REPRESENTATIONS OF THE ℓ^1-ALGEBRA OF AN INVERSE SEMIGROUP HAVING THE SEPARATION PROPERTY†

by BRUCE A. BARNES

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Introduction. Let S be a semigroup, and let $\ell^1(S)$ denote the ℓ^1-semigroup algebra of S. Beginning with the fundamental paper of E. Hewitt and H. Zuckerman [5], there has been a considerable amount of research done concerning the Banach algebra $\ell^1(S)$ in the case when S is abelian; see the bibliography [7]. However, until recently, there was very little information known concerning $\ell^1(S)$ when S was nonabelian and infinite. Now for certain classes of infinite nonabelian semigroups with involution, recent progress has been made in the study of the Banach *-algebra $\ell^1(S)$ and the *-representations of $\ell^1(S)$. In [2], B. Barnes and J. Duncan prove that $\ell^1(S)$ is Jacobson semisimple, study the spectrum of elements in $\ell^1(S)$, and construct and study *-representations of $\ell^1(S)$ when S is the free semigroup with a finite or countably infinite set of generators (and also in some cases where the generators satisfy certain relations). In [1], the present author considered the representation theory of $\ell^1(S)$ where S is an inverse semigroup. This paper is a sequel to [1].

In the remainder of this paper S is an inverse semigroup. If $a \in S$, we denote by a^* the unique element in S with the properties

$$aa^*a = a \quad \text{and} \quad a^*aa^* = a^*$$

(the usual notation for this element is a^{-1}). The map $a \to a^*$ on S lifts to an involution $f \to f^*$ on $\ell^1(S)$ defined by the rule

$$f^*(a) = \overline{f(a^*)} \quad (a \in S).$$

Thus, in this case, $\ell^1(S)$ is a Banach *-algebra. Since we consider only *-representations of $\ell^1(S)$ on Hilbert space, we use " representation " to mean automatically " *-representation on Hilbert space ". In [1] we constructed specific examples of irreducible representations of $\ell^1(S)$ for certain important inverse semigroups. We also developed a theory concerned with the irreducible *-representations of $\ell^1(S)$ determined by finite idempotents of S [1, §3]. An idempotent $e \in S$ is finite if eSe is a finite set. When e is a finite idempotent of S, there exists an ideal I of S such that e is primitive modulo I [1, Proposition 3.1]. This implies that there is a finite group G_e associated with e. Moreover, the irreducible representations of $\ell^1(G_e)$ determine in a natural way irreducible representations of $\ell^1(S)$. In this paper we prove that if S has a property that we call the separation property, then the family of irreducible *-representations of $\ell^1(S)$ determined by the finite idempotents of S is a separating family for $\ell^1(S)$; i.e. the intersection of the kernels of all the representations in the family is $\{0\}$. Our proof of this result provides a second proof of the theorem that the ℓ^1-algebra of an arbitrary inverse semigroup has a separating family of irreducible *-representations (this was first proved in [1, Theorem 2.3]).

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Also, we explicitly construct the set of all irreducible representations of $\ell^1(S)$ determined by finite idempotents for two interesting inverse semigroups of partial transformations. The first example is the symmetric inverse semigroup on a countably infinite set. In this case the set of finite groups G_e which induce irreducible representations of the ℓ^1-algebra is the set of all the finite symmetric groups. The second example is a semigroup of partial transformations for which the groups G_e are all trivial, with the result that the construction of the corresponding irreducible representations is especially easy to describe.

Now we introduce some notation. We denote the set of idempotents of S by $I(S)$. An element $e \in I(S)$ is finite if the set eSe is finite. The set of all finite idempotents of S we denote by $FI(S)$.

We use exclusively the complex number field which we denote by $\mathbf{C}$.

Let A be an algebra. An idempotent f in A is a minimal idempotent (abbreviation: m.i.) if Af is a minimal left ideal of A. If A is a normed algebra which is Jacobson semisimple, then f is a m.i. of A if and only if

$$fAf = \{\lambda f : \lambda \in \mathbf{C}\}$$

[6, p. 45]. We use the notation soc(A) to denote the socle of A [6, p. 46]. Now assume that A has an involution. A representation π of A on a Hilbert space H will often be denoted by the pair (π, H). If two representations (π, H) and (τ, K) are unitarily equivalent we write $(\pi, H) \approx (\tau, K)$.

If Λ is an index set, then $\ell^2(\Lambda)$ is the set of all complex functions f on Λ such that

$$\sum\{|f(\lambda)|^2 : \lambda \in \Lambda\} < \infty.$$

The set $\ell^2(\Lambda)$ is a Hilbert space with the usual operations and inner product [6, p. 296]. We let $\varphi(\mu)$ be the function in $\ell^2(\Lambda)$ that takes the value 1 at μ and is 0 elsewhere on Λ. Then $\{\varphi(\lambda) : \lambda \in \Lambda\}$ is an orthonormal basis for $\ell^2(\Lambda)$ which we call the standard basis. We note that there is an exception to this notation. As in [1], if S has a zero θ, then by convention we use the notation $\ell^2(S)$ to denote the space $\ell^2(S\backslash\{\theta\})$ as defined above. Similarly, by convention, we use the notation $\ell^1(S)$ to denote the space $\ell^1(S\backslash\{\theta\})$.

If X is a nonempty set, we denote the symmetric inverse semigroup on X by $\mathscr{I}_X$ [4, p. 29]. We let $\mathscr{F}_X$ be the set of all maps $a \in \mathscr{I}_X$ such that the domain of a is finite. If n is a positive integer, then $\mathscr{F}_n$ denotes the set of all $a \in \mathscr{I}_X$ such that the domain of a contains at most n elements. The set $\mathscr{F}_0$ consists of the empty map alone. If $b \in \mathscr{I}_X$, then we denote the domain and range of b by D_b and R_b, respectively.

The cardinality of a set B is denoted by $|B|$.

1. The separation property. In this section we introduce the separation property for a semigroup S and verify that any subsemigroup of $\mathscr{I}_X$ that contains every finite idempotent in $\mathscr{I}_X$ has the separation property.

