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TWO TYPES OF DUALITY IN MULTIOBJECTIVE FRACTIONAL PROGRAMMING

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In this paper, we are concerned with duality of a multiobjective fractional program. Two different dual problems are introduced with respect to the primal multiobjective fractional program. Under a mild assumption, we prove a weak duality theorem and a strong duality theorem for each type of duality. Finally, we explore some relations between these two types of duality.

1. INTRODUCTION

Multiobjective fractional programming duality has been of much interest in the last decade. Quite a number of publications appeared, such as references [**2, 3, 4, 6, 7, 8, 10, 12**]. Among them, Bector, Chandra and Husain introduced a dual problem of a certain quasidifferentiable multiobjective fractional programming problem based on the Kuhn-Tucker type optimality conditions in [**2**]. Bector, Chandra and Singh, in [**3**], used their linearisation technique to study the Schönfeld duality of a multiobjective fractional programming problem in which the objective functions are pseudo-convex and the constraints are linear. Egudo [**6**] studied both the Mond-Weir extension of the Bector dual analogy and the Schaible type vector dual of a multiobjective fractional programming problem where the components of the objective function vector have non-negative and convex numerators while the denominators are concave and positive. Weir in [**12**] gave a few results on the Mond-Weir type duality of a multiobjective fractional optimisation problem and also showed that for a properly efficient primal solution the dual solution is also properly efficient. Weir and Jeyakumar studied in [**13**] the Lagrange duality of a multiobjective fractional programming problem under an assumption of pre-invexity.

In this paper, we introduce two types of duality to a more general multiobjective fractional programming problem. Under a pre-invexity assumption, we prove a weak duality theorem and a strong duality theorem for each type of duality. Also we explore

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some relations between these two types of duality in multiobjective fractional programming. Finally, we indicate two possible extensions in the conclusions. It should be mentioned that the two types of duality introduced in this paper are different from those mentioned above.

2. Preliminaries

Let $x, y \in \mathbb{R}^m$. We denote

$$\begin{aligned} \mathbb{R}^m_+ &= \{x \in \mathbb{R}^m \mid x_i \geqslant 0,\ i = 1, \dots, m\}, \\ x \leqq y &\quad \text{if } y - x \in \mathbb{R}^m_+, \\ x \leqslant y &\quad \text{if } y - x \in \mathbb{R}^m_+ \setminus \{0\}, \\ x < y &\quad \text{if } y - x \in \operatorname{int}(\mathbb{R}^m_+). \end{aligned}$$

The multiobjective fractional programming problem considered in this paper can be formulated as

$$(\text{VFP}) \qquad \begin{cases} \text{V-minimise} & F(x) = \left(\dfrac{f_1(x)}{l_1(x)}, \dots, \dfrac{f_m(x)}{l_m(x)}\right)^T \\ \text{Subject to} & g(x) = \Big(g_1(x), \dots, g_p(x)\Big) \leqq 0 \\ & x \in X' \end{cases}$$

where X' is a nonempty open set in $\mathbb{R}^n$, each f_i, l_j and g_k are defined on $\mathbb{R}^n$, and each $l_j(x) \neq 0$ for any $x \in X \triangleq \{x \in X' \mid g(x) \leqq 0\}$.

Let $S \subset \mathbb{R}^m$. $\overline{y} \in S$ is called a weakly minimal efficient point in S if there is no $y \in S$ such that $y < \overline{y}$; $y^\circ \in S$ is called a weakly maximal efficient point in S if there is no $y \in S$ such that $y > y^\circ$. We denote by $\text{W-Min}\, S$ and $\text{W-Max}\, S$ the set of all the weakly minimal efficient point in S and the set of all the weakly maximal efficient points in S respectively. Put $F(X) = \{F(x) \mid x \in X\}$.

Definition 2.1: $\overline{x} \in X$ is called a weakly efficient solution of the multiobjective fractional programming problem (VFP) if $F(\overline{x}) \in \text{W-Min}\, F(X)$.

In order to derive duality theorems, first we quote the concept of a pre-invex vector valued function and three lemmas about their properties.

Suppose that $h : X' \to \mathbb{R}^m$.

Definition 2.2: h is called an $\mathbb{R}^m_+$-invex vector-valued function on X' if h is differentiable on X' and if there is an $\eta : X' \times X' \to \mathbb{R}^n$ such that for any $x, y \in X'$,

$$\Big[h(x) - h(y)\Big]^T \geqslant \Big[\eta(x, y)\Big]^T \nabla h(y)$$

where $\nabla h(y) = \left(\nabla h_1(y), \nabla h_2(y), \cdots, \nabla h_m(y)\right)$; h is called an $\mathbb{R}^m_+$-pre-invex vector-valued function on X' if there is an $\eta : X' \times X' \to \mathbb{R}^n$ such that for any $x, y \in X'$ and any $\lambda \in (0,1)$,

$$y + \lambda\eta(x,y) \in X' \quad \text{and} \quad \lambda h(x) + (1-\lambda)h(y) \geqslant h(y + \lambda\eta(x,y)).$$

REMARK 2.1. Invexity and pre-invexity are two extensions of convexity. In [**13**], Weir and Jeyakumar gave an example to illustrate that an invex function is not necessarily convex. For a detailed discussion, see [**5**] and [**13**].

The following lemma explores a relationship between invexity and pre-invexity.

LEMMA 2.1. [**13**] *If h is differentiable on X' and is an $\mathbb{R}^m_+$-pre-invex vector-valued function on X' (with respect to η), then h is an $\mathbb{R}^m_+$-invex vector-valued function on X' (with respect to η).*

With a proof similar to the one of [**13,** Theorem 1.2], we can prove the following lemma.

