\,\overline{g_i}(x,y)|z(\sigma,\tau)|]. \qquad (3.8)\end{aligned}$$

Now, a suitable application of the inequality given in Corollary 2.4 to (3.8) yields

$$|z(x,y)| \le \left[c^{(p-1)/p} + (p-1)\sum_{i=1}^{n} I_{\alpha_i,\beta_i}[\,\overline{g_i}(s,t)]\right]^{1/(p-1)}$$

for all $(x, y) \in \triangle$, where $\phi(x) = x - h_i(x)$, $\psi(y) = y - k_i(y)$ and

$$\overline{g_i}(\sigma,\tau) = M_i N_i g_i(\sigma + h_i(s), \tau + k_i(t)), \quad \sigma, s \in I_1, \quad \tau, t \in I_2.$$

In the following, we present an application of the inequality given in Section 2 to study the boundedness of the solutions of the initial boundary value problem for the hyperbolic partial delay differential equations of the form

$$\begin{aligned}&D_2(D_1\varphi(z(x,y)))\\ &\qquad = F[x, y, z(x-h_1(x), y-k_1(y)), \ldots, z(x-h_n(x), y-k_n(y))], \quad (3.9)\\ &\varphi(z(x,y_0)) = e_3(x), \quad \varphi(z(x_0,y)) = e_4(y), \quad e_3(x_0) = e_4(y_0) = 0, \qquad (3.10)\end{aligned}$$

where $\varphi \in C(R_+, R_+)$ is an increasing function with $\varphi(0) = 0$, $\varphi(\infty) = \infty$, $F \in C(\triangle \times R^n, R)$, $e_3 \in C^1(I_1, R_+)$, $e_4 \in C^1(I_2, R_+)$ and $h_i \in C^1(I_1, R_+)$, $k_i \in C^1(I_2, R_+)$ are nonincreasing and such that

$$\begin{gathered} x - h_i(x) \ge 0, \quad x - h_i(x) \in C^1(I_1, I_1),\\ y - k_i(y) \ge 0, \quad y - k_i(y) \in C^1(I_2, I_2),\\ h_i'(x) < 1, \quad k_i'(y) < 1, \quad h_i(x_0) = k_i(y_0) = 0\end{gathered}$$

for all $i = 1, \ldots, n$ and $x \in I_1$, $y \in I_2$.

The following theorem deals with a boundedness on the solution of (3.9).

THEOREM 3.2. *Assume that $F : \triangle \times R^n \to R$ is a continuous function for which there exists continuous nonnegative functions $f_i(x, y)$, $g_i(x, y)$ for all $i = 1, \ldots, n$ and $x \in I_1$, $y \in I_2$ such that*

$$\begin{cases} |F(x,y,u_1,\ldots,u_n)| \le \displaystyle\sum_{i=1}^{n}\{f_i(x,y)|u_i|w_1(|u_i|) + g_i(x,y)|u_i|w_2(|u_i|)\},\\ |e_3(x) + e_4(y)| \le c, \end{cases} \qquad (3.11)$$

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where c is a constant. Let

$$M_i = \max_{x \in I_1} \frac{1}{1 - h_i'(x)}, \quad N_i = \max_{y \in I_2} \frac{1}{1 - k_i'(y)}, \quad i = 1, \ldots, n. \tag{3.12}$$

If $z(x, y)$ is any solution of the problem (3.9) with the condition (3.10), then, for the case $w_2(u) \leq w_1(u)$,

$$|z(x, y)| \leq \varphi^{-1}\Big\{\Omega^{-1}\Big[H_1^{-1}\Big(H_1[\Omega(c)] + \sum_{i=1}^{n} I_{\phi_i,\psi_i}\big[\overline{f_i}(\sigma, \tau) + \overline{g_i}(\sigma, \tau)\big]\Big)\Big]\Big\},$$

where φ^{-1}, Ω^{-1}, H_1^{-1} are, respectively, inverse functions of φ, Ω, H_1 for all $(x, y) \in \triangle$, $\Omega(r)$, $H_1(z)$ are as in Theorem 2.6, $\phi(x) = x - h_i(x)$, $\psi(y) = y - k_i(y)$ and

$$\overline{f_i}(\sigma, \tau) = M_i N_i f_i(\sigma + h_i(s), \tau + k_i(t)),$$
$$\overline{g_i}(\sigma, \tau) = M_i N_i g_i(\sigma + h_i(s), \tau + k_i(t))$$

for all $\sigma, s \in I_1$ and $\tau, t \in I_2$.