DEFINITION 1.1. S has the separation property if for any finite subset $\{a_1, \ldots, a_n\}$ of distinct elements of S, there exists $e \in FI(S)$ such that $a_1 e \neq a_k e$ $(2 \leq k \leq n)$ and $a_1 e \neq \theta$ (if S has a zero, θ).

PROPOSITION 1.2. *Let S be any subsemigroup of $\mathscr{I}_X$ such that $FI(\mathscr{I}_X) \subset S$. Then S has the separation property.*

Proof. Let $\{a_1, \ldots, a_n\}$ be a set of distinct elements of S. Denote the domain of a_k as D_k, and let $A \backslash B$ denote the set difference of sets A and B. For each k, with $2 \leqq k \leqq n$, let

$$M_k = \{x \in D_1 \cap D_k : a_1(x) \neq a_k(x)\} \cup (D_1 \backslash D_k).$$

Note that if M_k is empty, then $D_1 \subset D_k$ and $a_1(x) = a_k(x)$ for all $x \in D_1$. In this case, $D_k \backslash D_1$ must be nonempty. Form a set Z as follows.

(1) If each M_k is empty ($2 \leqq k \leqq n$) then choose an element $y_k \in D_k \backslash D_1$ for each k and set

$$Z = \{y_2, \ldots, y_n\} \cup D_1.$$

(2) Assume M_j is nonempty for some j. If M_k is nonempty, then choose an element $y_k \in M_k$. If M_k is empty, then choose an element $y_k \in D_k \backslash D_1$. Then let

$$Z = \{y_2, \ldots, y_n\}.$$

Now let e be the finite idempotent map with domain Z; i.e. $D_e = Z$ and $e(x) = x$ for all $x \in Z$. It is easy to verify that $a_1 e \neq \theta$ and $a_1 e \neq a_k e$, $2 \leqq k \leqq n$.

We give two examples of interesting inverse subsemigroups of $\mathscr{I}_X$ that have the separation property. First, let $\mathcal{O}$ be a total order on X. We denote by $\mathscr{I}(X, \mathcal{O})$ the collection of all maps in $\mathscr{I}_X$ which are order-preserving with respect to the order $\mathcal{O}$. It is a straightforward exercise to show that $\mathscr{I}(X, \mathcal{O})$ is an inverse semigroup and that $I(\mathscr{I}_X) \subset \mathscr{I}(X, \mathcal{O})$. Thus, $\mathscr{I}(X, \mathcal{O})$ has the separation property by Proposition 1.2.

Now let $\sim$ be an equivalence relation on X. Let $\mathscr{I}(X, \sim)$ denote the collection of all maps $a \in \mathscr{I}_X$ with the property that for all $x \in D_a$, we have $x \sim a(x)$. Again, it is easy to show that $\mathscr{I}(X, \sim)$ is an inverse semigroup with the separation property.

The separation property on S implies a very useful property of the algebra $\ell^1(S)$; namely, that if $f \in \ell^1(S)$ and $fe = 0$ for all $e \in FI(S)$, then $f = 0$. As we shall see in §2, this property of $\ell^1(S)$ has a number of important ramifications concerning the structure and the representation theory of $\ell^1(S)$.

PROPOSITION 1.3. *Assume that S has the separation property. If $f \in \ell^1(S)$ and $fe = 0$ for all $e \in FI(S)$, then $f = 0$.*

Proof. Assume $fe = 0$ for all $e \in FI(S)$, and $f \neq 0$. Then f has the form

$$f = \sum \lambda_k a_k \quad (a_k \in S, \lambda_k \in \mathbf{C})$$

where $a_k \neq a_j$ for $k \neq j$, and $\lambda_k \neq 0$ for all k (the sum may be either finite or infinite). Choose n so large that

$$\sum_{k>n} |\lambda_k| < |\lambda_1|.$$

Since S has the separation property, there exists $e \in FI(S)$ such that $a_1 e \neq \theta$ and $a_1 e \neq a_k e$,

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$2 \leqq k \leqq n$. Let $M = \{k : a_k e = a_1 e\}$. Now $fe = 0$, so that $\sum_{k \in M} \lambda_k = 0$. Then

$$|\lambda_1| = |\sum_{k \in M, k \neq 1} \lambda_k|$$
$$\leqq \sum_{k > n} |\lambda_k| < |\lambda_1|.$$

This contradiction proves that $f = 0$.

There are examples where S does not have the separation property, but $l^1(S)$ does have the property that when $f \in l^1(S)$ and $fe = 0$ for all $e \in FI(S)$, then $f = 0$. The results in §2 depend only on this property of $l^1(S)$, and not on the stronger assumption that S has the separation property. The separation property does seem to be the most easily stated and the most readily verifiable assumption on a semigroup S that implies the desired property of $l^1(S)$.

2. The semigroup algebra when S has the separation property. In this section we prove that if S has the separation property, then the set of all irreducible representations of $l^1(S)$ which are determined by finite idempotents of S is a separating family for $l^1(S)$. At the same time we derive a second proof of the result that $l^1(S)$ has a faithful representation on Hilbert space when S is an arbitary inverse semigroup [**1**, Theorem 2.3]. This implies that $l^1(S)$ is an A^*-algebra in the sense of C. Rickart [**6**, p. 181]. It follows from [**6**, Theorem (4.1.19)] that $l^1(S)$ is Jacobson semisimple.

For the remainder of this section let π denote the left regular representation (LRR) of $l^1(S)$ on $l^2(S)$ as constructed in [**1**, §2]. As in [**1**], let $\{\varphi(b) : b \in S, b \neq \theta\}$ be the standard orthonormal basis for $l^2(S)$. If $a, b \in S \backslash \{\theta\}$, then by definition

$$\pi(a)\varphi(b) = \varphi(ab) \quad \text{if} \quad a^*ab = b,$$
$$\pi(a)\varphi(b) = 0 \quad \text{if} \quad a^*ab \neq b.$$

The only result we assume from [**1**] is that π is a representation of $l^1(S)$ on $l^2(S)$.