LEMMA 2.2. *Suppose that $f : X' \to \mathbb{R}^m$ and $g : X' \to \mathbb{R}^m$ are an $\mathbb{R}^m_+$-pre-invex vector-valued function and an $\mathbb{R}^p_+$-pre-invex vector-valued function on X' with respect to the same η, respectively. Then $Af + Bg$ is an $\mathbb{R}^k_+$-pre-invex vector-valued function on X' with respect to η for any $A \in \mathbb{R}^{k\times m}_+$ and any $B \in \mathbb{R}^{k\times p}_+$.*

For a pre-invex function, Weir and Jeyakumar proven the following interesting alternative theorem in [**13**].

LEMMA 2.3. *If h is an $\mathbb{R}^m_+$-pre-invex vector-valued function on X', then one and only one of the following two systems*

(i) $h(x) < 0$, $x \in X'$;

(ii) $\lambda^T h(x) \geqslant 0$ *for any* $x \in X'$, $\lambda \in \Lambda_+$ *has solutions, where* $\Lambda_+ = \{\lambda \in \mathbb{R}^m_+ \mid \sum_{i=1}^{m} \lambda_i = 1\}$.

Put $f(x) = (f_1(x), \cdots, f_m(x))$ and $l(x) = (l_1(x), \cdots, l_m(x))$ for any $x \in \mathbb{R}^n$. In the rest of this paper, we assume that the vector-valued functions f and g are an $\mathbb{R}^m_+$-pre-invex vector-valued function and an $\mathbb{R}^p_+$-pre-invex vector-valued function respectively, and l and $-l$ are both $\mathbb{R}^m_+$-pre-invex vector-valued functions, with respect to the same η.

3. DUALITY I

Define the vector-valued Lagrange function $L : X' \times \mathbb{R}^{m\times p}_+ \to \mathbb{R}^m$ of (VFP) by

$$L(x, U) = F(x) + [\operatorname{diag}(l_1(x), \cdots, l_m(x))]^{-1} U g(x),$$

where $\operatorname{diag}(l_1(x), \cdots, l_m(x))$ is the diagonal matrix consisting of $l_1(x), \cdots, l_m(x)$, and $\mathbb{R}_+^{m\times p}$ is the set of all the nonnegative matrices in $\mathbb{R}^{m\times p}$. For simplicity of notation, we put

$$U \circ g(x) = [\operatorname{diag}(l_1(x)), \cdots, l_m(x)]^{-1} U g(x).$$

We introduce the concept of a weak saddle point as follows.

DEFINITION 3.1: $(\overline{x}, \overline{U}) \in X' \times \mathbb{R}_+^{m\times p}$ is called a weak saddle point of the vector-valued Lagrange function L if

$$L(\overline{x}, \overline{U}) \in \text{W-Min}\left\{L(x, \overline{U}) \mid x \in X'\right\} \cap \text{W-Max}\left\{L(\overline{x}, U) \mid U \in \mathbb{R}_+^{m\times p}\right\}.$$

The following is a necessary and sufficient condition for a weak saddle point.

PROPOSITION 3.1. *$(\overline{x}, \overline{U}) \in X' \times \mathbb{R}_+^{m\times p}$ is a weak saddle point of L if and only if*

(1) $L(\overline{x}, \overline{U}) \in \text{W-Min}\{L(x, \overline{U}) \mid x \in X'\}$,
(2) $g(\overline{x}) \leqq 0$,
(3) $\overline{U} \circ g(\overline{x}) \not< 0$.

PROOF: First we suppose that $(\overline{x}, \overline{U})$ is a weak saddle point of L. By the definition of a weak saddle point of L, we know that (1) is satisfied and

$$L(\overline{x}, \overline{U}) \in \text{W-Max}\{L(\overline{x}, U) \mid U \in \mathbb{R}_+^{m\times p}\}.$$

The above expression implies

$$F(\overline{x}) + \overline{U} \circ g(\overline{x}) \not< F(\overline{x}) + U \circ g(\overline{x}), \quad \forall U \in \mathbb{R}_+^{m\times p}. \tag{3.1}$$

Let

$$D_1 = \{U \circ g(\overline{x}) - \overline{U} \circ g(\overline{x}) \mid U \in \mathbb{R}_+^{m\times p}\}.$$

It is easy to verify that D_1 is a nonempty convex set and, by (3.1),

$$D_1 \cap \operatorname{int}\left(\mathbb{R}_+^m\right) = \emptyset.$$

By the separation theorem of two convex sets (see Corollary 1 of Theorem 2.3.8 in [**1**]), there exists $\overline{u} \in \mathbb{R}^m \backslash \{0\}$ such that

$$\overline{\mu}^T \gamma \geqslant \overline{\mu}^T\left(U \circ g(\overline{x}) - \overline{U} \circ g(\overline{x})\right), \quad \forall \gamma \in \mathbb{R}_+^m, \quad \forall U \in \mathbb{R}_+^{m\times p}. \tag{3.2}$$

If we take $U = \overline{U}$ in the above inequality,

$$\overline{\mu}^T \gamma \geqslant 0 \quad \forall \gamma \in \mathbb{R}_+^m.$$

Hence, $\overline{\mu} \in \mathbb{R}^m_+ \backslash \{0\}$. Taking $\gamma = 0$ in (3.2), we have

$$\overline{\mu}^T\big(U \circ g(\overline{x}) - \overline{U} \circ g(\overline{x})\big) \leqslant 0, \quad \forall U \in \mathbb{R}^{m\times p}_+. \tag{3.3}$$

If (2) was not satisfied, that is, $g(\overline{x}) \nleqq 0$, then there would exist $\overline{\lambda} \in \mathbb{R}^p_+$ such that $\overline{\lambda}^T g(\overline{x}) > 0$. Without loss of any generality, we can assume

$$\overline{\lambda}^T g(\overline{x}) > \overline{\mu}^T\big(\overline{U} \circ g(\overline{x})\big). \tag{3.4}$$

