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POLYNOMIAL LIE SUBALGEBRAS OF THE INFINITE MATRIX LIE ALGEBRA

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ABSTRACT. We construct two classes of Lie subalgebras of the infinite matrix Lie algebra $\mathfrak{gl}_\infty(\mathbb{C})$ and prove that they are all simple Lie algebras.

1. **Introduction.** The infinite matrix Lie algebras $\mathfrak{gl}_\infty(\mathbb{C})$ and $\mathfrak{sl}_\infty(\mathbb{C}) = [\mathfrak{gl}_\infty(\mathbb{C}), \mathfrak{gl}_\infty(\mathbb{C})]$ have been discussed by many authors (for example [K], [KR]). In this paper we construct two classes of Lie subalgebras of $\mathfrak{sl}_\infty(\mathbb{C})$ and prove that they are all simple Lie algebras.

Recall that the Lie algebra $\mathfrak{gl}_\infty(\mathbb{C})$ is given by

$$\mathfrak{gl}_\infty(\mathbb{C}) = \sum_{i,j\in\mathbb{Z}} \mathbb{C}E_{ij}$$

with Lie bracket

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}.$$

Let $p(t) = a_0 + a_1 t + \cdots + a_r t^r$, $a_r \neq 0$. Define

$$E_{ij}^{p(t)} := \sum_{k=0}^{r} a_k E_{i,j+k}$$

and

$$\mathfrak{gl}_\infty\big(p(t)\big) := \sum_{i,j\in\mathbb{Z}} \mathbb{C}E_{ij}^{p(t)}.$$

Then $\mathfrak{gl}_\infty\big(p(t)\big)$ is a Lie subalgebra of $\mathfrak{gl}_\infty(\mathbb{C})$ and $\mathfrak{sl}_\infty\big(p(t)\big) := \big[\mathfrak{gl}_\infty\big(p(t)\big), \mathfrak{gl}_\infty\big(p(t)\big)\big]$ is a simple Lie algebra (see Section 1). Further, we define

$$F_{ij}^{p(t)} := \sum_{k,l=0}^{r} a_k a_l E_{i+k,j+l}$$

and

$$\widetilde{\mathfrak{gl}}_\infty\big(p(t)\big) := \sum_{i,j\in\mathbb{Z}} \mathbb{C}F_{ij}^{p(t)}.$$

Then $\widetilde{\mathfrak{gl}}_\infty\big(p(t)\big)$ is a Lie subalgebra of $\mathfrak{gl}_\infty\big(p(t)\big)$ and

$$\widetilde{\mathfrak{sl}}_\infty\big(p(t)\big) := \big[\widetilde{\mathfrak{gl}}_\infty\big(p(t)\big), \widetilde{\mathfrak{gl}}_\infty\big(p(t)\big)\big]$$

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is a simple Lie subalgebra (see Section 3). Since the $n \times n$ matrix Lie algebra $\mathfrak{gl}_n(\mathbb{C}) = \sum_{1 \le i,j \le n} \mathbb{C}E_{ij}$ is a Lie subalgebra of $\mathfrak{gl}_\infty(\mathbb{C})$, we naturally get Lie subalgebras

$$\mathfrak{gl}_n\big(p(t)\big) := \mathfrak{gl}_\infty\big(p(t)\big) \cap \mathfrak{gl}_n(\mathbb{C}),$$
$$\mathfrak{sl}_n\big(p(t)\big) := \mathfrak{sl}_\infty\big(p(t)\big) \cap \mathfrak{gl}_n(\mathbb{C}),$$
$$\widetilde{\mathfrak{gl}}_n\big(p(t)\big) := \widetilde{\mathfrak{gl}}_\infty\big(p(t)\big) \cap \mathfrak{gl}_n(\mathbb{C}),$$
$$\widetilde{\mathfrak{sl}}_n\big(p(t)\big) := \widetilde{\mathfrak{sl}}_\infty\big(p(t)\big) \cap \mathfrak{gl}_n(\mathbb{C}),$$

if $\deg\big(p(t)\big) < n$. We call $\mathfrak{gl}_\infty\big(p(t)\big)$, $\mathfrak{sl}_\infty\big(p(t)\big)$, $\widetilde{\mathfrak{gl}}_\infty\big(p(t)\big)$, $\widetilde{\mathfrak{sl}}_\infty\big(p(t)\big)$ the polynomial Lie subalgebras of $\mathfrak{gl}_\infty(\mathbb{C})$, and $\mathfrak{gl}_n\big(p(t)\big)$, $\mathfrak{sl}_n\big(p(t)\big)$, $\widetilde{\mathfrak{gl}}_n\big(p(t)\big)$, $\widetilde{\mathfrak{sl}}_n\big(p(t)\big)$ the polynomial Lie subalgebras of $\mathfrak{gl}_n(\mathbb{C})$.

This paper is organized as follows. In Section 1, we prove that $\mathfrak{sl}_\infty\big(p(t)\big)$ is a simple Lie algebra for every $p(t) \in \mathbb{C}[t]$. In Section 2, we discuss representations of $\mathfrak{gl}_\infty\big(p(t)\big)$. In Section 3, we prove that $\widetilde{\mathfrak{sl}}_\infty\big(p(t)\big)$ is a simple Lie algebra for every $p(t) \in \mathbb{C}[t]$, and in Section 4 we discuss $\mathfrak{gl}_n\big(p(t)\big)$ and $\widetilde{\mathfrak{gl}}_n\big(p(t)\big)$ in the cases $n = 2$ and $n = 3$.

In this paper, we denote the complex number field by $\mathbb{C}$ and denote the polynomial ring with one variable t over $\mathbb{C}$ by $\mathbb{C}[t]$.