LEMMA 2.1. *If $h \in l^1(S)$ and $h = \lambda_1 a_1 + \ldots + \lambda_m a_m$, where $a_k \neq a_j$ $(k \neq j)$ and $\lambda_k \neq 0$ $(1 \leqq k \leqq m)$, then $\pi(h) \neq 0$. As a consequence, if S is finite, then π is a faithful representation of $l^1(S)$ on $l^2(S)$.*

Proof. Let $e_k = a_k^* a_k$ $(1 \leqq k \leqq m)$. With respect to the usual ordering of idempotents in S, the set $\{e_1, \ldots, e_m\}$ contains a maximal element which we may assume to be e_1. Then

$$\pi(a_j)\varphi(e_1) = 0 \quad \text{if} \quad e_j \neq e_1, \quad \text{and}$$
$$\pi(a_j)\varphi(e_1) = \varphi(a_j e_1) = \varphi(a_j) \quad \text{if} \quad e_j = e_1.$$

Therefore,

$$\pi(h)\varphi(e_1) = \sum\{\lambda_j \varphi(a_j) : j \text{ such that } e_1 = e_j\} \neq 0.$$

When S is finite, then Lemma 2.1 implies that π is a faithful representation of $l^1(S)$ on $l^2(S)$. Thus $l^1(S)$ is Jacobson semisimple in this case. We state this result as a lemma.

LEMMA 2.2. *If S is finite, then $l^1(S)$ is Jacobson semisimple.*

The result in Lemma 2.2 also follows from a more general theorem [**4**, Theorem 5.26].

THEOREM 2.3. *Assume that S has the separation property. Let $A = \ell^1(S)$.*

(1) *If k is a m.i. of eAe for some $e \in FI(S)$, then Ak is a minimal left ideal of A.*

(2) *If $f \in A$ and $fAk = \{0\}$ whenever k is a m.i. of eAe for some $e \in FI(S)$, then $f = 0$.*

Proof. If $e \in FI(S)$, then by Lemma 2.2, $eAe = \ell^1(eSe)$ is Jacobson semisimple. Let k be a m.i. of eAe. Note that the existence of at least one such k is assured by the application of standard Wedderburn theory to the finite-dimensional, Jacobson semisimple algebra eAe. We prove that Ak is a minimal left ideal of A. Suppose $h = gk$, $g \in A$, $h \neq 0$. Then $he = gke = gk = h$. By Proposition 1.3 there exists $f \in FI(S)$ such that $h^*f \neq 0$. Then $fh \neq 0$ and $fh = fhe$ is contained in the finite dimensional algebra fAe. Thus fhe is a finite linear combination of maps in fSe. Lemma 2.1 implies that $\pi(fh) \neq 0$, and therefore, $\pi(h^*fh) \neq 0$. Then

$$h^*fh = eh^*fhe \in eAe.$$

Also, note that

$$h^*fh = (eh^*fge)k.$$

Since k is a m.i. of eAe, there exists $t \in eAe$ such that $th^*fh = k$. Therefore whenever $h \in Ak$, $h \neq 0$, then $Ah = Ak$. This proves (1).

The given condition in (2) implies by an easy argument that $fe = 0$, for all $e \in FI(S)$. Then $f = 0$ by Proposition 1.3.

COROLLARY 2.4. *Let S be an inverse semigroup. Then $\ell^1(S)$ has a faithful representation on Hilbert space. As a result, $\ell^1(S)$ is Jacobson semisimple.*

Proof. First assume that S has the separation property. Let J be the $*$-radical of $A = \ell^1(S)$ [**6**, Definition (4.4.9)]. Now we prove that $J = \{0\}$. Suppose $g \in J$, $g \neq 0$. By Theorem 2.3, there exists a minimal idempotent k of eAe for some $e \in FI(S)$ such that $gAk \neq \{0\}$. If $ghk \neq 0$, $h \in A$, then $Ak = Aghk \subset J$. But if π is the LRR of A, then by Lemma 2.1, $\pi(k) \neq 0$. It follows by [**6**, Theorem (4.6.7)] that $k \notin J$. This contradiction proves the result when S has the separation property.

To prove the general case, note that if S is an inverse semigroup, then S can be embedded as an inverse subsemigroup of $S' = \mathscr{I}_S$ [**4**, Theorem 1.20]. It follows that there is a $*$-isomorphism τ of $\ell^1(S)$ into $\ell^1(S')$. Since S' has the separation property [Proposition 1.2], then there is a faithful representation γ of $\ell^1(S')$ on Hilbert space. Then $\gamma \circ \tau$ is a faithful representation of $\ell^1(S)$ on Hilbert space. It follows from this and [**6**, Theorem (4.1.19)] that $\ell^1(S)$ is Jacobson semisimple.

Define

$$\mathscr{F}(S) = \{a \in S : a^*a \in FI(S)\}.$$

If $a \in \mathscr{F}(S)$, then

$$aa^*Saa^* = a(a^*a)(a^*Sa)(a^*a)a^*.$$

Therefore, since $(a^*a)S(a^*a)$ is finite, we have $aa^* \in FI(S)$. Thus, if $a \in \mathscr{F}(S)$, then $a^* \in \mathscr{F}(S)$. If $a \in \mathscr{F}(S)$ and $b \in S$, then

$$(ab)^*(ab)S(ab)^*(ab) \subset b^*(a^*a)S(a^*a)b,$$

and this last set is finite. Therefore $ab \in \mathscr{F}(S)$. This verifies that $\mathscr{F}(S)$ is an ideal of S.

PROPOSITION 2.5. *Assume that S has the separation property. Let $A = \ell^1(S)$. Then*

(1) *$\ell^1(\mathscr{F}(S))$ is the closure of* soc(A), *and*

(2) *the left annihilator of $\ell^1(\mathscr{F}(S))$ is $\{0\}$.*

The proof of this proposition is a routine application of Theorem 2.3.