Let $e \in \mathbb{R}^m_+$ satisfy $\overline{\mu}^T e = 1$. Set $\widetilde{U} = [\operatorname{diag}(l_1(x), \cdots, l_m(x))]e\overline{\lambda}^T$. We can easily show $\widetilde{U} \in \mathbb{R}^{m\times p}_+$ and

$$\begin{aligned}\overline{\mu}^T\Big(\widetilde{U} \circ g(\overline{x})\Big) &= \overline{\mu}^T[\operatorname{diag}(l_1(x), \cdots, l_m(x))]^{-1}\widetilde{U} g(\overline{x})\\ &= \overline{\mu}^T e\overline{\lambda}^T g(\overline{x}) = \overline{\lambda}^T g(\overline{x}).\end{aligned}$$

From (3.4), we have $\overline{\mu}^T\Big(\widetilde{U} \circ g(\overline{x})\Big) > \overline{\mu}^T\big(\overline{U} \circ g(\overline{x})\big)$, which contradicts (3.3). Therefore, (2) is satisfied. Letting $U = 0 \in \mathbb{R}^{m\times p}_+$ in (3.1), we get

$$\overline{U} \circ g(\overline{x}) \not< 0.$$

That is, (3) is satisfied.

Next, we show that $(\overline{x}, \overline{U})$ is a weak saddle point of L if (1) – (3) are satisfied. Suppose that (1) – (3) are satisfied. Since $g(\overline{x}) \leqq 0$,

$$U \circ g(\overline{x}) \leqq 0, \quad \forall U \in \mathbb{R}^{m\times p}_+.$$

Because $\overline{U} \circ g(\overline{x}) \not< 0$,

$$F(\overline{x}) + U \circ g(\overline{x}) \not> F(\overline{x}) + \overline{U} \circ g(\overline{x}), \quad \forall U \in \mathbb{R}^{m\times p}_+.$$

That is,

$$L(\overline{x}, \overline{U}) \in \text{W-Min}\{L(\overline{x}, U) \mid U \in \mathbb{R}^{m\times p}_+\}.$$

The above expression together with (1) implies that $(\overline{x}, \overline{U})$ is a weak saddle point of L. The proof is completed. $\square$

The next result indicates a relation between a weak saddle point of L and a weakly efficient solution of (VFP).

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PROPOSITION 3.2. *If $(\overline{x},\overline{U})$ is a weak saddle point of L and $\overline{U}\circ g(\overline{x})=0$, then $\overline{x}$ is a weak efficient solution of (VFP).*

PROOF: Suppose that $(\overline{x},\overline{U})$ is a weak saddle point L and that $\overline{U}\circ g(\overline{x})=0$. By Proposition 3.1, $\overline{x}\in X$. If $\overline{x}$ was not a weakly efficient solution of (VFP), there would exist an $x^\circ\in X$ such that $F(x^\circ)<F(\overline{x})$. Since $g(x^\circ)\leqq 0$ and $\overline{U}\in\mathbb{R}_+^{m\times p}$, $\overline{U}\circ g(x^\circ)=[\operatorname{diag}(l_1(x^\circ)),\cdots,l_n(x^\circ)]^{-1}\overline{U}g(x^\circ)\leqq 0$. Because $\overline{U}\circ g(\overline{x})=0$,

$$F(x^\circ)+\overline{U}\circ g(x^\circ)<F(\overline{x})+\overline{U}\circ g(\overline{x}).$$

This contradicts the fact $L(\overline{x},\overline{U})\in\text{W-Min}\{L(x,\overline{U})\mid x\in X'\}$. Therefore, $\overline{x}$ is a weakly efficient solution of (VFP). This completes the proof. ∎

Let $\omega=(\omega_1,\cdots,\omega_m)^T\in\mathbb{R}^m$ and put $H(x,\omega)=(f_1(x)-\omega_1 l_1(x),\cdots,f_m(x)-\omega_m l_m(x))^T$.

We say that the Slater constraint qualification is satisfied in (VFP) if there exists an $x\in X'$ such that $g(x')<0$. Under the assumption of the Slater constraint qualification, we show another relation between a weak saddle point and a weakly efficient solution.

THEOREM 3.1. *Suppose that the Slater constraint qualification is satisfied. If $\overline{x}$ is a weakly efficient solution of (VFP), then there exists $\overline{U}\in\mathbb{R}_+^{m\times p}$ such that $(\overline{x},\overline{U})$ is a weak saddle point of L and $U\circ g(\overline{x})=0$.*

PROOF: Let $\overline{\omega}=F(\overline{x})$. Thus, $H(\overline{x},\overline{\omega})=0$. Since $\overline{x}$ is a weakly efficient solution of (VFP), it is easy to show that

$$H(\lambda,\overline{\omega})\not< H(\overline{x},\overline{\omega})=0,\quad \forall x\in X. \tag{3.5}$$

In fact, if (3.5) was not satisfied, then there would exist an $x^\circ\in X$ such that $H(x^\circ,\overline{\omega})<0$, that is,

$$f_i(x^\circ)-\overline{\omega}_i l_i(x^\circ)<0,\quad i=1,\cdots,m.$$

So

$$F_i(x^\circ)=\frac{f_i(x^\circ)}{l_i(x^\circ)}<\overline{\omega}_i\quad i=1,\cdots,m.$$

Hence, $F(x^\circ)<\overline{\omega}=F(\overline{x})$. This contradicts the assumption that $\overline{x}$ is a weakly efficient solution to (VFP). From (3.5), the system

$$(H(x,\overline{\omega}),g(x))<0,\quad x\in X'$$

has no solution. Because f, g and $\pm l$ are an $\mathbb{R}_+^m$-pre-invex vector-valued function, an $\mathbb{R}_+^p$-pre-invex vector-valued function and $\mathbb{R}_+^m$-pre-invex vector-valued functions on X'

with respect to the same η, by Lemma 2.2, $(H(\cdot,\overline{\omega}), g(\cdot))$ is $\mathbb{R}_+^m \times \mathbb{R}_+^p$-pre-invex on X'. According to Lemma 2.3, there exists $(\overline{\mu},\overline{\lambda}) \in \mathbb{R}_+^m \times \mathbb{R}_+^p$, $(\overline{\mu},\overline{\lambda}) \neq 0$ such that

$$\overline{\mu}^T H(x,\overline{\omega}) + \overline{\lambda}^T g(\overline{x}) \geqslant 0, \quad \forall x \in X'. \tag{3.6}$$