2. **First class polynomial Lie subalgebras of $\mathfrak{gl}_\infty(\mathbb{C})$.** We discuss the first class of polynomial Lie algebras of $\mathfrak{gl}_\infty(\mathbb{C})$ in this section. $\mathfrak{gl}_\infty(\mathbb{C}) = \sum_{i,j\in\mathbb{Z}} \mathbb{C}E_{ij}$ is an infinite dimensional Lie algebra with Lie bracket

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}.$$

Let $p(t) = a_0 + a_1 t + \cdots + a_r t^r \in \mathbb{C}[t]$, where $a_r \neq 0$. Define $E_{ij}^{p(t)} := \sum_{k=0}^r a_k E_{i,j+k}$, and $\mathfrak{gl}_\infty\big(p(t)\big) := \sum_{i,j\in\mathbb{Z}} \mathbb{C}E_{ij}^{p(t)}$. Then we have the following:

PROPOSITION 1. *$\mathfrak{gl}_\infty\big(p(t)\big)$ is a Lie subalgebra of $\mathfrak{gl}_\infty(\mathbb{C})$.*

PROOF.

$$\begin{aligned}
[E_{ij}^{p(t)}, E_{kl}^{p(t)}] &= \sum_{m,n=0}^{r} a_m a_n [E_{i,j+m}, E_{k,l+n}] \\
&= \sum_{m,n=0}^{r} a_m a_n (\delta_{j+m,k} E_{i,l+n} - \delta_{l+n,i} E_{k,j+m} \\
&= \Big(\sum_{m=0}^{r} a_m \delta_{j+m,k}\Big) E_{il}^{p(t)} - \Big(\sum_{n=0}^{r} a_n \delta_{l+n,i}\Big) E_{kj}^{p(t)} \in \mathfrak{gl}_\infty\big(p(t)\big)
\end{aligned}$$

for all $i, j, k, l \in \mathbb{Z}$. ■

REMARKS. (1) If $p(t)$ is a non-zero constant, then

$$\mathfrak{gl}_\infty\big(p(t)\big) = \mathfrak{gl}_\infty(\mathbb{C}).$$

(2) If $p(t) = \sum_m a_m t^m \in \mathbb{C}[t]$, and $q(t) = \sum_n b_n t^n \in \mathbb{C}[t]$, and we use the notation $(E_{ij}^{p(t)})^{q(t)} = \sum_n b_n E_{i,j+n}^{p(t)}$, then

$$(E_{ij}^{p(t)})^{q(t)} = (E_{ij}^{q(t)})^{p(t)} = E_{ij}^{p(t)q(t)}.$$

(3) Without loss of generality,we assume in the rest of the paper that $a_0 = 1$.

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PROPOSITION 2. *Let* $p(t) = 1 + a_1 t + \cdots + a_r t^r$, $q(t) = 1 + b_1 t + \cdots + b_s t^s$, $a_r \neq 0$, $b_s \neq 0$. *Then*

(a) $\mathfrak{gl}_\infty\big(q(t)\big) \subseteq \mathfrak{gl}_\infty\big(p(t)\big)$ *iff* $p(t) \mid q(t)$.

(b) If $\mathrm{g.c.d.}\,\big(p(t), q(t)\big) = 1$, *then*

$$\mathfrak{gl}_\infty\big(p(t)q(t)\big) = \mathfrak{gl}_\infty\big(p(t)\big) \cap \mathfrak{gl}_\infty\big(q(t)\big).$$

In particular, if

$$p(t) = \prod_{j=1}^{l} (1 + \alpha_j t)^{k_j},$$

where $\alpha_1, \ldots, \alpha_l$ *are distinct, then*

$$\mathfrak{gl}_\infty\big(p(t)\big) = \bigcap_{i=1}^{l} \mathfrak{gl}_\infty\big((1 + \alpha_j t)^{k_j}\big).$$

PROOF. (a) It is clear that if $p(t) \mid q(t)$, then $\mathfrak{gl}_\infty\big(q(t)\big) \subseteq \mathfrak{gl}_\infty\big(p(t)\big)$. Conversely, if $\mathfrak{gl}_\infty\big(q(t)\big) \subseteq \mathfrak{gl}_\infty\big(p(t)\big)$, we have

$$E_{i0}^{q(t)} = \sum_k c_k E_{ik}^{p(t)} = E_{i0}^{p(t)r(t)},$$

where $r(t) = \sum_{k \geq 0} c_k t^k \in \mathbb{C}[t]$. So $p(t) \mid q(t)$.

(b) Let

$$X \in \mathfrak{gl}_\infty\big(p(t)\big) \cap \mathfrak{gl}_\infty\big(q(t)\big).$$

We may write

$$X = \sum_{m \leq k \leq n, m_1 \leq l \leq n_1} c_{kl} E_{kl}^{p(t)} = \sum_k E_{k,m_1}^{p(t)c_k(t)},$$

where $c_k(t) = \sum_l c_{kl} t^l \in \mathbb{C}[t]$. On the other hand, we may write $X = \sum_k E_{k,m_2}^{q(t)d_k(t)}$, where $d_k(t) = \sum_l d_{kl} t^l \in \mathbb{C}[t]$. So for every k, $q(t)d_k(t) = p(t)c_k(t)$. But $\mathrm{g.c.d.}\,\big(p(t), q(t)\big) = 1$, so $q(t) \mid c_k(t)$ and hence

$$X \in \mathfrak{gl}_\infty\big(p(t)q(t)\big).$$

Now (b) follows at once. ∎

PROPOSITION 3. *Let* $p(t) = 1 + a_1 t + \cdots + a_r t^r$, $a_r \neq 0$. *Define*

$$\mathfrak{sl}_\infty\big(p(t)\big) = \big[\mathfrak{gl}_\infty\big(p(t)\big), \mathfrak{gl}_\infty\big(p(t)\big)\big].$$

Then $\mathfrak{sl}_\infty\big(p(t)\big)$ *is a simple Lie algebra and*

$$\mathfrak{gl}_\infty\big(p(t)\big) = \mathbb{C}E_{00}^{p(t)} \ltimes \mathfrak{sl}_\infty\big(p(t)\big).$$

PROOF. We first prove the following

CLAIM 1.

$$\mathfrak{sl}_\infty\big(p(t)\big) = \sum_{j\in\mathbb{Z}, i\neq j,\dots,j+r} \mathbb{C}E_{ij}^{p(t)} + \sum_{i\in\mathbb{Z}} \mathbb{C}(E_{ii}^{p(t)} - E_{i-1,i-1}^{p(t)}) + \sum_{i\in\mathbb{Z}, s=1,\dots,r} \mathbb{C}(a_s E_{ii}^{p(t)} - E_{i+s,i}^{p(t)}).$$

PROOF OF CLAIM 1. We have

(1) $$[E_{ij}^{p(t)}, E_{kl}^{p(t)}] = \Big(\sum_{m=0}^{r} a_m \delta_{j+m,k}\Big) E_{il}^{p(t)} - \Big(\sum_{n=0}^{r} a_n \delta_{l+n,i}\Big) E_{kj}^{p(t)}.$$

For $i < l$ or $i > l + r$, $\sum_n a_n \delta_{l+n,i} = 0$. So

$$\sum_{l\in\mathbb{Z}, i\neq l,\dots,l+r} \mathbb{C}E_{il}^{p(t)} \subseteq \mathfrak{sl}_\infty\big(p(t)\big).$$