PROOF. It is easy to see that the solution $z(x, y)$ of the problem (3.9) satisfies the equivalent integral equation:

$$\begin{aligned}\varphi(z(x, y)) &= \int_{x_0}^{x}\int_{y_0}^{y} F(s, t, z(s - h_1(s), t - k_1(t)), \ldots, z(s - h_n(s), t - k_n(t)))\, dt\, ds \\ &\quad + e_3(x) + e_4(y).\end{aligned}$$

From (3.11) and making the change of variables, we have

$$\begin{aligned}\varphi(|z(x, y)|) &\leq c + \sum_{i=1}^{n} I_{x,y}\Big[f_i(x, y)|z(s - h_i(s), t - k_i(t))| \\ &\qquad \times w_1(|z(s - h_i(s), t - k_i(t))|) \\ &\qquad + g_i(x, y)|z(s - h_i(s), t - k_i(t))| w_2(|z(s - h_i(s), t - k_i(t))|)\Big] \\ &\leq c + \sum_{i=1}^{n} I_{\phi_i,\psi_i}\Big\{\overline{f_i}(\sigma, \tau)|z(\sigma, \tau)| w_1(|z(\sigma, \tau)|) \\ &\qquad + \overline{g_i}(x, y)|z(\sigma, \tau)| w_2(|z(\sigma, \tau)|)\Big\}.\end{aligned} \tag{3.13}$$

Now, a suitable application of the inequality given in Theorem 2.6 (1) to (3.13) yields the desired result. This completes the proof. □

REMARK 2. Consider the partial differential equation (3.9) with the initial boundary condition (3.10). Assume that $F : \triangle \times R^n \to R$ is a continuous function for which

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there exists continuous nonnegative functions $g_i(x, y)$ such that

$$|F(x, y, u_1, \dots, u_n)| \le \sum_{i=1}^{n} g_i(x, y)|u_i|. \tag{3.14}$$

Let M_i and N_i be functions defined by (3.12). If $z(x, y)$ is any solution of the problem (3.9) with the condition (3.10), then the solution $z(x, y)$ can be written as

$$\begin{aligned}\varphi(z(x, y)) \\ &= \int_{x_0}^{x}\int_{y_0}^{y} F(s, t, z(s-h_1(s), t-k_1(t)), \dots, z(s-h_n(s), t-k_n(t)))\,dt\,ds \\ &\quad + e_3(x) + e_4(y).\end{aligned} \tag{3.15}$$

From (3.14) and (3.15) making the change of variables, we have

$$\begin{aligned}\varphi(|z(x, y)|) &\le c + \int_{x_0}^{x}\int_{y_0}^{y}\sum_{i=1}^{n} g_i(x, y)|z(s-h_i(s), t-k_i(t))|\,dt\,ds \\ &\le c + \sum_{i=1}^{n} I_{\phi_i,\psi_i}\big[\,\overline{g_i}(x, y)|z(\sigma, \tau)|\big].\end{aligned} \tag{3.16}$$

Now, a suitable application of the inequality given in Theorem 2.3 to (3.16) yields

$$|z(x, y)| \le \varphi^{-1}\Big\{\Omega^{-1}\Big(\Omega(c) + \sum_{i=1}^{n} I_{\phi_i,\psi_i}\big[\,\overline{g_i}(\sigma, \tau)\big]\Big)\Big\}$$

for all $(x, y) \in \triangle$, where $\phi(x) = x - h_i(x)$, $\psi(y) = y - k_i(y)$ and

$$\overline{g_i}(\sigma, \tau) = M_i N_i g_i(\sigma + h_i(s), \tau + k_i(t)), \quad \sigma, s \in I_1, \ \tau, t \in I_2.$$

Acknowledgements

This paper was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD) (KRF-2007-313-C00040).

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