Since $g(\overline{x}) \leqq 0$ and $\overline{\lambda} \in \mathbb{R}_+^p$, we have $\overline{\lambda}^T g(\overline{x}) \leqslant 0$. From (3.6), $\overline{\lambda}^T g(\overline{x}) \geqslant 0$. Hence,

$$\overline{\lambda}^T g(\overline{x}) = 0. \tag{3.7}$$

If $\overline{\mu} = 0$, then $\overline{\lambda} \in \mathbb{R}_+^p \backslash \{0\}$. By the assumption that the Slater constraint qualification is satisfied, there is an $x' \in X'$ such that $g(x') < 0$. So $\overline{\lambda}^T g(x') < 0$. But if we let $x = x'$ in (3.6), we get $\overline{\lambda}^T g(x') \geqslant 0$, a contradiction. Therefore, $\overline{\mu} \in \mathbb{R}_+^m \backslash \{0\}$. Let $e \in \mathbb{R}_+^m$ satisfy $\overline{\mu}^T e = 1$. Set $\overline{U} = e\overline{\lambda}^T$. Obviously, $\overline{U} \in \mathbb{R}_+^{m \times p}$, and $\overline{\mu}^T \overline{U} = \overline{\lambda}^T$. By (3.7),

$$\overline{U} \circ g(\overline{x}) = 0.$$

By Proposition 3.1, if $(\overline{x},\overline{U})$ was not a weak saddle point of L, then

$$L(\overline{x},\overline{U}) \notin \text{W-Min}\{L(x,\overline{U}) \mid x \in X'\}.$$

There would exist an $\widetilde{x} \in X'$ such that $L(\widetilde{x},\overline{U}) < L(\overline{x},\overline{U})$, that is,

$$\frac{f_i(\widetilde{x}) + \overline{u}_i^T g(\widetilde{x})}{l_i(\widetilde{x})} < \frac{f_i(\overline{x})}{l_i(\overline{x})} = \overline{\omega}_i, \quad i = 1, \dots, m$$

where $(\overline{u}_1, \cdots, \overline{u}_m)^T = \overline{U}$. Hence,

$$f_i(\widetilde{x}) - \overline{\omega}_i l_i(\widetilde{x}) + \overline{u}_i^T g(\widetilde{x}) < 0, \quad i = 1, \cdots, m$$

that is,

$$H(\widetilde{x},\overline{\omega}) + \overline{U} g(\widetilde{x}) < 0.$$

Therefore,

$$\overline{\mu}^T H(\widetilde{x},\overline{\omega}) + \overline{\lambda}^T g(\widetilde{x}) = \overline{\mu}^T \Big(H(\widetilde{x},\overline{\omega}) + \overline{U} g(\widetilde{x}) \Big) < 0.$$

This contradicts (3.6). Hence, $(\overline{x},\overline{U})$ is a weak saddle point of L. The proof is completed. $\square$

To derive the duality for (VFP), we are required to show the following proposition.

PROPOSITION 3.3. *Suppose that f, g and l are continuously differentiable on X'. If $(\overline{x},\overline{U}) \in X' \times \mathbb{R}_+^{m\times p}$ is a weak saddle point of L, then there exists a $\overline{\lambda} \in \Lambda_+$ such that*

$$\nabla_x L(\overline{x},\overline{U})\overline{\lambda} = 0.$$

PROOF: Suppose that $(\overline{x},\overline{U})$ is a weak saddle point of L. Since $L(\overline{x},\overline{U}) \in$ W-Min$\{L(x,\overline{U}) \mid x \in X'\}$,

$$\left[\eta(x,\overline{x})\right]^T \nabla_x L(\overline{x},\overline{U}) \not< 0, \quad \forall x \in X'. \tag{3.8}$$

In fact, if it was not the case, there would exist an $x^\circ \in X'$ such that

$$L'_x\Big(\big(\overline{x},\eta(x^\circ,\overline{x})\big),\overline{U}\Big) = \left[\eta(x^\circ,\overline{x})\right]^T \nabla_x L(\overline{x},\overline{U}) < 0, \tag{3.9}$$

where $L'_x\Big((\overline{x},\eta(x^\circ,\overline{x})),\overline{U}\Big)$ is the directional derivative of $L(\cdot,\overline{U})$ at $\overline{x}$ along the direction $\eta(x^\circ,\overline{x})$. Since X' is open, from (3.9), there exists a sufficiently small $\lambda > 0$ such that $\overline{x} + \lambda\eta(x^\circ,\overline{x}) \in X'$ and

$$L\Big(\overline{x} + \lambda\eta(x^\circ,\overline{x}),\overline{U}\Big) < L(\overline{x},\overline{U}).$$

This contradicts the fact $L(\overline{x},\overline{U}) \in$ W-Min$\{L(x,\overline{U}) \mid x \in X'\}$. Hence, (3.8) holds. By (3.8) and the Gordan Theorem (see [1]), there exists a $\overline{\lambda} \in \Lambda_+$ such that

$$\nabla_x L(\overline{x},\overline{U})\overline{\lambda} = 0.$$

This completes the proof. ∎

Let

$$T = \Big\{(x,U) \in X' \times \mathbb{R}_+^{m\times p} \mid \text{ there exists } \ \lambda \in \Lambda_+ \text{ such that } \nabla_x L(x,U)\lambda = 0\Big\}.$$

In the sequel, we always assume that T is not empty.

REMARK 3.1. We only need to assume that $\{L(x,U) \mid x \in X'\}$ is $\mathbb{R}_+^m$-compact for each fixed $U \in \mathbb{R}_+^{m\times p}$. By Proposition 3.3, $T \neq \emptyset$. Refer to [9] and [11] for a discussion on $\mathbb{R}_+^m$-compactness.