Let $A = \ell^1(S)$. Denote by $\mathscr{M}$ the set of all elements $k \in A$ such that k is a m.i. of eAe for some $e \in FI(S)$ and $k = k^*$. If $k \in \mathscr{M}$, then by Theorem 2.3 (1) we have that k is a selfadjoint m.i. of A. There is a standard method for constructing an irreducible representation of A on a Hilbert space using the left ideal Ak, where k is a selfadjoint m.i. of A [**6**, Theorem (4.10.3)]. We denote this representation by π_k and the representation space by H_k. In this construction, Ak is naturally imbedded in H_k. If $\langle \cdot, \cdot \rangle$ is the inner product on H_k and $f, g, h \in A$, then $\langle \pi_k(f)gk, hk \rangle$ is the unique scalar λ such that $kh^*fgk = \lambda k$. It is easily verified that

$$\ker(\pi_k) = \{f \in A : fAk = \{0\}\}.$$

PROPOSITION 2.6. *Assume that S has the separation property.*

(1) *The collection of irreducible representations $\{\pi_k : k \in \mathscr{M}\}$ is a separating family for A.*

(2) *If (π, H) is an irreducible representation of A such that $\ell^1(\mathscr{F}(S)) \not\subset \ker(\pi)$, then there exist $k \in \mathscr{M}$ such that $(\pi, H) \approx (\pi_k, H_k)$.*

Proof. Part (1) follows from [**6**, Lemma (4.10.1)] and Theorem 2.3.

Now we prove (2). There exists $k \in \mathscr{M}$ such that $k \notin \ker(\pi)$. Then $\pi(k)$ is a nonzero projection on H. Choose $x_0 \in H$, with $\| x_0 \| = 1$, such that $\pi(k)x_0 = x_0$. Consider the positive functionals α and β defined on A by the equations

$$\beta(f) = (\pi(f)x_0, x_0) \quad (f \in A),$$

$$\alpha(f) = \langle \pi_k(f)k, k \rangle \quad (f \in A).$$

Note that $\alpha(f) = \lambda$, where $kfk = \lambda k$. By [**6**, Lemma (4.5.8)], (π, H) and (π_k, H_k) are equivalent to the representations of A determined by the positive functionals β and α, respectively. If $f \in A$, then

$$\begin{aligned}\beta(f) &= (\pi(f)x_0, x_0) \\ &= (\pi(f)\pi(k)x_0, \pi(k)x_0) \\ &= (\pi(kfk)x_0, x_0) \\ &= (\alpha(f)\pi(k)x_0, x_0) \\ &= \alpha(f).\end{aligned}$$

Therefore, $\pi \approx \pi_k$.

If S is an inverse subsemigroup of $\mathscr{I}_X$ for some set X and $FI(\mathscr{I}_X) \subset S$, then $\mathscr{F}(S)$ is the set of all maps in S that have a finite domain; i.e. $\mathscr{F}(S) = \mathscr{F}_X \cap S$. Also, S has the separation property by Proposition 3.1, so that Propositions 2.5 and 2.6 apply to $\ell^1(S)$. In particular, the closure of the socle of $\ell^1(S)$ is $\ell^1(\mathscr{F}_X \cap S)$, and if (π, H) is an irreducible representation of $\ell^1(S)$

such that $l^1(\mathscr{F}_X \cap S) \not\subset \ker(\pi)$, then π is equivalent to a representation of $l^1(S)$ of the form π_k. These remarks apply to the examples $\mathscr{I}(X, \mathscr{O})$ and $\mathscr{I}(X, \sim)$ mentioned in §1.

3. Construction of certain irreducible representations of the symmetric inverse semigroup on a set. Let $\mathbf{N}$ be the set of positive integers. In this section we explicitly construct certain irreducible representations of $l^1(\mathscr{I}_{\mathbf{N}})$ that are induced by the known irreducible matrix representations of the finite symmetric groups. Moreover, we show that the set of representations constructed is a separating family for $l^1(\mathscr{I}_{\mathbf{N}})$. A similar construction can be effected for $l^1(\mathscr{I}_X)$, where X is a set of any given cardinality. To carry out the procedure in this general case, some total order must be designated for X. In the case at hand, $X = \mathbf{N}$, we use the natural ordering on $\mathbf{N}$.

Fix a positive integer n. Let S_n be the symmetric group acting on the set of numbers $M = \{1, 2, \ldots, n\}$. When dealing with S_n or the representations of S_n, we use the notation and terminology in [3]. Fix a set of positive integers $\{m_1, m_2, \ldots, m_k\}$ with the properties $m_1 \geqq m_2 \geqq \ldots \geqq m_k$ and $n = m_1 + \ldots + m_k$. Let F be the frame [3, p. 111] corresponding to the collection $\{m_1, \ldots, m_k\}$. Denote by Δ_F the set of standard tableaux with entries from M [3, p. 118]. In [3, Chapter IV], an irreducible matrix representation of S_n is constructed using the set Δ_F. This representation lifts to an irreducible algebra representation of $l^1(S_n)$ on some vector space. We use Young's orthogonal matrix representation of S_n [3, pp. 133–135] which lifts to an irreducible representation of $l^1(S_n)$ on the Hilbert space $l^2(\Delta_F)$.

We allow a tableau in the frame F to have any set of n distinct positive integers as entries. This is a more general concept of tableau than that in [3], where entries are restricted to elements of M. If T is a tableau, then let $\{T\}$ denote the set of entries in T. When $a \in \mathscr{I}_{\mathbf{N}}$ and T is a tableau with $\{T\} \subset D_a$, we let aT be the tableau formed by replacing each entry m in T by $a(m)$. Let Ω_F be the set of all tableaux of the form ψT where $T \in \Delta_F$ and ψ is an order-preserving map such that $D_\psi = \{T\}$. The tableaux in Ω_F are called the standard tableaux (for the given frame F). The standard tableaux are exactly those tableaux T with entries in $\mathbf{N}$ with the property that the numbers increase in every row of T from left to right and in every column of T downward.

