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COHOMOLOGY OF INDUCED MODULES IN RINGS OF DIFFERENTIAL OPERATORS

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1. Introduction

1.1. Let K be a field of characteristic zero and let $\Delta=\{\delta_1,\dots,\delta_n\}$ be a set of commuting K-derivations of the commutative Noetherian K-algebra R. Let $S=R[X_1,\dots,X_n]$ be the corresponding ring of differential operators, so $[X_i,r]=X_ir-rX_i=\delta_i(r)$, and $[X_i,X_j]=0$, for $1\leqq i,\ j\leqq n$. Let M be a maximal ideal of R with R/M of finite dimension over K. The purpose of this note is to describe the groups

$$E^*:=\{\operatorname{Ext}^i_S(S/SM,S/SM):i\geqq 0\}.$$

The dimension s over R/M of the subspace of $\operatorname{Hom}_{R/M}(M/M^2,R/M)$ generated by the image of Δ is called *the differential codimension of* M *with respect to* Δ [3, Proposition 2.1]. Let $V=\{v\in S:Mv\subseteq SM\}$, *the idealiser* of SM, and put $V_0=V/SM$; note that the groups E^* are (V_0-V_0)—bimodules.

1.2. The result we shall prove is the following.

Theorem. *Let the hypotheses and notation be as in* 1.1.

(i) *Let* M_1 *and* M_2 *be distinct maximal ideals of* R. *Then* $\operatorname{Ext}^*_S(S/SM_1,S/SM_2)=0$.

(ii) V_0 *is a polynomial algebra in* $(n-s)$ *variables over* R/M.

(iii) *There is a sequence* $\{x_1,\dots,x_s\}$ *of elements of* M, *whose images in* R_M *form an* R_M*-sequence, such that, as right* V_0*-modules,*

$$\operatorname{Ext}^i_S(S/SM,S/SM)\cong\operatorname{Ext}^i_{\bar R}(R/M,R/M)\otimes_R V_0,$$

for all $i\geqq 0$, *where* $\bar R=R/\langle x_1,\dots,x_s\rangle$. *If* R *is* M*-adically complete, then* $\bar R$ *is isomorphic to the centraliser in* R *of* Δ.

1.3. There is a special case of the above result which is well-known, (although we have not been able to find an explicit statement of it in the literature). Namely, let X be a non-singular affine variety over an algebraically closed field K characteristic 0, let $R=\mathcal{O}(X)$ be the ring of regular functions on X, and let M be a maximal ideal of R. Let D be the ring of differential operators on X. Then $\operatorname{Ext}^p_D(D/DM,D/DM)$ is K for $p=0$, and 0 for $p>0$. For, we may replace R by its M-adic completion (as in 3.1); in doing so we replace D by $\hat{D}=K[[X_1,\dots,X_n]][\partial/\partial X_1,\dots,\partial/\partial X_n]$ [**2**, Ch. 3, Lemma 1.5], and the result for $\hat{D}$ can be obtained by a much simplified version of the proof given below—essentially only 2.1 and 3.1 are needed.

1.4. Routine arguments using Shapiro's Lemma [**1**, page 109] or [**5**, Theorem 11.65] show that, if $R/M\cong K'$, (so $\dim_K K'=m<\infty$ by hypothesis), we can, in proving 1.1 (ii) and (iii), replace R by $R'=K'\otimes_K R$ and S by $S'=K'\otimes_K S$, so that $K'\otimes_K(R/M)$ is the direct sum of m copies of K'. We leave the details of these reductions to the reader. We shall assume throughout Sections 3 and 4 that $R/M=K$.

1.5. The proof of the theorem is organised as follows. A result on injective hulls of simple R-modules is obtained in Section 2, and this is used to handle linearly independent derivations in Section 3, first for the easier case where R is complete, and then in general. The main result is proved in Section 4.

2. The injective hull of the residue field of a local ring

Lemma 2.1. *Let K be a field, let $X_1,\dots,X_s$ be commuting indeterminates, and put $A=K[[X_1,\dots,X_s]]$. Let $D=A[Y_1,\dots,Y_s]$, where $[Y_i,Y_j]=0$ and $[Y_i,a]=\partial a/\partial X_i$, for $1\leqq i$, $j\leqq s$ and $a\in A$. Let M be the maximal ideal of A. Then D/DM is isomorphic to $E_A(K)$ as A-modules.*

Proof. By [**1**, p. 173, ex. 32], $E_A(K)$ is isomorphic to $K[Y_1,\dots,Y_s]$, where $X_j.p(Y)=\partial p/\partial Y_j$ for $1\leqq j\leqq s$ and $p(Y)\in K[Y_1,\dots,Y_s]$. Now D/DM has K-basis afforded by the monomials in $\{Y_1,\dots,Y_s\}$; and if τ is one such monomial, $X_j(\tau+DM)=(-\partial\tau/\partial Y_j+DM)$ for $1\leqq j\leqq s$. The result follows.