Now we can define our first dual problem of the multiobjective fractional programming problem (VFP) as follows:

$$\text{(VFD1)} \qquad \begin{cases} \text{V-maximise} & L(x,U) \\ \text{subject to} & (x,U) \in T. \end{cases}$$

DEFINITION 3.1: $(\overline{x},\overline{U}) \in T$ is called a weakly efficient solution of the dual problem (VFD1) if

$$L(\overline{x},\overline{U}) \in \text{W-Max}\{L(x,U) \mid (x,U) \in T\}.$$

The following result is a weak duality theorem.

Theorem 3.2. *If x is a feasible solution of (VFP) and $(\overline{x},\overline{U})$ is a feasible solution of (VFD1), then*

$$F(x) \not< L(\overline{x},\overline{U}).$$

Proof: Since $(\overline{x},\overline{U}) \in T$, there exists a $\overline{\lambda} \in \Lambda_+$ such that

$$\nabla_x L(\overline{x},\overline{U})\overline{\lambda} = 0.$$

So

$$\begin{aligned}
[\eta(x,\overline{x})]^T \nabla F(\overline{x})\overline{\lambda} &= -[\eta(x,\overline{x})]^T \nabla_x (\overline{U} \circ g(\overline{x}))\overline{\lambda} \\
&= -[\eta(x,\overline{x})]^T \sum_{i=1}^{m} \overline{\lambda}_i \nabla \left[\frac{\overline{u}_i^T g(\overline{x})}{l_i(\overline{x})}\right] \\
&= -[\eta(x,\overline{x})]^T \sum_{i=1}^{m} \overline{\lambda}_i \frac{l_i(\overline{x})\nabla g(\overline{x})\overline{u}_i - \overline{u}_i^T g(\overline{x})\nabla l_i(\overline{x})}{l_i^2(\overline{x})}.
\end{aligned}$$

Because g (respectively, $\pm l$) is differentiable and $\mathbb{R}^p_+$-pre-invex (respectively, $\mathbb{R}^m_+$-pre-invex) on X' with respect to η, by Lemma 2.1,

$$\begin{aligned}
[\eta(x,\overline{x})]^T \nabla F(\overline{x})\overline{\lambda} &\geqslant \sum_{i=1}^{m} \overline{\lambda}_i \frac{l_i(\overline{x})\overline{u}_i^T [g(\overline{x}) - g(x)] - \overline{u}_i^T g(\overline{x})[l_i(\overline{x}) - l_i(\overline{x})]}{l_i^2(\overline{x})} \\
&= \sum_{i=1}^{m} \overline{\lambda}_i \frac{l_i(x)}{l_i(\overline{x})} \left[\frac{\overline{u}_i^T g(\overline{x})}{l_i(\overline{x})} - \frac{\overline{u}_i^T g(x)}{l_i(x)}\right] \\
&\geqslant \sum_{i=1}^{m} \overline{\lambda}_i \frac{l_i(x)}{l_i(x)} \frac{\overline{u}_i^T g(\overline{x})}{l_i(\overline{x})}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
[\eta(x,\overline{x})]^T \nabla F(\overline{x})\overline{\lambda} &= [\eta(x,\overline{x})]^T \sum_{i=1}^{m} \overline{\lambda}_i \nabla \left[\frac{f_i(\overline{x})}{l_i(\overline{x})}\right] \\
&= \sum_{i=1}^{m} \overline{\lambda}_i [\eta(x,\overline{x})]^T \frac{l_i(\overline{x})\nabla f_i(\overline{x}) - f_i(\overline{x})\nabla l_i(\overline{x})}{l_i^2(\overline{x})}.
\end{aligned}$$

Since f and $\pm l$ are differentiable and $\mathbb{R}^m_+$-pre-invex on X' with respect to the same η, by Lemma 2.1,

$$\begin{aligned}
[\eta(x,\overline{x})]^T \nabla F(\overline{x})\overline{\lambda} &\leqslant \sum_{i=1}^{m} \overline{\lambda}_i \frac{l_i(\overline{x})[f_i(x) - f_i(\overline{x})] - f_i(\overline{x})[l_i(x) - l_i(\overline{x})]}{l_i^2(\overline{x})} \\
&= \sum_{i=1}^{m} \overline{\lambda}_i \frac{l_i(x)}{l_i(\overline{x})} \left[\frac{f_i(x)}{l_i(x)} - \frac{f_i(\overline{x})}{l_i(\overline{x})}\right].
\end{aligned}$$

Hence,

$$\sum_{i=1}^{m}\overline{\lambda}_i\frac{l_i(x)}{l_i(\overline{x})}\left[\frac{f_i(x)}{l_i(x)}-\frac{f_i(\overline{x})}{l_i(\overline{x})}\right]\geqslant\sum_{i=1}^{m}\overline{\lambda}_i\frac{l_i(x)}{l_i(\overline{x})}\frac{\overline{u}_i^Tg(\overline{x})}{l_i(\overline{x})},$$

$$\sum_{i=1}^{m}\overline{\lambda}_i\frac{l_i(x)}{l_i(\overline{x})}\frac{f_i(x)}{l_i(x)}\geqslant\sum_{i=1}^{m}\overline{\lambda}_i\frac{l_i(x)}{l_i(\overline{x})}\frac{f_i(\overline{x})-\overline{u}_i^Tg(\overline{x})}{l_i(\overline{x})},$$

that is,

$$\sum_{i=1}^{m}\overline{\lambda}_i\frac{l_i(x)}{l_i(\overline{x})}F_i(x)\geqslant\sum_{i=1}^{m}\overline{\lambda}_i\frac{l_i(x)}{l_i(\overline{x})}L_i(\overline{x},\overline{U}).$$

Therefore,

$$F(x)\not<L(\overline{x},\overline{U}).$$

The proof is completed. ∎

From the above theorem, we have immediately

Corollary 3.1. *Suppose that x is feasible for (VFP) and $(\overline{x},\overline{U})$ is feasible for (VFD1). If $F(x)=L(\overline{x},\overline{U})$, then x and $(\overline{x},\overline{U})$ are weakly efficient solutions of (VFP) and (VFD1) respectively.*

With this corollary and Theorem 3.1, we can prove the following strong duality theorem.