Let $i = l, k = j = i - 1$ in (1). We get

$$\sum_i \mathbb{C}(E_{ii}^{p(t)} - E_{i-1,i-1}^{p(t)}) \subseteq \mathfrak{sl}_\infty\big(p(t)\big).$$

Finally, let $i = j = l$ in (1), we get

$$\sum_{i\in\mathbb{Z}, m=1,\dots,r} \mathbb{C}(a_m E_{ii}^{p(t)} - E_{i+m,i}^{p(t)}) \subseteq \mathfrak{sl}_\infty\big(p(t)\big).$$

But

$$E_{00}^{p(t)} \notin \mathfrak{sl}_\infty\big(p(t)\big).$$

In fact, if

$$E_{00}^{p(t)} = E_{00} + a_1 E_{01} + \cdots + a_r E_{0r} \in \mathfrak{sl}_\infty\big(p(t)\big),$$

then $E_{00} \in \mathfrak{sl}_\infty(\mathbb{C})$. This is a contradiction. Since

$$\mathfrak{gl}_\infty\big(p(t)\big) = \mathbb{C}E_{00}^{p(t)} + \sum_{j\in\mathbb{Z}, i\neq j,\dots,j+r} \mathbb{C}E_{ij}^{p(t)} + \sum_{i\in\mathbb{Z}} \mathbb{C}(E_{ii}^{p(t)} - E_{i-1,i-1}^{p(t)}) + \sum_{i\in\mathbb{Z}, s=1,\dots,r} \mathbb{C}(a_s E_{ii}^{p(t)} - E_{i+s,i}^{p(t)}),$$

we proved the Claim 1.

Now let $N \neq 0$ be an ideal of $\mathfrak{sl}_\infty\big(p(t)\big)$, and let $0 \neq X \in N$. We may write

$$X = \sum_{k=m}^{n} E_{kl}^{p(t)c_k(t)},$$

where $c_k(t) \in \mathbb{C}[t]$. Choose $q \gg 0$. Then $E_{qn}^{p(t)} \in \mathfrak{sl}_\infty\big(p(t)\big)$, and

$$\begin{aligned} X_1 &= [E_{qn}^{p(t)}, X] \\ &= [E_{qn}^{p(t)}, E_{nl}^{p(t)c_n(t)}] \\ &= \sum_{j=m_1}^{n_1} c_j [E_{qn}^{p(t)}, E_{n,l+j}^{p(t)}] \\ &= \sum_{j=m_1}^{n_1} c_j E_{q,l+j}^{p(t)} \in N, \end{aligned}$$

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where $c_n(t) = \sum_{j=m_1}^{n_1} c_j t^j$.

Let $k \gg q$. Then

$$[X_1, E^{p(t)}_{n_1+r+1,k}] = \sum_{j=m_1}^{n_1} c_j [E^{p(t)}_{q,l+j}, E^{p(t)}_{n_1+l+r,k}]$$
$$= c_{n_1} [E^{p(t)}_{q,l+n_1}, E^{p(t)}_{n_1+l+r,k}]$$
$$= c_{n_1} a_r E^{p(t)}_{qk} \in N.$$

So we have that $E^{p(t)}_{qk} \in N$ for all $k \gg q \gg 0$.

CLAIM 2. *$E^{p(t)}_{ij} \in N$ for all $i, j \in \mathbb{Z}$ with $i \neq j, \ldots, j+r$.*

PROOF OF THE CLAIM 2. For every $j \in \mathbb{Z}$, and $i \neq j, \ldots, j+r$, choose k, q such that $k \gg q \gg j$ and $k \gg q \gg i$. Then $E^{p(t)}_{qk} \in N$, and $E^{p(t)}_{iq} \in \mathfrak{sl}_\infty\big(p(t)\big)$. Hence

$$[E^{p(t)}_{iq}, E^{p(t)}_{qk}] = E^{p(t)}_{ik} \in N.$$

But $E^{p(t)}_{kj} \in \mathfrak{sl}_\infty\big(p(t)\big)$, so we have

$$[E^{p(t)}_{ik}, E^{p(t)}_{kj}] = E^{p(t)}_{ij} \in N.$$

Now it is easy to prove the Proposition. Since $E^{p(t)}_{i,i+1} \in N$, and $a_1 E^{p(t)}_{ii} - E^{p(t)}_{i+1,i} \in \mathfrak{sl}_\infty\big(p(t)\big)$,

$$[a_1 E^{p(t)}_{ii} - E^{p(t)}_{i+1,i}, E^{p(t)}_{i,i+1}] = a_1 E^{p(t)}_{i,i+1} - (E^{p(t)}_{i+1,i+1} - E^{p(t)}_{ii}) \in N.$$

So

$$E^{p(t)}_{i+1,i+1} - E^{p(t)}_{ii} \in N$$

for all i. For $a_s E^{p(t)}_{ii} - E^{p(t)}_{i+s,i} \in \mathfrak{sl}_\infty\big(p(t)\big)$, choose $k \gg i$. Then

$$[E^{p(t)}_{ii} - E^{p(t)}_{kk}, a_s E^{p(t)}_{ii} - E^{p(t)}_{i+s,i}] = -[E^{p(t)}_{ii}, E^{p(t)}_{i+s,i}]$$
$$= -(a_s E^{p(t)}_{ii} - E^{p(t)}_{i+s,i}) \in N.$$

So we proved that $N = \mathfrak{sl}_\infty\big(p(t)\big)$ and hence $\mathfrak{sl}_\infty\big(p(t)\big)$ is a simple Lie algebra.

$$\mathfrak{gl}_\infty\big(p(t)\big) = \mathbb{C}E^{p(t)}_{00} \ltimes \mathfrak{sl}_\infty\big(p(t)\big)$$

is clear. We completed the proof. ∎

3. **Representations of $\mathfrak{gl}_\infty\big(p(t)\big)$.** In this section we discuss some representations of $\mathfrak{gl}_\infty\big(p(t)\big)$. First let $V = \sum_{k \in \mathbb{Z}} \mathbb{C}v_k$. V is a representation of $\mathfrak{gl}_\infty\big(p(t)\big)$ if we define

$$E^{p(t)}_{ij} \cdot v_k = \Big(\sum_{m=0}^{r} a_m \delta_{j+m,k}\Big) v_i,$$

where $p(t) = 1 + a_1 t + \cdots + a_r t^r$.

PROPOSITION 1. *V is an irreducible representation of* $\mathfrak{gl}_\infty\big(p(t)\big)$.

