CORRESPONDENCE.

ANNUITY-VALUES ON MAKEHAM'S HYPOTHESIS.

To the Editors of the Journal of the Institute of Actuaries.

SIRS,—The review (*J.I.A.*, vol. l, p. 320) of my paper "Su una "relazione fra l'annualità vitalizia di gruppo e l'annualità semplice, "nell'ipotesi di Makeham," suggests the following observations:

(1). The reader might suppose that I had repeated—although without being aware of the fact—McClintock's analysis in "On the "computation of Annuities on Mr. Makeham's Hypothesis" (*J.I.A.*, vol. xviii, p. 242). But that is not the case either in form or in substance.

By means of the B-function—which McClintock did not consider—I endeavour to establish whether, and if so on what conditions, continuous annuity-values for m joint lives can be obtained by the formula

$$\bar{a}_{x_1x_2 \ldots x_m} = \frac{1}{p_m \log c}\left[F(1-p_m,\ q_m) - c^{q_m} q_m^{p_m} \Gamma(1-p_m)\right] \quad . \quad (1)$$

where F is a hypergeometrical function and Γ the Γ-function.

Neither van der Belt, to whom the corresponding formula for $m=1$ is attributed (Enc. des Sciences Mathematiques, T.I., vol. 4, p. 531), nor McClintock, whose priority ought to be recognized—nor, so far as I know, any other writer—ever suggested the generalization and conditions established by me.

(2). Formula (1) is valid so long as the function F is convergent, and so also is the formula

$$a_{xy}=\mathrm{K}a_x \ . \ . \ . \ (2)$$

which can be readily deduced from (1). The coefficient K depends on the functions F and Γ, and the calculation of its numerical value can be carried to any degree of approximation.

In the case of the *Text-Book* $3\frac{1}{2}$ per-cent Table, formulas (1) and (2) are applicable so long as the number of lives m does not exceed 9. If $m=2$ the formulas are applicable so long as w does not exceed 67, where $2c^w=c^x+c^y$; and if $m=3$ they are applicable so long as w does not exceed 63, where $3c^w=c^x+c^y+c^z$. The resulting coefficient K is always positive.

It follows that the reviewer's statement that the formula "seems to be inapplicable (the numerator becoming negative) to "such a practical case as the evaluation of $a_{60.60.60}$", cannot refer to formulas (1) and (2), and I do not know how it can refer to formula IV′, because the numerator of this formula is not negative under the stated conditions, and I had anticipated the reviewer by stating that the formula is available for values of q_m "abbastanza piccoli" (rather small) and consequently not for greater values of m and older ages.

(3). When commutation-tables for a particular mortality-table are not available, and it is not convenient for any reason to undertake the rather laborious work of tabulation, it seems undeniable that formula (1) with the indicated limitations, and the passage from $\bar{a}_x$ to a_x, will suffice for all requirements. Further, if one has a single-life commutation-table or the single-life annuity-values for all ages, and it is inconvenient to construct commutation-tables for two or more lives of equal ages, formula (2)—which admits, by the simple process of calculating the coefficient K, of the passage to joint-life annuities—is not to be despised.

The coefficient K can of course be represented by several approximate expressions. For instance, I gave a first approximation applicable to the RF $3\frac{1}{2}$ per-cent Table. The corresponding approximation for the *Text-book* $3\frac{1}{2}$ per-cent Table—to pass from a_x to a_{xx}—would be

$$\mathrm{K}=\frac{\cdot 12+\cdot 49f-\cdot 31(1+2f)f^{\cdot 51}}{\cdot 14+\cdot 25f-\cdot 22(1+f)f^{\cdot 45}}$$

where $$f=\cdot 00105+\cdot 000096x+\cdot 0000044x^2$$

and it can hardly be said that this involves "a somewhat laborious calculation."

(4) Finally, the writer of the review appears to give the preference to the formula $a_{xy}=a'_w$. But for the application of this

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formula it is necessary to have annuity-values for all ages and for several rates of interest, or to seek the assistance of other more or less approximate formulas. Therefore, generally, one must be content with an approximation which has not been proved to be satisfactory in all cases in its results. Formula (2) on the other hand is always applicable when the single-life annuity-value is given at the same rate of interest; and formula (1) is applicable in every case without any preliminary tabulation.