Theorem 3.3. *Suppose that the Slater constraint qualification is satisfied. If $\overline{x}$ is a weakly efficient solution of (VFP), then there exists $\overline{U}\in\mathbb{R}_+^{m\times p}$ such that $(\overline{x},\overline{U})$ is a weakly efficient solution of (VFD1) and $F(\overline{x})=L(\overline{x},\overline{U})$.*

Proof: Suppose that $\overline{x}$ is a weakly efficient solution of (VFP). By Theorem 3.1, there exists $\overline{U}\in\mathbb{R}_+^{m\times p}$ such that $(\overline{x},\overline{U})$ is a weak saddle point of L and $\overline{U}\circ g(\overline{x})=0$. Hence, $F(\overline{x})=L(\overline{x},\overline{U})$. By Proposition 3.3, $(\overline{x},\overline{U})$ is a feasible solution of (VFD1). From Corollary 3.1, $(\overline{x},\overline{U})$ is a weakly efficient solution of (VFD1). This completes the proof. ∎

4. Duality II

In this section, we present two duality theorems in the sense of another type of duality for the multiobjective fractional programming problem (VFP). First, we introduce our second type of dual problem of (VFP). Let

$$G(x,\omega,U)=H(x,\omega)+Ug(x)$$

and

$$T'=\{(x,\omega,U)\in X'\times\mathbb{R}_+^m\times\mathbb{R}_+^{m\times p}\mid \text{ there exists }\lambda\in\Lambda_+\text{ such that }\nabla_xG(x,\omega,U)=0\}.$$

Define a vector optimisation problem as follows

$$\text{(VFD2)}\qquad \begin{cases} \text{V-maximise} & \omega = (\omega_1, \cdots, \omega_m)^T \\ \text{subject to} & (x, \omega, U) \in T' \\ & G(x, \omega, U) \leqq 0. \end{cases}$$

The problem (VFD2) can be viewed as a dual problem of (VFP) since we can prove the following weak duality theorem and strong duality theorem for them.

THEOREM 4.1. *Suppose that x is feasible for (VFP) and that $(\overline{x}, \overline{\omega}, \overline{U})$ is feasible for (VFD2). We have*

$$F(x) \not< \overline{\omega}.$$

PROOF: Since x is a feasible solution for (VFP) and $(\overline{x}, \overline{\omega}, \overline{U})$ is a feasible solution for (VFD2), there exists $\overline{\lambda} \in \Lambda_+$ such that

$$\nabla_x G(\overline{x}, \overline{\omega}, \overline{U})\overline{\lambda} = 0 \tag{4.1}$$

and $g(\overline{x}) \leqq 0$. Because f and $\pm l$ are differentiable and $\mathbb{R}^m_+$-pre-invex on X' with respect to the same η, by Lemma 2.2, $\overline{\lambda}^T H(\cdot, \overline{\omega})$ is differentiable and $\mathbb{R}^1_+$-pre-invex on X' with respect to the same η. From Lemma 2.1 and (4.1),

$$\begin{aligned} \overline{\lambda}^T H(x, \overline{\omega}) - \overline{\lambda}^T H(\overline{x}, \overline{\omega}) &\geqslant [\eta(x, \overline{x})]^T \nabla_x \left(\overline{\lambda}^T H(\overline{x}, \overline{\omega})\right) \\ &= [\eta(x, \overline{x})]^T \nabla_x H(\overline{x}, \overline{\omega})\overline{\lambda} \\ &= -[\eta(x, \overline{x})]^T \nabla_x \left(\overline{U} g(\overline{x})\right)\overline{\lambda} \\ &= -[\eta(x, \overline{x})]^T \nabla_x \left(\overline{\lambda}^T \overline{U} g(\overline{x})\right). \end{aligned}$$

Since g is differentiable and $\mathbb{R}^p_+$-pre-invex on X' with respect to the same η, by Lemmas 2.1 and 2.2, $\overline{\lambda}^T \overline{U} g$ is differentiable and $\mathbb{R}^1_+$-pre-invex on X' with respect to the same η. So

$$\overline{\lambda}^T H(x, \overline{\omega}) - \overline{\lambda}^T H(\overline{x}, \overline{\omega}) \geqslant -\left(\overline{\lambda}^T \overline{U} g(x) - \overline{\lambda}^T \overline{U} g(\overline{x})\right) \geqslant \overline{\lambda}^T \overline{U} g(\overline{x}).$$

Hence,

$$\overline{\lambda}^T H(x, \overline{\omega}) \geqslant \overline{\lambda}^T G(\overline{x}, \overline{\omega}, \overline{U}).$$

Since $G(\overline{x}, \overline{\omega}, \overline{U}) \geqslant 0$,

$$H(x, \overline{\omega}) \not< 0.$$

Therefore, $F(x) \not< \overline{\omega}$. The proof is completed. $\square$

An immediate corollary of this weak duality theorem is as follows.