If K and J are any two finite subsets of $\mathbf{N}$ such that $|K| = |J|$, then there exists a unique order-preserving (abbreviation: o.p.) map in $\mathscr{F}_{\mathbf{N}}$ with domain K and range J. We use this fact in what follows. Also, we identify S_n as a subset of $\mathscr{I}_{\mathbf{N}}$ by identifying $\sigma \in S_n$ with the map $a \in \mathscr{I}_{\mathbf{N}}$ having domain $M = \{1, 2, \ldots, n\}$ and taking values $a(m) = \sigma(m)$, $1 \leqq m \leqq n$.

If $a \in \mathscr{I}_{\mathbf{N}}$, and K is a subset of $\mathbf{N}$ such that $K \subset D_a$, then $a|K$ denotes the restriction map determined by restricting a to K.

DEFINITION 3.1. Let a be a map in $\mathscr{I}_{\mathbf{N}}$. Assume that $T \in \Omega_F$ and $\{T\} \subset D_a$. Let

(i) χ be the unique o.p. map with domain M and range $\{aT\}$,

(ii) ψ be the unique o.p. map with domain $\{T\}$ and range M, and

(iii) $\sigma = \chi^*(a|\{T\})\psi^* \in S_n$.

Then

$$a|\{T\} = \chi\sigma\psi.$$

We call the product $\chi\sigma\psi$ the standard decomposition of $a|\{T\}$.

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Let e_M be the map in $\mathscr{I}_{\mathrm{N}}$ with domain M taking values $e_M(k) = k$, $k \in M$. Then e_M is the identity of S_n.

LEMMA 3.2. *Let a, b be maps in $\mathscr{I}_{\mathrm{N}}$. Assume $T \in \Omega_F$ and $\{T\} \subset D_b$ and $\{bT\} \subset D_a$. Let $b \,|\, \{T\} = \chi\sigma\psi$ and $a \,|\, \{bT\} = \chi_1\sigma_1\psi_1$ be the corresponding standard decompositions. Then $\psi_1\chi = e_M$ and $ab \,|\, \{T\}$ has standard decomposition*

$$ab \,|\, \{T\} = \chi_1\sigma_1\sigma\psi.$$

Proof. By definition χ is unique o.p. map with domain M and range $\{bT\}$ and ψ_1 is the unique o.p. map with domain $\{bT\}$ and range M. It follows that $\psi_1\chi = e_M$. Then

$$\begin{aligned} ab \,|\, \{T\} &= (a \,|\, \{bT\})(b \,|\, \{T\}) \\ &= \chi_1\sigma_1\psi_1\chi\sigma\psi \\ &= \chi_1\sigma_1 e_M \sigma\psi \\ &= \chi_1\sigma_1\sigma\psi. \end{aligned}$$

Recall that Δ_F is the set of standard tableaux with entries from $\{1, 2, \ldots, n\}$ relative to the frame F. Let γ be Young's orthogonal representation of S_n on the finite Hilbert space $\ell^2(\Delta_F)$; see [3, pp. 133–135] where γ is explicitly constructed in terms of matrices with respect to the basis $\{\varphi(T) : T \in \Delta_F\}$. If $T \in \Omega_F$ and χ is a o.p. map with $\{T\} \subset D_\chi$, then $\chi T \in \Omega_F$. When $\sum_{k=1}^{m} \lambda_k \varphi(T_k)$ is any finite sum in $\ell^2(\Omega_F)$, define

$$\chi\left(\sum_{k=1}^{m} \lambda_k \varphi(T_k)\right) = \sum_{k=1}^{m} \lambda_k \varphi(\chi T_k) \in \ell^2(\Omega_F).$$

Now we define $\pi_\gamma(a)$ for $a \in \mathscr{I}_{\mathrm{N}}$. First we define $\pi_\gamma(a)$ on the basis $\{\varphi(T) : T \in \Omega_F\}$.

Let $T \in \Omega_F$. Then

(1) if $\{T\} \not\subset D_a$, let $\pi_\gamma(a)\varphi(T) = 0$;
(2) if $\{T\} \subset D_a$, and $a \,|\, \{T\} = \chi\sigma\psi$ is the standard decomposition of $a \,|\, \{T\}$, let

$$\pi_\gamma(a)\varphi(T) = \chi(\gamma(\sigma)\varphi(\psi T)).$$

It is not difficult to verify that the natural linear extension of $\pi_\gamma(a)$ to the inner product space of finite linear combinations of $\{\varphi(T) : T \in \Omega_F\}$ is a bounded operator with norm at most 1.

Thus $\pi_\gamma(a)$ extends in the usual way to a bounded linear operator on $\ell^2(\Omega_F)$ with $\|\pi_\gamma(a)\| \leqq 1$. If $f \in \ell^1(\mathscr{I}_{\mathrm{N}})$, $f = \sum \lambda_k a_k$, then let

$$\pi_\gamma(f) = \sum \lambda_k \pi_\gamma(a_k).$$

Then $\pi_\gamma(f)$ is a bounded linear operator on $\ell^2(\Omega_F)$. In fact, $\|\pi_\gamma(f)\| \leqq \|f\|$, $f \in \ell^1(\mathscr{I}_{\mathrm{N}})$. Now we prove that $f \to \pi_\gamma(f)$ is a representation of $\ell^1(\mathscr{I}_{\mathrm{N}})$ on $\ell^2(\Omega_F)$. We do this in (3.3) and (3.4) below.