Lemma 2.2. *Let R_1,R_2 be commutative Noetherian rings containing a subfield K. For $i=1,2$, let M_i be an ideal of R_i with $R_i/M_i\cong K$, and set $E_i=E_{R_i}(R_i/M_i)$. Let $R=R_1\otimes_K R_2$ and $E=E_1\otimes_K E_2$ (so E is an R-module in the obvious way). Then $E=E_R(R_1/M_1\otimes R_2/M_2)$.*

Proof. Let $G=\operatorname{Hom}_{R_1}(R_1/M_1,-)$, so if X is an R-module, $GX=\{x\in X:M_1x=0\}$, and G is a left exact functor from R-modules to R_2-modules. Note that if X is R-injective, then GX is R_2-injective. Let $F=\operatorname{Hom}_{R_2}(R_2/M_2,-)$, a left exact functor from R_2-modules to abelian groups.

Apply the five term exact sequence of cohomology [**5**, Theorem 11.2] to obtain

$$\operatorname{Ext}^1_{R_2}(R_2/M_2,\operatorname{Hom}_{R_1}(R_1/M_1,E))\to\operatorname{Ext}^1_R(R/I,E)\to\operatorname{Hom}_{R_2}(R_2/M_2,\operatorname{Ext}^1_{R_1}(R_1/M_1,E))$$

where $I=M_1R+M_2R$. The two outside groups are zero, and hence so is $\operatorname{Ext}^1_R(R/I,E)$. Since E is clearly an essential extension of R/I, this is sufficient to ensure that E is injective, by the Artin–Rees theorem [**6**, p. 255, Theorem 4′].

Corollary 2.3. *Continue with the notation of 2.2. Let $0 \to R_1/M_1 \to \mathbf{E}_1^*$ be a minimal injective resolution of R_1-modules. Then*

$$(*)\ 0 \to R_1/M_1 \otimes_K E_2 \to \mathbf{E}_1^* \otimes_K E_2$$

is a minimal R-injective resolution of E_2, where E_2 is viewed as an R-module with $M_1E_2 = 0$.

Proof. Each term in $\mathbf{E}_1^*$ is a finite direct sum of copies of E_1, so each term of $\mathbf{E}_1^* \otimes E_2$ is R-injective by 2.2. The sequence (*) is exact because K is a field, so it is an R-injective resolution of E_2. If $d_i: E_1^i \to E_1^{i+1}$, then $\operatorname{socle}(E_1^{i+1}) \subseteq \operatorname{im} d_i$ by hypothesis, so

$$\operatorname{socle}(E_1^{i+1} \otimes E_2) = \operatorname{socle}(E_1^{i+1}) \otimes E_2 \subseteq \operatorname{im}(d_i \otimes 1);$$

hence (*) is a minimal resolution.

3. Linearly independent derivations

3.1. Throughout Sections 3 and 4 the notation will be that introduced in 1.1. As noted in 1.4, we shall assume that $R/M = K$. Moreover, let $\hat{R}$ denote the M-adic completion of R. The derivations $\{\delta_i\}$ extend to a set of commuting derivations of $\hat{R}$, which we denote by the same notation. We can thus form the ring of differential operators $\hat{R}[\mathbf{X};\Delta] := \hat{S}$.

Let V [resp. $\hat{V}$] be the idealiser of SM in S [resp. of $\hat{S}M$ in $\hat{S}$], and let $V_0 = V/SM$ [resp. $\hat{V}_0 = \hat{V}/\hat{S}M$]. Using the fact that each element of S/SM is killed by a power of M, it is easy to check that $\hat{V} = V + \hat{S}M$, so that, as rings, $V_0 \cong \hat{V}_0$.

Lemma. *Let the notation be as in* 1.1 *and above.*

(i) *Let M_1 and M_2 be distinct maximal ideals of R. Then* $\operatorname{Ext}_S^i(S/SM_1, S/SM_2) = 0$.

(ii) *As left R- and right V_0-modules,*

$$\begin{aligned}\operatorname{Ext}_S^*(S/SM, S/SM) &\cong \operatorname{Ext}_R^*(R/M, S/SM)\\ &\cong \operatorname{Ext}_{\hat{R}}^*(\hat{R}/\hat{R}M, \hat{S}/\hat{S}M)\\ &\cong \operatorname{Ext}_{\hat{S}}^*(\hat{S}/\hat{S}M, \hat{S}/\hat{S}M).\end{aligned}$$

Similar identifications can be made with $S_M := R_M[X;\Delta]$ in place of $\hat{S}$.