I am, Sirs, &c.,

F. INSOLERA.

R. Ist. Sup. di Commercio, Turin,
23 *January* 1918.

[We are glad to publish Prof. Insolera's letter, but we do not think that there was anything in our review to suggest that his analysis was the same as McClintock's. The reviewer's statement that Prof. Insolera had used McClintock's method (*i.e.*, the method of evaluating the integral for $\bar{a}$ in an infinite series) for the purpose of obtaining the ratio of $a_{xyz} \ldots$ to a_x appears to be in accordance with the facts. McClintock did not restrict his investigation to the case of a single life; he indicated that the method could be applied, by a simple modification, to m lives, and his formula, modified accordingly, is identical in substance with Prof. Insolera's generalised formula (1). With regard to the conditions of applicability of the formula, the limitations imposed by Prof. Insolera do not appear to be necessary. Although the hypergeometrical series, in its general form, is divergent if x is >1, the special type of hypergeometrical series entering into McClintock's and Prof. Insolera's formula is convergent for all values of Σc^x. It follows that the formula is valid for lives of any ages, and it may be extended to any number of lives by further integrations by parts. The objection to the formula is not that it is subject to any limitations in theory, but that it involves an impracticable amount of calculation—owing to the slow convergency of the series—except for young lives. We are indebted to Mr. G. J. Lidstone for the information that when $\Sigma c^x \log_e 1/g$ is large a good result can be obtained by means of Schlömilch's series for the incomplete Γ-function (*see* Bromwich's "*Introduction to the Theory of Infinite Series*"). There would seem to remain, however, a considerable interval between the age at which McClintock's formula ceases to be of practical utility and that at which the Schlömilch series becomes applicable.

The statement quoted by Prof. Insolera in (2) refers to his formula IV′. This formula is given for $a_{x_1 x_2 \dots x_m}$ without any explicit limitation of its applicability, and the words at the beginning of the investigation "quando si abbia da fare con gruppi di pochi elementi, cosi che q_m sia abbastanza piccolo" (when one has to do with combinations of a few lives so that q_m is sufficiently small) would not, we think, lead the ordinary reader to suppose that the formula does not apply to the calculation of a joint-life annuity on three lives of 60.

With regard to the approximation to K given in (3) it should be borne in mind that the expression is derived from formula IV′ and is of limited application. It appears to give $a_{65\,.\,65}=6{\cdot}375$, the true value being 5·486. The approximate formula $a_{xy}=a_w+\log_e s(\text{I}a)_w$, gives (without using tables at more than one rate of interest) the correct result 5·486. —Eds. *J.I.A.*].

MORTALITY AMONG NEUTRALS IN WAR-TIME.

To the Editors of the Journal of the Institute of Actuaries.

DEAR SIRS,—Those Members of the Institute who read Professor Hersch's paper "La Mortalité chez les Neutres en Temps de Guerre", reviewed in *J.I.A.*, vol. l, p. 72, will remember that in this paper the author endeavoured to answer the question: "Which classes "of a population are most seriously affected by the indirect effect "of a War?"

The method adopted by the author was to consider the increase of mortality due to a War as the *absolute difference* between the mortality experienced in a time of War and the normal mortality of a time of peace, and to compare the results thus obtained for the different age groups. The method was, in fact, equivalent to a comparison of $q_{ng}-q_n$ age-group by age-group, where q_{ng} represents the mortality from all causes, including the indirect effect of a War, and q_n the normal mortality.

The same subject was dealt with by Mr. J. W. Nixon in his paper "War and National Vital Statistics with Special Reference "to the Franco-Prussian War"—Journal of the Royal Statistical Society, vol. lxxix, part 4. In this paper the author contended that the proper method of comparison was to compare, not the absolute, but the percentage increase in mortality, *i.e.*, not $q_{ng}-q_n$, but

$$\frac{q_{ng}-q_n}{q_n}$$