(3.3). $$\pi_\gamma(fg) = \pi_\gamma(f)\pi_\gamma(g) \quad (f, g \in \ell^1(\mathscr{I}_{\mathrm{N}})).$$

Proof. It suffices to prove that $\pi_\gamma(ab)\varphi(T) = \pi_\gamma(a)\pi_\gamma(b)\varphi(T)$ when $a, b \in \mathscr{I}_N$ and $T \in \Omega_F$. If $\{T\} \not\subset D_b$ or $\{bT\} \not\subset D_a$, then $\{T\} \not\subset D_{ab}$ so that, by (1), $\pi_\gamma(a)\pi_\gamma(b)\varphi(T) = 0 = \pi_\gamma(ab)\varphi(T)$. Suppose that $\{T\} \subset D_b$ and $\{bT\} \subset D_a$. Let $b \mid \{T\} = \chi\sigma\psi$ and $a \mid \{bT\} = \chi_1\sigma_1\psi_1$ be the corresponding standard decompositions. Then, by Lemma 3.2, we have $\psi_1\chi = e_M$ and $ab \mid \{T\}$ has standard decomposition $\chi_1\sigma_1\sigma\psi$. Then

$$\begin{aligned}\pi_\gamma(ab)\varphi(T) &= \chi_1(\gamma(\sigma_1\sigma)\varphi(\psi T)) \\ &= \chi_1\gamma(\sigma_1)e_M\gamma(\sigma)\varphi(\psi T) \\ &= (\chi_1\gamma(\sigma_1)\psi_1)(\chi\gamma(\sigma)\psi)\varphi(T) \\ &= \pi_\gamma(a)\pi_\gamma(b)\varphi(T).\end{aligned}$$

(3.4). $$\pi_\gamma(f^*) = \pi_\gamma(f)^*, \quad f \in l^1(\mathscr{I}_N).$$

Proof. It suffices to show that

$$(\pi_\gamma(a)\varphi(T), \varphi(S)) = (\varphi(T), \pi_\gamma(a^*)\varphi(S)) \tag{1}$$

whenever $a \in \mathscr{I}_N$ and $T, S \in \Omega_F$.

Assume first that $\{T\} \not\subset D_a$. In this case $\pi_\gamma(a)\varphi(T) = 0$, and we prove that $(\varphi(T), \pi_\gamma(a^*)\varphi(S)) = 0$. If $\{S\} \not\subset D_{a^*}$, then $\pi_\gamma(a^*)\varphi(S) = 0$. Suppose that $\{S\} \subset D_{a^*} = R_a$. Let $a^* \mid \{S\} = \chi_1\tau\psi_1$ be the standard decomposition of this map. Since $R_{\chi_1} = \{a^*S\} \subset D_a$, we have $R_{\chi_1} \neq \{T\}$. Therefore

$$(\varphi(T), \varphi(\chi_1 W)) = 0, \quad \text{for all} \quad W \in \Delta_F. \tag{2}$$

By definition

$$\pi_\gamma(a^*)\varphi(S) = \chi_1\gamma(\tau)\varphi(\psi_1 S).$$

Also,

$$\gamma(\tau)\varphi(\psi_1 S) = \sum_{k=1}^{m} \lambda_k \varphi(S_k)$$

where $S_k \in \Delta_F$, $\lambda_k \in \mathbf{C}$. Then

$$\pi_\gamma(a^*)\varphi(S) = \sum_{k=1}^{m} \lambda_k \varphi(\chi_1 S_k).$$

Therefore by (2), $(\varphi(T), \pi_\gamma(a^*)\varphi(S)) = 0$. This proves that (1) holds when $\{T\} \not\subset D_a$. Similarly, (1) holds when $\{S\} \not\subset D_{a^*}$.

Now assume that $\{T\} \subset D_a$ and $\{S\} \subset D_{a^*}$. Let

$$a \mid \{T\} = \chi\sigma\psi \quad \text{and} \quad a^* \mid \{S\} = \chi_1\tau\psi_1$$

be the standard decompositions of these maps. There exist

$$T_k \in \Delta_F, \lambda_k \in \mathbf{C} \quad (1 \leqq k \leqq p),$$

$$S_j \in \Delta_F, \mu_j \in \mathbf{C} \quad (1 \leqq j \leqq q),$$

such that

$$\gamma(\sigma)\varphi(\psi T) = \lambda_1\varphi(T_1)+\ldots+\lambda_p\varphi(T_p),$$
$$\gamma(\tau)\varphi(\psi_1 T) = \mu_1\varphi(S_1)+\ldots+\mu_q\varphi(S_q).$$

Note that $\{\chi T_k\} = \{aT\}$ and $\{\chi_1 S_j\} = \{a^*S\}$ for all k,j. Thus, if $\{aT\} \neq \{S\}$, then $\{T\} \neq \{a^*S\}$, so that $(\pi_\gamma(a)\varphi(T), \varphi(S)) = 0 = (\varphi(T), \pi_\gamma(a^*)\varphi(S))$. Assume that $\{aT\} = \{S\}$. Then $\{T\} = \{a^*S\}$. In this case, from the definitions of χ_1, ψ_1, χ, and ψ, we have

$$\chi = \psi_1^* \quad \text{and} \quad \chi_1^* = \psi.$$

Also, $a^* \,|\, \{S\} = (a \,|\, \{T\})^*$. Then

$$\tau = \chi_1^*(a^* \,|\, \{S\})\psi_1^* = \psi(a \,|\, \{T\})^*\chi$$
$$= (\chi^*(a \,|\, \{T\})\psi^*)^* = \sigma^*.$$

Note that whenever $W, V \in \Omega_F$, and ξ is an o.p. map from $\{W\}$ to $\{V\}$, then $(\xi\varphi(W), \varphi(V)) = (\varphi(W), \xi^*\varphi(V))$. Therefore,

$$\begin{aligned}(\pi_\gamma(a)\varphi(T), \varphi(S)) &= (\chi\gamma(\sigma)\psi\varphi(T), \varphi(S))\\ &= (\gamma(\sigma)\psi\varphi(T), \chi^*\varphi(S))\\ &= (\psi\varphi(T), \gamma(\sigma^*)\chi^*\varphi(S))\\ &= (\varphi(T), \psi^*\gamma(\sigma^*)\chi^*\varphi(S))\\ &= (\varphi(T), \chi_1\gamma(\tau)\psi_1\varphi(S))\\ &= (\varphi(T), \pi_\gamma(a^*)\varphi(S)).\end{aligned}$$

This completes the verification of (1) in all cases.