Proof. By [**1**, p. 109] or [**5**, Theorem 11.65].

$$\operatorname{Ext}_S^i(S/SM_1, S/SM_2) \cong \operatorname{Ext}_R^i(R/M_{1,R}|S/SM_2).$$

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Each element of the right hand module is annihilated by M_1 and by some power of M_2, so (i) follows at once. The above isomorphism also yields the first and third isomorphisms of (ii). Since

$$\operatorname{Ext}^*_{\hat{R}}(\hat{R}/\hat{R}M, \hat{S}/\hat{S}M) \cong \hat{R} \otimes_R \operatorname{Ext}^*_R(R/M, \hat{S}/\hat{S}M)$$

$$\cong \operatorname{Ext}^*_R(R/M, S/SM),$$

by [5, Theorem 11.65] for the first isomorphism and the comments above the statement of the lemma for the second, we also obtain the second isomorphism of (ii).

3.2. Each $\delta_i \in D$ induces an element δ_i^* of $\operatorname{Hom}_R(M/M^2, R/M)$. Let $s = \dim_K \operatorname{span}\{\delta_i^* : 1 \leqq i \leqq n\}$; this is *the differential codimension of M with respect to Δ* [3, Proposition 2.1]. Renumber Δ so that $\delta_1^*, \ldots, \delta_s^*$ are linearly independent, and put $T = R[X_1, \ldots, X_s]$. Choose $x_1, \ldots, x_s \in M$ with images in M/M^2 forming part of a dual basis to $\delta_1^*, \ldots, \delta_s^*$. Put

$$I = \{r \in R : cr \in \langle x_1, \ldots, x_s \rangle, \quad \text{for some } c \in R \backslash M\}, \text{ and } \bar{R} = R/I.$$

Proposition. $\operatorname{Ext}^*_T(T/TM, T/TM) \cong \operatorname{Ext}^*_{\bar{R}}(\bar{R}/M, \bar{R}/M)$.

Proof. Assume first that R is complete. Put $R_1 = \{r \in R : \delta_i(r) = 0, 1 \leqq i \leqq s\}$. Since $\operatorname{char} K = 0$, $R = R_1[[x_1, \ldots, x_s]]$ by [4, Section 4, Theorem 2 and remark at end of section]. Being an image of R, R_1 is a complete local ring with maximal ideal M_1, say. Put $Q = M_1 R$.

Thus $TQ = QT$ is an ideal of T and $R = R_1 \otimes_K R_2$ where $R_2 = R/Q = K[[x_1, \ldots, x_s]]$, so that $T/TQ = K[[x_1, \ldots, x_s]]\,[X_1, \ldots, X_s]$. Let M_2 be the ideal $\langle x_1, \ldots, x_s \rangle$ of R_2. By 2.1, TTM is the $K[[x_1, \ldots, x_s]]$-injective hull of $K = R_2/M_2$. Therefore, in the notation of 2.3,

$$\begin{aligned}
\operatorname{Ext}^*_T(T/TM, T/TM) &= \operatorname{Ext}^*_R(R/M, T/TM), && \text{by [5, 11.65]},\\
&= \operatorname{socle}(\mathbf{E}_1^* \otimes_K T/TM), && \text{by 2.3}\\
&= \operatorname{socle}(\mathbf{E}_1^* \otimes_K K)\\
&= \operatorname{Ext}^*_{R_1}(R_1/M_1, R_1/M_1). && (1)
\end{aligned}$$

Now drop the hypothesis that R is complete. The elements $x_1, \ldots, x_s$ of $\hat{R}$ chosen above can be taken in R, so that $R\langle x_1, \ldots, x_s \rangle \cap R = I$ [6, p. 257, Corollary 2]. Thus the subring R_1 of $\hat{R}$ defined above is just the $M/\langle x_1, \ldots, x_s \rangle$-adic completion of $\bar{R}$ [6, p. 258 Corollary 2], and so

$$\operatorname{Ext}^*_{R_1}(R_1/M_1, R_1/M_1) = \operatorname{Ext}^*_{\bar{R}}(\bar{R}/M, \bar{R}/M). \qquad (2)$$

The result follows from (1), (2) and 3.1.

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4. The main result

4.1. We retain the next three paragraphs the notations of 1.1, 3.1 and 3.2. For $s+1 \leqq i \leqq n$, $1 \leqq j \leqq s$ there exist elements r_{ij} of K such that

$$\delta_i^* = \sum_{j=1}^{s} r_{ij}\delta_j^*.$$

Set $Y_i = X_i - \sum_{j=1}^{s} r_{ij}X_j \in S$, for $s+1 \leqq i \leqq n$. Since

$$\left(\delta_i - \sum_{j=1}^{s} r_{ij}\delta_j\right)(M) \subseteq M, \tag{3}$$

for $i \geqq s+1$,

$$[Y_i, M] \subseteq M. \tag{4}$$

It follows that the subring $U = \langle SM, Y_{s+1}, \ldots, Y_n \rangle$ of S is contained in the idealiser V of SM. Since the monomials $\{X^I : I = (i_1, \ldots, i_s) \in \mathbb{N}^s\}$ form a free right generating set for S/SM as a right U/SM-module, it follows easily that

$$V = \langle SM, Y_{s+1}, \ldots, Y_n \rangle$$

so that

$$V_0 := V/SM = K[Y_{s+1}, \ldots, Y_n]$$

is a polynomial algebra over K (proving (ii) of the theorem).