PROOF. Let $0 \neq U \subseteq V$ be a subrepresentation and $0 \neq X = \sum_{k=m}^{n} c_k v_k \in U$. Then for every $i \in \mathbb{Z}$,

$$E^{p(t)}_{i,m-r} \cdot X = c_m E^{p(t)}_{i,m-r} \cdot v_m = c_m a_r v_i \in U.$$

Hence $v_i \in U$, and $U = V$. ∎

Next we consider $\mathfrak{gl}_\infty(\mathbb{C})$ as a $\mathfrak{gl}_\infty\big(p(t)\big)$-module. Assume that $p(t) = (1+\alpha_1 t)\cdots(1+\alpha_r t)$, and $p_i(t) = (1+\alpha_1 t)\cdots(1+\alpha_i t)$, where $i = 0, 1, \ldots, r$. then

$$\mathfrak{gl}_\infty\big(p(t)\big) = \mathfrak{gl}_\infty\big(p_r(t)\big) \subset \mathfrak{gl}_\infty\big(p_{r-1}(t)\big) \subset \cdots \subset \mathfrak{gl}_\infty\big(p_0(t)\big) = \mathfrak{gl}_\infty(\mathbb{C})$$

is a sequence of $\mathfrak{gl}_\infty\big(p(t)\big)$-modules.

PROPOSITION 2. *The map*

$$\frac{\mathfrak{gl}_\infty\big(p_i(t)\big)}{\mathfrak{gl}_\infty\big(p_{i+1}(t)\big)} \longrightarrow V$$

$$E^{p_i(t)}_{k0} + \mathfrak{gl}_\infty\big(p_{i+1}(t)\big) \longrightarrow v_k$$

is a $\mathfrak{gl}_\infty\big(p(t)\big)$*-module isomorphism,* $i = 0, 1, \ldots, r-1$.

PROOF. Fix i. Assume that $p(t) = p_i(t)q_i(t)$, where $p_i(t) = \sum_m b_m t^m$, and $q_i(t) = \sum_n c_n t^n$. Then

$$\begin{aligned}
\operatorname{ad}(E^{p(t)}_{uv}) \cdot \Big(E^{p_i(t)}_{k0} + \mathfrak{gl}_\infty\big(p_{i+1}(t)\big)\Big) &= [E^{p(t)}_{uv}, E^{p_i(t)}_{k0}] + \mathfrak{gl}_\infty\big(p_{i+1}(t)\big) \\
&= \sum_n c_n [E^{p_i(t)}_{u,v+n}, E^{p_i(t)}_{k0}] + \mathfrak{gl}_\infty\big(p_{i+1}(t)\big) \\
&= \sum_n c_n \Big(\sum_m b_m \delta_{v+m+n,k} E^{p_i(t)}_{u0} - \sum_m b_m \delta_{m,u} E^{p_i(t)}_{k,v+n}\Big) \\
&\quad + \mathfrak{gl}_\infty\big(p_{i+1}(t)\big) \\
&= \Big(\sum_j a_j \delta_{j+v,k}\Big) E^{p_i(t)}_{u0} - \Big(\sum_m b_m \delta_{m,u}\Big) E^{p(t)}_{k,v} \\
&\quad + \mathfrak{gl}_\infty\big(p_{i+1}(t)\big) \\
&= \Big(\sum_j a_j \delta_{j+v,k}\Big)\Big(E^{p_i(t)}_{u0} + \mathfrak{gl}_\infty\big(p_{i+1}(t)\big)\Big).
\end{aligned}$$

We complete the proof. ∎

Finally, recall that $\mathfrak{gl}_\infty(\mathbb{C})$ has the triangular decomposition:

$$\mathfrak{gl}_\infty(\mathbb{C}) = \sum_{i>j} \mathbb{C}E_{ij} + \sum_{i\in\mathbb{Z}} \mathbb{C}E_{ii} + \sum_{i<j} \mathbb{C}E_{ij},$$

where $\sum_{i<j} \mathbb{C}E_{ij}$ and $\sum_{i>j} \mathbb{C}E_{ij}$ are Lie subalgebras of $\mathfrak{gl}_\infty(\mathbb{C})$, and $\sum_{i\in\mathbb{Z}} \mathbb{C}E_{ii}$ is an abelian Lie subalgebra of $\mathfrak{gl}_\infty(\mathbb{C})$.

Let $U(\mathfrak{g})$ be the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. Then we have

$$U\big(\mathfrak{gl}(\mathbb{C})\big) \cong U\Big(\sum_{i>j} \mathbb{C}E_{ij}\Big) \otimes U\Big(\sum_{i\in\mathbb{Z}} \mathbb{C}E_{ii}\Big) \otimes U\Big(\sum_{i<j} \mathbb{C}E_{ij}\Big).$$

Let $\Lambda = \{\lambda_i \mid i \in \mathbb{Z}\}$ be a set of complex numbers.

$$J(\Lambda) := \Big\langle \sum_{i<j} \mathbb{C}E_{ij}, E_{ii} - \lambda_i \Big\rangle,$$

the left ideal of $U\big(\mathfrak{gl}_\infty(\mathbb{C})\big)$ generated by $\sum_{i>j} \mathbb{C}E_{ij}$ and $E_{ii} - \lambda_i$, $i \in \mathbb{Z}$. Then

$$M(\Lambda) := \frac{U\big(\mathfrak{gl}_\infty(\mathbb{C})\big)}{J(\Lambda)}$$

is a left $\mathfrak{gl}_\infty(\mathbb{C})$-module. Hence it is a $\mathfrak{gl}_\infty\big(p(t)\big)$-module by restriction.

PROPOSITION 3. *Let $N \subseteq M(\Lambda)$. Then N is a $\mathfrak{gl}_\infty\big(p(t)\big)$-module iff N is a $\mathfrak{gl}_\infty(\mathbb{C})$-module.*

PROOF. Suppose that $N \neq 0$ is a $\mathfrak{gl}_\infty\big(p(t)\big)$-module. Let $0 \neq X \in N$. Then $E_{ij}^{p(t)} \cdot X \in N$. Since $N \subseteq M(\Lambda)$, so for $i \ll j$, $E_{ij} \cdot X = \cdots = E_{i,j+r-1} \cdot X = 0 \in N$. Then

$$E_{i,j-1} \cdot X = (E_{i,j-1} + a_1 E_{ij} + \cdots + a_r E_{i,j+r-1}) \cdot X = E_{i,j-1}^{p(t)} \cdot X \in N.$$

By induction on $j - i$, we see that $E_{ij} \cdot X \in N$, for all $i,j \in \mathbb{Z}$. Hence $\mathfrak{gl}_\infty(\mathbb{C}) \cdot N \subset N$. ■