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Corollary 4.1. *Suppose that x is feasible for(VFP) and $(\overline{x},\overline{\omega},\overline{U})$ is feasible for (VFD2). If $F(x)=\overline{\omega}$, then x and $(\overline{x},\overline{\omega},\overline{U})$ are weakly efficient solutions of (VFP) and (VFD2), respectively.*

Theorem 4.2. *Suppose that the Slater constraint qualification is satisfied. If $\overline{x}$ is a weakly efficient solution of(VFP), then there exist $\overline{\omega}\in\mathbb{R}^m$ and $\overline{U}\in\mathbb{R}_+^{m\times p}$ such that $(\overline{x},\overline{\omega},\overline{U})$ is a weakly efficient solution of (VFD2) and $F(\overline{x})=\overline{\omega}$.*

Proof: Define two multiobjective programming problems as follows:

$$(P_\omega)\qquad \begin{cases} \text{V-minimise} & H(x,\omega) \\ \text{subject to} & x\in X \end{cases}$$

and

$$(D_\omega)\qquad \begin{cases} \text{V-maximise} & G(x,\omega,U) \\ \text{subject to} & (x,U)\in T(\omega), \end{cases}$$

where

$$T(\omega)=\left\{(x,U)\in X'\times\mathbb{R}_+^{m\times p} \mid \text{ there exists } \lambda\in\Lambda_+ \text{ such that } \nabla_x G(x,\omega,U)\lambda=0\right\}.$$

Suppose that $\overline{x}$ is a weakly efficient solution of (VFP). Let $\overline{\omega}=F(\overline{x})$. Obviously, $H(\overline{x},\overline{\omega})=0$. With the same technique as in the first part of the proof of Theorem 3.1, we can prove that $\overline{x}$ is a weakly efficient solution of $(P_{\overline{\omega}})$. Since f and $\pm l$ are differentiable and $\mathbb{R}_+^m$-pre-inevx on X' with respect to the same η, by Lemma 2.2, for any fixed $\omega\in\mathbb{R}^m$, $H(\cdot,\omega)$ is differentiable and $\mathbb{R}_+^m$-pre-invex on X'. Applying Theorem 3.3 to the special multiobjective fractional programming problem $(P_{\overline{\omega}})$ (where each denominator function is 1), we know that there exists $\overline{U}\in\mathbb{R}_+^{m\times p}$ such that $(\overline{x},\overline{U})$ is a weakly efficient solution of $(D_{\overline{\omega}})$ and $G(\overline{x},\overline{\omega},\overline{U})=H(\overline{x},\overline{\omega})=0$. It is obvious that $(\overline{x},\overline{\omega},\overline{U})$ is feasible for (VFD2). If $(\overline{x},\overline{\omega},\overline{U})$ was not a weakly efficient solution of (VFD2), then there would exist a feasible solution (x',ω',U') of (VFD2) such that $\overline{\omega}<\omega$. Hence,

$$H(\overline{x},\omega')<H(\overline{x},\overline{\omega})=0.$$

Obviously, $\overline{x}$ is also a feasible solution of $(P_{\overline{\omega}})$ and (x',U') is a feasible solution of $(D_{\omega'})$. By applying Theorem 3.2 to $(P_{\omega'})$ and $(D_{\omega'})$, we have

$$H(\overline{x},\omega')\not< G(x',\omega',U').$$

Hence, $G(x'\omega',U')\ngeq 0$ which contradicts the fact that (x',ω',U') is feasible for (VFD2). Therefore, $(\overline{x},\overline{\omega},\overline{U})$ is a weakly efficient solution of (VFD2) and $F(\overline{x})=\overline{\omega}$. The proof is completed. ∎

5. Relations between duality I and Duality II

In the previous two sections, we introduced two types of duality for the multi-objective fractional programming problem (VFP) and proved a weak duality theorem and a strong duality theorem for each type of duality. In this section, we discuss some relations between these two types of duality.

First we prove a lemma.

Proposition 5.1. *(i) If $(x, U) \in T$ and $\omega = F(x) + U \circ g(x)$, then $(x, \omega, U) \in T'$;*

(ii) if $(x, \omega, U) \in T'$ and $G(x, \omega, U) = 0$, then $(x, U) \in T$.

Proof: (i) Suppose that $(x, U) \in T$ and $\omega = F(x) + U \circ g(x)$. Since $(x, U) \in T$, there exists $\lambda \in \Lambda_+$ such that

$$\begin{aligned} 0 = \nabla_x L(x, U)\lambda &= \sum_{i=1}^{m} \lambda_i \nabla_x \Big(\frac{f_i(x) + u_i^T g(x)}{l_i(x)} \Big) \\ &= \sum_{i=1}^{m} \lambda_i \frac{[\nabla f_i(x) + \nabla g(x) u_i] l_i(x) - [f_i(x) + u_i^T g(x)] \nabla l_i(x)}{l_i^2(x)} \\ &= \sum_{i=1}^{m} \lambda_i \frac{1}{l_i(x)} [\nabla f_i(x) - \omega_i \nabla l_i(x) + \nabla g(x) u_i]. \end{aligned}$$

Hence, $(x, \omega, U) \in T'$.

(ii) Suppose that $(x, \omega, U) \in T'$ and $G(x, \omega, U) = 0$. Because $(x, \omega, U) \in T'$, there exists $\lambda \in \Lambda_+$ such that

$$\nabla_x G(x, \omega, U)\lambda = 0.$$

Since $G(x, \omega, U) = 0$,

$$\omega = F(x) + U \circ g(x).$$

So,

$$\begin{aligned} 0 &= \nabla_x G(x, \omega, U)\lambda \\ &= \sum_{i=1}^{m} \lambda_i [\nabla f_i(x) - \omega_i \nabla l_i(x) + \nabla g(x) u_i] \\ &= \sum_{i=1}^{m} \lambda_i l_i(x) \frac{[\nabla f_i(x) + \nabla g(x) u_i] l_i(x) - [f_i(x) + u_i^T g(x)] \nabla l_i(x)}{l_i^2(x)} \\ &= \sum_{i=1}^{m} \lambda_i l_i(x) \Big[\nabla_x \Big(\frac{f_i(x) + u_i^T g(x)}{l_i(x)} \Big) \Big] \\ &= \sum_{i=1}^{m} \lambda_i l_i(x) \nabla_x l_i(x, U). \end{aligned}$$

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Therefore, $(x, U) \in T$. The proof is completed. $\square$