Lemma 4.2. (i) *With the notation of* 4.1,

$$S/SM \cong T/TM \otimes_K V_0 = T/TM \otimes_R V_0$$

as $(T - V_0)$*-bimodules.*

(ii) *Let W be a finitely generated left T-module. Then, as right V_0-modules,*

$$\mathrm{Ext}_T^*(W, T/TM) \otimes_R V_0 \cong \mathrm{Ext}_S^*(S \otimes_T W, S/SM).$$

Proof. (i) It is easily checked that the map $\psi : S/SM \to T/TM \otimes_K V_0$ given by $\psi((X^I + SM)v) = (X^I + TM) \otimes v$, $(v \in V_0)$, is a well-defined bimodule isomorphism. The second equality holds since T/TM and V_0 are annihilated by M.

(ii) Let C be a left T-module. Define a map

$$\Theta : \mathrm{Hom}_T(C, T/TM) \otimes_R V_0 \to \mathrm{Hom}_S(S \otimes_T C, S/SM)$$

by setting, for $s \in S$, $v \in V_0$, $c \in C$ and $f \in \mathrm{Hom}_T(C, T/TM)$,

$$\Theta(f \otimes v)(s \otimes c) = s\psi^{-1}(f(c) \otimes v).$$

Routine checks confirm that $\mathrm{im}\,\Theta$ consists of S-homomorphisms, and that Θ is a homomorphism of right V_0-modules.

We note next that

$$\textit{when } C = T^{(n)} \textit{ is free, } \Theta \textit{ is an isomorphism.} \tag{6}$$

For in this case, as right V_0-modules,

$$\mathrm{Hom}_T(C, T/TM) \otimes_R V_0 \cong (T/TM \otimes_R V_0)^{(n)},$$

and

$$\mathrm{Hom}_S(S \otimes_T C, S/SM) \cong (S/SM)^{(n)},$$

and one sees that Θ is just the sum of n copies of the isomorphism of (i).

The proof now continues along familiar lines. Let W be a finitely generated left T-module, and let

$$\mathbf{F}_* \to W \to 0 \tag{7}$$

be a resolution of W by finitely generated free T-modules. Apply $\mathrm{Hom}_T(-, T/TM) \otimes_K V_0$ to (7); as K is a field the resulting complex has cohomology

$$\mathrm{Ext}_T^*(W, T/TM) \otimes_K V_0. \tag{8}$$

On the other hand, if we apply $\mathrm{Hom}_S(S \otimes_T -, S/SM)$ to (7), then since S is a free T-module we get the complex $\mathrm{Hom}_S(S \otimes \mathbf{F}_*, S/SM)$, with cohomology

$$\mathrm{Ext}_S^*(S \otimes_T W, S/SM). \tag{9}$$

Notice that the groups (8) can be denoted

$$\mathrm{Ext}_T^*(W, T/TM) \otimes_R V_0, \tag{10}$$

because $\mathrm{Hom}_T(\mathbf{F}_*, T/TM)M = \mathrm{Ext}_T^*(W, T/TM)M = MV_0 = 0$. The desired isomorphism now follows from (9), (10) and the isomorphism

$$\mathrm{Hom}_T(\mathbf{F}_*, T/TM) \otimes_R V_0 \cong \mathrm{Hom}_S(S \otimes_T \mathbf{F}_*, S/SM)$$

of right V-modules given by Θ.

4.3. Proof of Theorem 1.2. Parts (i) and (ii) have already been proved in 3.1(i) and 4.1. By Lemma 4.2(ii) and Proposition 3.2,

$$\mathrm{Ext}_S^*(S/SM, S/SM) \cong \mathrm{Ext}_T^*(T/TM, T/TM) \otimes_R V_0$$

$$\cong \mathrm{Ext}_{R/I}^*(R/M, R/M) \otimes_R V_0,$$

where $I = \{\tau \in R : c\tau \in \langle x_1, \ldots, x_s \rangle,\ c \in R \setminus M\}$, and these are isomorphisms of right V_0-modules. Moreover, both $\mathrm{Ext}_{R/I}^*(R/M, R/M)$ and $\mathrm{Ext}_R^*(R/M, R/M)$ are isomorphic to $\mathrm{Ext}_{R_M}^*(R_M/M_M, R_M/M_M)$, so (iii) follows.

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