4. **Second class polynomial Lie subalgebras of $\mathfrak{gl}_\infty(\mathbb{C})$.** In this section we discuss the second class polynomial Lie subalgebras of $\mathfrak{gl}_\infty(\mathbb{C})$. We assume that $p(t) = 1 + a_1 t + \cdots + a_r t^r$, where $a_r \neq 0$. We define

$$F_{ij}^{p(t)} := \sum_k a_k E_{i+k,j}^{p(t)} = \sum_{k,l} a_k a_l E_{i+k,j+l}$$

and

$$\widetilde{\mathfrak{gl}}_\infty\big(p(t)\big) := \sum_{i,j\in\mathbb{Z}} \mathbb{C}F_{ij}^{p(t)}.$$

PROPOSITION 1. *$\widetilde{\mathfrak{gl}}_\infty\big(p(t)\big)$ is a Lie subalgebra of $\mathfrak{gl}_\infty\big(p(t)\big)$.*

PROOF.

$$(1) \qquad [F_{ij}^{p(t)}, F_{kl}^{p(t)}] = \Big(\sum_{m,n=0}^{r} a_m a_n \delta_{j+n,k+m}\Big) F_{il}^{p(t)} - \Big(\sum_{m,n=0}^{r} a_m a_n \delta_{l+m,i+n}\Big) F_{kj}^{p(t)} \qquad ■$$

It is clear that if $p(t), q(t) \in \mathbb{C}[t]$, then $\widetilde{\mathfrak{gl}}_\infty\big(p(t)\big) \subset \widetilde{\mathfrak{gl}}_\infty\big(q(t)\big)$ iff $q(t) \mid p(t)$. Now we define

$$\widetilde{\mathfrak{sl}}_\infty\big(p(t)\big) := \big[\widetilde{\mathfrak{gl}}_\infty\big(p(t)\big), \widetilde{\mathfrak{gl}}_\infty\big(p(t)\big)\big].$$

We prove that $\widetilde{\mathfrak{sl}}_\infty\big(p(t)\big)$ is a simple Lie algebra.

First we see from (1) that if $i < l - r$ or $i > l + r$, then $\sum_{m,n=0}^{r} a_m a_n \delta_{l+m,i+n} = 0$. In this case if we choose $j = 0$, $k = r$, then $a_r F_{il}^{p(t)} \in \widetilde{\mathfrak{sl}}_\infty\big(p(t)\big)$. Hence

$$F_{il}^{p(t)} \in \widetilde{\mathfrak{sl}}_\infty\big(p(t)\big) \tag{2}$$

for all $l \in \mathbb{Z}$, $i < l - r$ or $i > l + r$.

Next let $i = l$ and $j = k$ in (1). Then

$$\sum_{m=0}^{r} a_m^2 (F_{ii}^{p(t)} - F_{jj}^{p(t)}) \in \widetilde{\mathfrak{sl}}_\infty\big(p(t)\big) \tag{3}$$

for all $i, j \in \mathbb{Z}$.

Finally let $i = l = j$ in (1). We get

$$\Big(\sum_{m,n=0}^{r} a_m a_n \delta_{i+m,k+n}\Big) F_{ii}^{p(t)} - \Big(\sum_{m=0}^{r} a_m^2\Big) F_{ki}^{p(t)} \in \widetilde{\mathfrak{sl}}_\infty\big(p(t)\big) \tag{4}$$

for all $i, k \in \mathbb{Z}$.

We consider two different cases:

CASE 1. $\sum_{m=0}^{r} a_m^2 \neq 0$.

In this case $F_{00}^{p(t)} \notin \widetilde{\mathfrak{sl}}_\infty\big(p(t)\big)$. In fact,

$$F_{00}^{p(t)} = \sum_{m=0}^{r} a_m^2 E_{mm} + \sum_{m \neq n} a_m a_n E_{mn}.$$

If $F_{00}^{p(t)} \in \widetilde{\mathfrak{sl}}_\infty\big(p(t)\big) \subseteq \mathfrak{sl}_\infty(\mathbb{C})$, then $\sum_m a_m^2 E_{mm} \in \mathfrak{sl}_\infty(\mathbb{C})$, and hence $\sum_m a_m^2 = 0$. This is a contradiction. So in this case, we have the following lemma.

LEMMA 1.

$$\widetilde{\mathfrak{sl}}_\infty\big(p(t)\big) = \sum_{j \in \mathbb{Z}, i \neq j-r, \ldots, j+r} \mathbb{C} F_{ij}^{p(t)} + \sum_{i \in \mathbb{Z}} \mathbb{C}(F_{ii}^{p(t)} - F_{i-1,i-1}^{p(t)})$$
$$+ \sum_{i \in \mathbb{Z}, i-r \leq s \leq i+r} \mathbb{C}\Big(\Big(\sum_{m,n} a_m a_n \delta_{i+m,s+n}\Big) F_{ii}^{p(t)} - \Big(\sum_m a_m^2\Big) F_{si}^{p(t)}\Big).$$

PROPOSITION 2. *If $\sum_{m=0}^{r} a_m^2 \neq 0$, then $\widetilde{\mathfrak{sl}}_\infty\big(p(t)\big)$ is a simple Lie algebra.*

PROOF. Let $N \neq 0$ be an ideal of $\widetilde{\mathfrak{sl}}_\infty\big(p(t)\big)$, and

$$0 \neq X = \sum_{i=m,\ldots,n, j=m_1,\ldots,n_1} c_{ij} F_{ij}^{p(t)} \in N.$$

Let $q \gg 0$. Then $F_{q,n+r}^{p(t)} \in \widetilde{\mathfrak{sl}}_\infty\big(p(t)\big)$. Hence

$$\begin{aligned} X_1 &= [F_{q,n+r}^{p(t)}, X] \\ &= \Big[F_{q,n+r}^{p(t)}, \sum_{j=m_1}^{n_1} c_{nj} F_{nj}^{p(t)}\Big] \\ &= \sum_{j=m_1}^{n_1} c_{nj} a_r F_{qj}^{p(t)} \in N. \end{aligned}$$

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Let $k \gg q$. Then $F^{p(t)}_{n_1+r,k} \in \widetilde{\mathfrak{sl}}_\infty\big(p(t)\big)$. Hence

$$[X_1, F^{p(t)}_{n_1+r,k}] = a_r c_{n,n_1}[F^{p(t)}_{q,n_1}, F^{p(t)}_{n_1+r,k}] \\ = a_r^2 c_{n,n_1} F^{p(t)}_{qk} \in N.$$

Now for $i \in \mathbb{Z}$, choose $k \gg q \gg i$. Then $F^{p(t)}_{qk} \in N$ and $F^{p(t)}_{i,q+r} \in \widetilde{\mathfrak{sl}}_\infty\big(p(t)\big)$. This implies that

$$[F^{p(t)}_{i,q+r}, F^{p(t)}_{qk}] = a_r F^{p(t)}_{ik} \in N.$$

For every $j \in \mathbb{Z}$, with $j < i - r$ or $j > i + r$. Choose $k \gg j + r$. Then $F^{p(t)}_{k+r,j} \in \widetilde{\mathfrak{sl}}_\infty\big(p(t)\big)$. Hence

$$[F^{p(t)}_{ik}, F^{p(t)}_{k+r,j}] = a_r F^{p(t)}_{ij} \in N.$$

In summary,we proved that

$$F^{p(t)}_{ij} \in N \tag{5}$$

for all $i,j \in \mathbb{Z}$, with $i < j - r$ or $i > j + r$.