Let n be a positive integer. Denote by J the closed ideal $\ell^1(\mathscr{F}_{n-1})$ of A. Let Q_J be the natural quotient map of A onto A/J. The algebra A/J is isomorphic to $\ell^1(S/\mathscr{F}_{n-1})$. If h is a selfadjoint m.i. of A/J, then (π_h, H_h) is an irreducible representation of A/J (the notation here is the same as that used in §2 in the remarks preceding Proposition 2.6). Therefore, $f \to (\pi_h \circ Q_J)(f)$ is an irreducible representation of A on H_h. Representations of this sort are discussed in [**1**, §3].

As before, we have S_n identified with the set of all one-to-one maps of M onto M. Then $\ell^1(S_n)$ is a subalgebra of A, and in fact, $\ell^1(S_n) \subset e_M A e_M$.

LEMMA 3.5. *Let h' be a selfadjoint m.i. of $\ell^1(S_n)$. Assume that (τ, K) is a representation of A with the properties that*

(i) $\ell^1(\mathscr{F}_{n-1}) \subset \ker(\tau)$, *and*
(ii) *there exists $x_0 \in K$ such that $\tau(Ah')x_0$ is dense in K.*

Let $h = Q_J(h')$, where $J = \ell^1(\mathscr{F}_{n-1})$. Then h is a selfadjoint m.i. of A/J and

$$(\tau, K) \approx (\pi_h \circ Q_J, H_h).$$

Proof. If $g \in A$, then $e_M g e_M = f + k$ for some $f \in \ell^1(S_n)$ and some $k \in J$. Therefore,

$$h'gh' = h'e_M g e_M h' = h'fh' + h'kh'$$
$$= \lambda h' + h'kh',$$

where λ is a scalar and $h'kh' \in J$. Consequently, $hQ_J(g)h = Q_J(h'gh') = \lambda h$. This proves that h is a selfadjoint m.i. of A/J. Now the lemma follows from [**1**, Theorem 3.2].

THEOREM 3.6. *Let $\mathscr{R}$ be the collection of all the representations $(\pi_\gamma, l^2(\Omega_F))$, where γ is any irreducible representation of $l^1(S_n)$ and n is any positive integer.*

(1) *Any representation in $\mathscr{R}$ is an irreducible representation of $l^1(\mathscr{I}_{\mathrm{N}})$.*

(2) *If (τ, K) is an irreducible representation of $l^1(\mathscr{I}_{\mathrm{N}})$ such that $l^1(\mathscr{F}_{\mathrm{N}}) \not\subset \ker(\tau)$, then (τ, K) is equivalent to some representation in $\mathscr{R}$.*

(3) *The collection $\mathscr{R}$ is a separating family of irreducible representations for $l^1(\mathscr{I}_{\mathrm{N}})$.*

Proof. First we prove that the representations in $\mathscr{R}$ are irreducible. Let n be a positive integer, and let γ be an irreducible representation of $l^1(S_n)$ on $l^2(\Delta_F)$, where F is the corresponding frame. Choose h' a selfadjoint m.i. of $l^1(S_n)$ such that $\gamma(h') \neq 0$. Choose $T \in \Delta_F$ such that $\gamma(h')\varphi(T) \neq 0$. Since γ is irreducible on $l^2(\Delta_F)$, there exists $f \in l^1(S_n)$ such that $\gamma(fh')\varphi(T) = \varphi(T)$. Now we prove that $\varphi(T)$ is cyclic vector for π_γ in $l^2(\Omega_F)$. Assume $R_1 \in \Omega_F$. There exist $T_1 \in \Delta_F$ and an order-preserving map $\psi_1 : M \to \{R_1\}$ such that $\psi_1\varphi(T_1) = \varphi(\psi_1 T_1) = \varphi(R_1)$. There exists $g \in l^1(S_n)$ such that $\pi_\gamma(g)\varphi(T) = \varphi(T_1)$. Thus, $\pi_\gamma(\psi_1 g)\varphi(T) = \varphi(R_1)$. Suppose $R_k \in \Omega_F$, $1 \leqq k \leqq m$. Consider the vector $\lambda_1\varphi(R_1) + \ldots + \lambda_m\varphi(R_m)$. By the previous argument, we can choose $f_k \in l^1(\mathscr{I}_{\mathrm{N}})$ $(1 \leqq k \leqq m)$ such that

$$\pi_\gamma(f_k)\varphi(T) = \varphi(R_k) \quad (1 \leqq k \leqq m).$$

Thus,

$$\pi_\gamma(k)\varphi(T) = \lambda_1\varphi(R_1) + \ldots + \lambda_m\varphi(R_m),$$

where $k = \lambda_1 f_1 + \ldots + \lambda_m f_m$. It follows from Lemma 3.5 that $(\pi_\gamma, l^2(\Omega_F))$ is irreducible.

Let (τ, K) be a representation of $l^1(\mathscr{I}_{\mathrm{N}})$ such that $\mathscr{F}_{\mathrm{N}} \not\subset \ker(\tau)$. There exists $e \in FI(\mathscr{I}_{\mathrm{N}})$ such that $\tau(e) \neq 0$. We may assume that e is a minimal element in the set

$$\{f \in FI(\mathscr{I}_{\mathrm{N}}) : \tau(f) \neq 0\}$$

with respect to the usual ordering on $I(\mathscr{I}_{\mathrm{N}})$. Suppose $f, g \in FI(\mathscr{I}_{\mathrm{N}})$ and $|D_f| = |D_g|$. Then there exists $a \in \mathscr{I}_{\mathrm{N}}$ such that $f = a^*a$ and $g = aa^*$. Therefore,

$$\tau(f) = 0 \Leftrightarrow \tau(a) = 0 \Leftrightarrow \tau(g) = 0.$$

It follows from the choice of e that $\tau(f) = 0$ if $f \in FI(\mathscr{I}_{\mathrm{N}})$ and $|D_f| < |D_e|$, while $\tau(f) \neq 0$ if $|D_f| = |D_e|$. Let $n = |D_e|$. By the previous argument, the ideal $\ker(\tau) \cap \mathscr{I}_{\mathrm{N}}$ of $\mathscr{I}_{\mathrm{N}}$ is $\mathscr{F}_{n-1}$. Therefore, $l^1(\mathscr{F}_{n-1}) \subset \ker(\tau)$. Also, $\tau(e_M) \neq 0$ where $M = \{1, 2, \ldots, n\}$. Consequently there exists a selfadjoint m.i. h' of $l^1(S_n)$ such that $\tau(h') \neq 0$. Let γ be the irreducible representation of $l^1(S_n)$ on $l^2(\Delta_F)$ determined by h', where F is the corresponding frame. Let $J = l^1(\mathscr{F}_{n-1})$, and let $h = Q_J(h')$. Then, by Lemma 3.5,

$$(\tau, K) \approx (\pi_h \circ Q_J, H_h) \approx (\pi_\gamma, l^2(\Omega_F)).$$

This proves (2).