With the above proposition, we can prove

Theorem 5.1. (i) *If $(\overline{x}, \overline{\omega}, \overline{U})$ is a weakly efficient solution of (VFD2) and $G(\overline{x}, \overline{\omega}, \overline{U}) = 0$, then $(\overline{x}, \overline{U})$ is a weakly efficient solution of (VFD1);*

(ii) *if $(\overline{x}, \overline{U})$ is a weakly efficient solution and $\overline{\omega} = F(\overline{x}) + \overline{U} \circ g(\overline{x}) \in F(X)$, then $(\overline{x}, \overline{\omega}, \overline{U})$ is a weakly efficient solution of (VFD2).*

Proof: (i) Suppose that $(\overline{x}, \overline{\omega}, \overline{U})$ is a weakly efficient solution of (VFD2) and $G(\overline{x}, \overline{\omega}, \overline{U}) = 0$. By Proposition 5.1 (ii), $(\overline{x}, \overline{U})$ is feasible for (VFD1). If $(\overline{x}, \overline{U})$ was not a weakly efficient solution of (VFD1), there would exist $(x', U') \in T$ such that

$$L(\overline{x}, \overline{U}) < L(x', U').$$

Let $\omega' = L(x', U')$. By Proposition 5.1(i), (x', ω', U') is feasible for (VFD2). Because $G(\overline{x}, \overline{\omega}, \overline{U}) = 0$, $\overline{\omega} = L(\overline{x}, \overline{U})$. So $\overline{\omega} < \omega'$. This contradicts that $(\overline{x}, \overline{\omega}, \overline{U})$ is a weakly efficient solution of (VFD2). Hence, $(\overline{x}, \overline{D})$ is a weakly efficient solution of (VFD1).

(ii) Suppose that $(\overline{x}, \overline{U})$ is a weakly efficient solution and $\overline{\omega} = F(\overline{x}) + \overline{U} \circ g(\overline{x}) \in F(X)$. By Proposition 5.1(i), $(\overline{x}, \overline{\omega}, \overline{U})$ is feasible for (VFD2). If $(\overline{x}, \overline{\omega}, \overline{U})$ was not a weakly efficient solution of (VFD2), there would exist a feasible solution (x', ω', U') of (VFD2) such that

$$\overline{\omega} < \omega'.$$

Because $\overline{\omega} \in F(X)$, there would exist an $x^\circ \in X$ such that $\overline{\omega} = F(x^\circ)$. So

$$F(x^\circ) < \omega'.$$

This contradicts Theorem 4.1. Therefore, $(\overline{x}, \overline{\omega}, \overline{U})$ is a weakly efficient solution of (VFD2). The proof is completed. $\square$

Remark 5.1. We do not assume that $\overline{\omega} \in F(\overline{x}) + \overline{U} \circ g(\overline{x}) \in \text{W-Min}\, F(x)$ in the above theorem. The requirement is quite mild. The dimensions of the decision variables in the two dual problems are different. But Proposition 5.1 gives a sufficient condition for the two components (x, U) of a feasible solution (x, ω, U) of the second dual problem (VFD2) to be feasible for the first dual problem (VFD1). One can find a similar condition to guarantee that the vector (x, ω, U) made by a feasible solution (x, U) of the first dual problem (VFD1) with $\omega = F(x) + U \circ g(x)$ is also feasible to (VFD2).

6. Conclusion

Two problems might be interesting for further investigation. The first one is

Can one establish such a theory for the multiobjective fractional programming problem (VFP) in the sense of an efficient solution instead of a weakly efficient solution?

The second problem is related to a possible extension of the results in this paper. In (VFP), the domination structure is determined by $\mathbb{R}^m_+$. If the domination structure is determined by a convex cone (see [**9**]), how to extend the results to a more general case? It seems possible and not very difficult.

References

[**1**] M.S. Bazaraa and C.M. Shetty, *Nonlinear programming: theory and algorithms* (John Wiley and Sons, New York, New York, 1979).

[**2**] C.R. Bector, S. Chandra and I. Husain, 'Optimality conditions and duality in subdifferentiable multiobjective fractional programming', *J. Optim. Theory Appl.* **79** (1993), 105–125.

[**3**] C.R. Bector, S. Chaudra and C. Singh, 'A linearization approach to multiobjective programming duality', *J. Math. Anal. Appl.* **175** (1993), 268–279.

[**4**] L. Coladas, Z. Li and S. Wang, 'Dualidad en programacion fracional multiobjetive', in *Proceedings of XXI Congreso Nacional de Estadistica e Investigacion Operativa, Calella, Barcelona April 18–21*, 1994, pp. 280–281.

[**5**] B.D. Craven and B.M. Glover, 'Invex functions and duality', *J. Austral. Math. Soc. Ser. A* **39** (1985), 1–20.

[**6**] R.R. Egudo, 'Multiobjective fractional duality', *Bull. Austral. Math. Soc.* **37** (1988), 367–378.

[**7**] R.R. Egudo and M.A. Hanson, 'Multiobjective duality with invexity', *J. Math. Anal. Appl.* **126** (1987), 469–477.

[**8**] Y. Hu and F. Zhang, 'Two dual types of multiobjective programming', *Acta Math. Appl. Sinica* **14** (1991), 128–135.

[**9**] Y. Sawaragi, H. Nakayama and T. Tanino, *Theory of multiobjective optimization* (Academic Press, New York, New York, 1985).

[**10**] C. Singh, 'A class of multiple-criteria fractional programming problem', *J. Math. Anal. Appl.* **115** (1986), 202–213.

[**11**] S. Wang, 'Theory of the conjugate duality in multiobjective optimization', *J. Systems Sci. Math. Sci.* **4** (1984), 303-312.

[**12**] T. Weir, 'A duality theorem for a multiple objective fractional optimization problem', *Bull. Austral Math. Soc.* **34** (1986), 415–425.

[**13**] T. Weir and V. Jeyakumar, 'A class of nonconvex functions and mathematical programming', *Bull. Austral. Math. Soc.* **38** (1988), 177–189.

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