Next let $j = i + r + s$, where $s \geq 1$. Then by (5), $F^{p(t)}_{i,i+r+s}, F^{p(t)}_{i+r+s,i} \in N$ and

$$[F^{p(t)}_{i,i+r+s}, F^{p(t)}_{i+r+s,i}] = \Big(\sum_m a_m^2\Big)(F^{p(t)}_{ii} - F^{p(t)}_{i+r+s,i+r+s}) \in N.$$

From this we get

$$F^{p(t)}_{ii} - F^{p(t)}_{i-1,i-1} = (F^{p(t)}_{ii} - F^{p(t)}_{i+r+1,i+r+1}) - (F^{p(t)}_{i-1,i-1} - F^{p(t)}_{i+r+1,i+r+1}) \in N. \tag{6}$$

Finally, for $i \in Z$, $i - r \leq s \leq i + r$,

$$Y_1 := \Big(\sum_{m,n=0}^{r} a_m a_n \delta_{i+m,s+n}\Big)F^{p(t)}_{ii} - \Big(\sum_{m=0}^{r} a_m^2\Big)F^{p(t)}_{si} \in \widetilde{\mathfrak{sl}}_\infty\big(p(t)\big).$$

For all $i,j \in \mathbb{Z}$,

$$Y_2 := \Big(\sum_{m,n=0}^{r} a_m a_n \delta_{i+m,s+n}\Big)(F^{p(t)}_{ii} - F^{p(t)}_{jj}) \in \widetilde{\mathfrak{sl}}_\infty\big(p(t)\big).$$

So

$$Y_1 - Y_2 = \Big(\sum_{m,n=0}^{r} a_m a_n \delta_{i+m,s+n}\Big)F^{p(t)}_{jj} - \Big(\sum_m a_m^2\Big)F^{p(t)}_{si} \in \widetilde{\mathfrak{sl}}_\infty\big(p(t)\big)$$

for all $i,j \in \mathbb{Z}$, and $i - r \leq s \leq i + r$.

Let $q \gg j \gg i$. Then $F^{p(t)}_{qq} - F^{p(t)}_{ii} \in N$. Hence

$$[F^{p(t)}_{qq} - F^{p(t)}_{ii}, Y_1 - Y_2] = \Big(\sum_{m=0}^{r} a_m^2\Big)[F^{p(t)}_{ii}, F^{p(t)}_{si}] \\ = \Big(\sum_m a_m^2\Big)\bigg(\Big(\sum_{m,n=0}^{r} a_m a_n \delta_{i+m,s+n}\Big)F^{p(t)}_{ii} - \Big(\sum_{m=0}^{r} a_m^2\Big)F^{p(t)}_{si}\bigg) \in N.$$

So

(7) $$\Big(\sum_{m,n=0}^{r} a_m a_n \delta_{i+m,s+n}\Big) F_{ii}^{p(t)} - \Big(\sum_{m=0}^{r} a_m^2\Big) F_{si}^{p(t)} \in N$$

for all $i \in \mathbb{Z}$, and $i - r \leq s \leq i + r$.

Now we see from Lemma 1 that $N = \widetilde{\mathfrak{sl}}_\infty\big(p(t)\big)$ and hence complete the proof. ∎

CASE 2. $\sum_{m=0}^{r} a_m^2 = 0$.

We see from (2) and (4) that in this case $F_{ii}^{p(t)} \in \widetilde{\mathfrak{sl}}_\infty\big(p(t)\big)$ and $F_{ij}^{p(t)} \in \widetilde{\mathfrak{sl}}_\infty\big(p(t)\big)$ for all $j \in \mathbb{Z}$, $i < j - r$ or $i > j + r$.

Let $k = j + r$ in (1). We get

$$a_r F_{il}^{p(t)} - \Big(\sum_{m,n=0}^{r} a_m a_n \delta_{l+m,i+n}\Big) F_{kj}^{p(t)} \in \widetilde{\mathfrak{sl}}_\infty\big(p(t)\big).$$

Then

$$a_r F_{i+s,l+s}^{p(t)} - \Big(\sum_{m,n=0}^{r} a_m a_n \delta_{l+m,i+n}\Big) F_{kj}^{p(t)} \in \widetilde{\mathfrak{sl}}_\infty\big(p(t)\big)$$

for all $s \in \mathbb{Z}$ and hence

(8) $$F_{il}^{p(t)} - F_{i+1,l+1}^{p(t)} \in \widetilde{\mathfrak{sl}}_\infty\big(p(t)\big)$$

for all $i, l \in \mathbb{Z}$.

LEMMA 2. *If* $\sum_{m=0}^{r} a_m^2 = 0$, *then*

$$\widetilde{\mathfrak{sl}}_\infty\big(p(t)\big) = \sum_{j\in\mathbb{Z}, i\neq j-r,\ldots,j+r} \mathbb{C}F_{ij}^{p(t)} + \sum_{i\in\mathbb{Z}, -r\leq s\leq r, \sum_m a_m a_{m+s}=0} \mathbb{C}F_{i,i+s}^{p(t)}$$
$$+ \sum_{i\in\mathbb{Z}, -r\leq s\leq r, \sum_m a_m a_{m+s}\neq 0} \mathbb{C}(F_{i,i+s}^{p(t)} - F_{i+1,i+s+1}^{p(t)}).$$