Part (3) follows from (2) and Proposition 2.6.

4. A separating family of irreducible representations of $\mathscr{I}(\mathbf{N}, \mathcal{O})$. Let $\mathbf{N}$ be the positive integers, and let $\mathcal{O}$ be the usual total order on $\mathbf{N}$. Recall that $\mathscr{I}(\mathbf{N}, \mathcal{O})$ is the inverse subsemigroup of $\mathscr{I}_{\mathbf{N}}$ consisting of those maps $a \in \mathscr{I}_{\mathbf{N}}$ which are order-preserving. For the remainder of this section we let S denote $\mathscr{I}(\mathbf{N}, \mathcal{O})$. As another application of the results in section 2 and the results in [1, §3], we construct the set of all irreducible representations $\ell^1(S)$ that are determined by finite idempotents of S. In this particular case, the construction is quite easy.

Fix n a positive integer. Let Γ_n be the set of all subsets K of $\mathbf{N}$ such that $|K| = n$. Now we define a representation π_n of $\ell^1(S)$ on the Hilbert space $\ell^2(\Gamma_n)$. Let $\{\varphi(K) : K \in \Gamma_n\}$ be the standard orthonormal basis of this Hilbert space. If $a \in S$ and $T \in \Gamma_n$, define

$$\pi_n(a)\varphi(T) = \varphi(aT), \quad \text{if} \quad T \subset D_a,$$
$$\pi_n(a)\varphi(T) = 0, \quad \text{if} \quad T \not\subset D_a.$$

It is easy to show that $a \to \pi_n(a)$ is a representation of S on $\ell^2(\Gamma_n)$. This semigroup representation extends to a representation of $\ell^1(S)$ on $\ell^2(\Gamma_n)$ in the usual way. We denote this extension by π_n also. We have $\pi_n \neq \pi_m$ if $n \neq m$.

THEOREM 4.1. (1) *For each $n \geqq 1$, the representation $(\pi_n, \ell^2(\Gamma_n))$ is irreducible.*

(2) *If (τ, K) is an irreducible representation of $\ell^1(S)$ such that $\ell^1(\mathscr{F}_{\mathbf{N}} \cap S) \not\subset \ker(\tau)$, then $(\tau, K) \approx (\pi_n, \ell^2(\Gamma_n))$, for some n.*

(3) *The collection $(\pi_n, \ell^2(\Gamma_n))$ $(n \geqq 1)$ is a separating family of irreducible representations for $\ell^1(S)$.*

Proof. Fix a positive integer n. Let $A = \ell^1(S)$, and let $J = \ell^1(\mathscr{F}_{n-1} \cap S)$. Choose $e \in FI(S)$ such that $|D_e| = n$. Note that if $a \in S$, then either $eae = e$ or $eae \in \mathscr{F}_{n-1}$. Therefore, if $g \in A$, then $ege = \lambda e + k$, where $\lambda \in \mathbf{C}$ and $k \in J$. Let $h = Q_J(e)$. Then

$$hQ_J(g)h = Q_J(ege) = Q_J(\lambda e + k) = \lambda h.$$

This proves that h is a selfadjoint m.i. of A/J. Let $T = D_e$. Now we prove that $\varphi(T)$ is a cyclic vector for π_n. Suppose $R_j \in \Gamma_n$ and $\lambda_j \in \mathbf{C}$ $(1 \leqq j \leqq m)$. There exist $\psi_j \in S$ such that T is the domain of ψ_j and $\psi_j T = R_j$ for $1 \leqq j \leqq m$. Let $f = \lambda_1 \psi_1 + \ldots + \lambda_m \psi_m \in \ell^1(S)$. Then

$$\pi_n(f)\varphi(T) = \sum_{j=1}^{m} \lambda_j \varphi(\psi_j T)$$
$$= \sum_{j=1}^{m} \lambda_j \varphi(R_j).$$

Therefore $\varphi(T)$ is cyclic for π_n. It follows from [1, Theorem 3.2] that $(\pi_n, \ell^2(\Gamma_n)) \approx (\pi_h \circ Q_J, H_h)$. This proves (1).

Now assume that (τ, K) is an irreducible representation of $\ell^1(S)$ such that $\ell^1(\mathscr{F}_{\mathbf{N}} \cap S) \not\subset \ker(\tau)$. Then there exists $e \in FI(S)$ such that $\tau(e) \neq 0$. We may assume that e is a minimal element of the set

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$$\{f \in FI(S) : \tau(f) \neq 0\}.$$

Then, as in the proof of part (2) of Theorem 3.6, it follows that $\ell^1(\mathscr{F}_{n-1} \cap S) \subset \ker(\tau)$ where $n = |D_e|$. Again, let $J = \ell^1(\mathscr{F}_{n-1} \cap S)$. A repetition of the proof of part (1) shows that $h = Q_J(e)$ is a selfadjoint m.i. of A/J, and that $(\pi_n, \ell^2(\Gamma_n)) \approx (\pi_h \circ Q_J, H_h)$. But also, by [**1**, Theorem 3.2] we have $(\tau, K) \approx (\pi_h \circ Q_J, H_h)$. This proves (2).

Part (3) is an immediate consequence of Proposition 1.2, Proposition 2.6, and (2).

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