PROOF. We claim that $F_{0r}^{p(t)} \notin \widetilde{\mathfrak{sl}}_\infty\big(p(t)\big)$. In fact, since

$$\begin{aligned} F_{0r}^{p(t)} = {} & a_0^2 E_{0r} + a_0 a_1 E_{0,r+1} + \cdots + a_0 a_r E_{0,2r} \\ & + a_0 a_1 E_{1r} + a_1^2 E_{1,r+1} + \cdots + a_1 a_r E_{1,2r} \\ & + \cdots \\ & + a_0 a_r E_{rr} + a_1 a_r E_{r,r+1} + \cdots + a_r^2 E_{r,2r}, \end{aligned}$$

if

$$F_{0r}^{p(t)} \in \widetilde{\mathfrak{sl}}_\infty\big(p(t)\big) \subseteq \mathfrak{sl}_\infty(\mathbb{C}),$$

then $a_r E_{rr} \in \mathfrak{sl}_\infty(\mathbb{C})$. This is a contradiction. Similarly we have $F_{0,-r}^{p(t)} \notin \widetilde{\mathfrak{sl}}_\infty\big(p(t)\big)$. Now for $-r \leq s \leq r$,

$$[F_{0s}^{p(t)}, F_{0r}^{p(t)}] = \Big(\sum_{m,n=0}^{r} a_m a_n \delta_{m+s,n}\Big) F_{0r}^{p(t)} - a_r F_{0s}^{p(t)} \in \widetilde{\mathfrak{sl}}_\infty\big(p(t)\big).$$

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But $F_{0r}^{p(t)} \notin \widetilde{\mathfrak{sl}}_\infty\big(p(t)\big)$, so $F_{0s}^{p(t)} \in \widetilde{\mathfrak{sl}}_\infty\big(p(t)\big)$ iff $\sum_{m=0}^{r} a_m a_{m+s} = 0$. If $F_{0s}^{p(t)} \in \widetilde{\mathfrak{sl}}_\infty\big(p(t)\big)$, then by (8), $F_{i,i+s}^{p(t)} \in \widetilde{\mathfrak{sl}}_\infty\big(p(t)\big)$ for all $i \in \mathbb{Z}$. ∎

PROPOSITION 3. *If $\sum_m a_m^2 = 0$, then $\widetilde{\mathfrak{sl}}_\infty\big(p(t)\big)$ is a simple Lie algebra.*

PROOF. Let $N \neq 0$ be an ideal of $\widetilde{\mathfrak{sl}}_\infty\big(p(t)\big)$. An argument similar to the one of Proposition 2 shows that

$$F_{ij}^{p(t)} \in N$$

for all $i,j \in \mathbb{Z}$, with $i < j - r$ or $i > j + r$.

For $-r \leq s \leq r$,

$$\begin{aligned}[F_{ij}^{p(t)}, F_{j+s,l}^{p(t)}] &= \Big(\sum_{m,n=0}^{r} a_m a_n \delta_{m,s+n}\Big) F_{il}^{p(t)} - \Big(\sum_{m,n=0}^{r} a_m a_n \delta_{l+m,i+n}\Big) F_{j+s,j}^{p(t)} \\ &= \Big(\sum_{m=0}^{r} a_m a_{m-s}\Big) F_{il}^{p(t)} - \Big(\sum_{m=0}^{r} a_m a_n \delta_{l+m,i+n}\Big) F_{j+s,j}^{p(t)}\end{aligned} \tag{9}$$

Let $i = l \gg j$ and $s = r$ in (9). We get $F_{ii}^{p(t)} \in N$ for all $i \in \mathbb{Z}$.

Assume that $\sum_m a_m a_{m-s} = 0$. For $j \in \mathbb{Z}$, choose $i = l + r \ll j$. Then from (9), $-a_r F_{j+s,j}^{p(t)} \in N$. So $F_{i,i-s}^{p(t)} \in N$ for all $i \in \mathbb{Z}$.

Assume that $\sum_{m=0}^{r} a_m a_{m+s} \neq 0$. Then for $i \gg k$, $F_{i,k+s}^{p(t)}, F_{k,i+s}^{p(t)} \in N$, and hence

$$\begin{aligned}[F_{i,k+s}^{p(t)}, F_{k,i+s}^{p(t)}] &= \Big(\sum_{m,n=0}^{r} a_m a_n \delta_{k+s+m,k+n}\Big) F_{i,i+s}^{p(t)} - \Big(\sum_{m,n=0}^{r} a_m a_n \delta_{i+s+m,i+n}\Big) F_{k,k+s}^{p(t)} \\ &= \Big(\sum_{m=0}^{r} a_m a_{m+s}\Big)(F_{i,i+s}^{p(t)} - F_{k,k+s}^{p(t)}) \in N.\end{aligned}$$

Then from

$$F_{i,i+s}^{p(t)} - F_{k,k+s}^{p(t)} \in N,$$

and

$$F_{i+1,i+1+s}^{p(t)} - F_{k,k+s}^{p(t)} \in N,$$

where $i \gg k$, we get

$$F_{i,i+s}^{p(t)} - F_{i+1,i+1+s}^{p(t)} \in N,$$

for all $i \in \mathbb{Z}$. ∎

5. **Polynomial Lie subalgebras of $\mathfrak{gl}_n(\mathbb{C})$.** $\mathfrak{gl}_n(\mathbb{C}) = \sum_{1 \leq i,j \leq n} \mathbb{C} E_{ij}$ is a Lie subalgebra of $\mathfrak{gl}_\infty(\mathbb{C})$. We define for $p(t) = 1 + a_1 t + \cdots + a_r t^r$, where $r < n$, the following Lie subalgebras of $\mathfrak{gl}_n(\mathbb{C})$:

$$\begin{aligned}\mathfrak{gl}_n\big(p(t)\big) &:= \mathfrak{gl}_\infty\big(p(t)\big) \cap \mathfrak{gl}_n(\mathbb{C}),\\ \mathfrak{sl}_n\big(p(t)\big) &:= \mathfrak{sl}_\infty\big(p(t)\big) \cap \mathfrak{gl}_n(\mathbb{C}),\\ \widetilde{\mathfrak{gl}}_n\big(p(t)\big) &:= \widetilde{\mathfrak{gl}}_\infty\big(p(t)\big) \cap \mathfrak{gl}_n(\mathbb{C}),\\ \widetilde{\mathfrak{sl}}_n\big(p(t)\big) &:= \widetilde{\mathfrak{sl}}_\infty\big(p(t)\big) \cap \mathfrak{gl}_n(\mathbb{C}).\end{aligned}$$

We call them the polynomial Lie subalgebras of $\mathfrak{gl}_n(\mathbb{C})$. In this section we discuss the cases $n = 2$ and $n = 3$. We will see that $\mathfrak{sl}_n\big(p(t)\big)$ and $\widetilde{\mathfrak{sl}}_n\big(p(t)\big)$ are not simple Lie algebras in general.

(a) $n = 2, p(t) = 1 + at, (a \neq 0)$.

$$\mathfrak{gl}_2\big(p(t)\big) = \mathbb{C}E_{11}^{p(t)} + \mathbb{C}E_{21}^{p(t)},$$

where

$$E_{11}^{p(t)} = E_{11} + aE_{12} = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix},$$

$$E_{21}^{p(t)} = E_{21} + aE_{22} = \begin{pmatrix} 0 & 0 \\ 1 & a \end{pmatrix}.$$

Since

$$[E_{11}^{p(t)}, E_{21}^{p(t)}] = aE_{11}^{p(t)} - E_{21}^{p(t)},$$

if we set

$$X = E_{11}^{p(t)} - \frac{a}{2}E_{21}^{p(t)},$$

and

$$Y = E_{11}^{p(t)} - \frac{1}{a}E_{21}^{p(t)},$$

then

$$\mathfrak{gl}_2\big(p(t)\big) = \mathbb{C}X + \mathbb{C}Y,$$

where $[X, Y] = Y$.

(b) $n = 3, p(t) = 1 + at + bt^2, (b \neq 0)$.

$$\mathfrak{gl}_3\big(p(t)\big) = \mathbb{C}E_{11}^{p(t)} + \mathbb{C}E_{21}^{p(t)} + \mathbb{C}E_{31}^{p(t)},$$

where

$$E_{11}^{p(t)} = \begin{pmatrix} 1 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_{21}^{p(t)} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & a & b \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_{31}^{p(t)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & a & b \end{pmatrix}.$$

It is easy to see that

$$\mathfrak{sl}_3\big(p(t)\big) = \mathbb{C}(aE_{11}^{p(t)} - E_{21}^{p(t)}) + \mathbb{C}(bE_{11}^{p(t)} - E_{31}^{p(t)})$$

is an abelian Lie algebra and

$$\mathfrak{gl}_3\big(p(t)\big) = \mathbb{C}E_{11}^{p(t)} \ltimes \mathfrak{sl}_3\big(p(t)\big).$$

(c) $n = 3, p(t) = 1 + at, (a \neq 0)$.

$$\mathfrak{gl}_3\big(p(t)\big) = \sum_{i=1,2,3, j=1,2} \mathbb{C}E_{ij}^{p(t)},$$

where

$$E_{11}^{p(t)} = \begin{pmatrix} 1 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{12}^{p(t)} = \begin{pmatrix} 0 & 1 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$E_{21}^{p(t)} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & a & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{22}^{p(t)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 0 \end{pmatrix},$$
$$E_{31}^{p(t)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & a & 0 \end{pmatrix}, \quad E_{32}^{p(t)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & a \end{pmatrix}.$$

We see that

$$\mathfrak{sl}_3\big(p(t)\big) = \mathbb{C}(E_{11}^{p(t)} - E_{22}^{p(t)}) + \mathbb{C}E_{12}^{p(t)} + \mathbb{C}E_{21}^{p(t)} + \mathbb{C}E_{31}^{p(t)} + \mathbb{C}E_{32}^{p(t)}$$

is a Lie algebra of dimension 5 which is not simple, and

$$\mathfrak{gl}_3\big(p(t)\big) = \mathbb{C}E_{11}^{p(t)} \ltimes \mathfrak{sl}_3\big(p(t)\big).$$

(d) $n = 3, p(t) = 1 + at, (a \neq 0)$.

$$\widetilde{\mathfrak{gl}}_3\big(p(t)\big) = \sum_{i,j=1,2} \mathbb{C}F_{ij}^{p(t)},$$

where

$$F_{11}^{p(t)} = \begin{pmatrix} 1 & a & 0 \\ a & a^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_{12}^{p(t)} = \begin{pmatrix} 0 & 1 & a \\ 0 & a & a^2 \\ 0 & 0 & 0 \end{pmatrix},$$
$$F_{21}^{p(t)} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & a & 0 \\ a & a^2 & 0 \end{pmatrix}, \quad F_{22}^{p(t)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & a \\ 0 & a & a^2 \end{pmatrix}.$$

(i) If $1 + a^2 = 0$, then

$$\widetilde{\mathfrak{sl}}_3\big(p(t)\big) = \mathbb{C}F_{11}^{p(t)} + \mathbb{C}(F_{12}^{p(t)} - F_{21}^{p(t)}) + \mathbb{C}F_{22}^{p(t)} \cong \mathfrak{sl}_2(\mathbb{C})$$

is a simple Lie algebra and

$$\widetilde{\mathfrak{gl}}_3\big(p(t)\big) = \mathbb{C}F_{12}^{p(t)} \ltimes \widetilde{\mathfrak{sl}}_3\big(p(t)\big).$$

(ii) If $1 + a^2 \neq 0$, then $F_{11}^{p(t)} \notin \widetilde{\mathfrak{sl}}_3\big(p(t)\big)$. But

$$X := F_{12}^{p(t)} - F_{21}^{p(t)} = \begin{pmatrix} 0 & 1 & a \\ -1 & 0 & a^2 \\ -a & -a^2 & 0 \end{pmatrix} \in \widetilde{\mathfrak{sl}}_3\big(p(t)\big),$$

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and

$$Y := F_{11}^{p(t)} - F_{22}^{p(t)} = \begin{pmatrix} 1 & a & 0 \\ a & a^2-1 & -a \\ 0 & -a & -a^2 \end{pmatrix} \in \widetilde{\mathfrak{sl}}_3\big(p(t)\big).$$

So we have

$$[X, Y] = 2Z,$$

where

$$Z = \begin{pmatrix} a & -1 & -a-a^3 \\ -1 & -a-a^3 & -a^4 \\ -a-a^3 & -a^4 & a^3 \end{pmatrix} \in \widetilde{\mathfrak{sl}}_3\big(p(t)\big).$$

Since X, Y, Z are linear independent,

$$\widetilde{\mathfrak{sl}}_3\big(p(t)\big) = \mathbb{C}X + \mathbb{C}Y + \mathbb{C}Z.$$

Moreover,

$$[X, Z] = -2(1 + a^2 + a^4)Y,$$
$$[Y, Z] = -2(1 + a^2 + a^4)X.$$

So if $1 + a^2 + a^4 \neq 0$, then $\widetilde{\mathfrak{sl}}_3\big(p(t)\big) \cong \mathfrak{sl}_2(\mathbb{C})$. If $1 + a^2 + a^4 = 0$, then $\widetilde{\mathfrak{sl}}_3\big(p(t)\big)$ is isomorphic to the 3-dimensional Heisenberg algebra.

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