Can. J. Math., Vol. XXXIV, No. 6, 1982, pp. 1365–1377

PRIME NUMBERS IN SHORT INTERVALS AND A GENERALIZED VAUGHAN IDENTITY

D. R. HEATH-BROWN

1. Introduction. Many problems involving prime numbers depend on estimating sums of the form $\sum \Lambda(n)f(n)$, for appropriate functions $f(n)$, (here, as usual, $\Lambda(n)$ is the von Mangoldt function). Three distinct general methods have been used to estimate such sums. The earliest is due to Vinogradov (see [**13**, Chapter 9]); the second involves zero-density bounds for Dirichlet L-functions (see [**8**, Chapters 15 and 16] for example); and the third, due to Vaughan (see [**12**] for example) uses an arithmetical identity as will be explained later. The second and third methods are much simpler to apply than the first. On the other hand Vinogradov's technique is at least as powerful as Vaughan's and occasionally more so. In many cases Vaughan's identity yields better bounds than the use of zero-density estimates, but sometimes they are worse. The object of this paper is to present a simple extension of Vaughan's method which is essentially as powerful as any of the techniques mentioned above, to discuss its general implications, and to apply it to the proof of the following result of Huxley [**4**], which has previously only been within the scope of the zero density method.

THEOREM. *For any fixed ϑ in the range $7/12 < \vartheta \leqq 1$, and $y = x^{\vartheta}$,*

$$\sum_{x-y<n\leqq x} \Lambda(n) \sim y. \tag{1}$$

It follows that there is always a prime number between $x - y$ and x, if x is large enough.

One significant feature of our proof is that there is no explicit mention of zeros of $\zeta(s)$; it is not necessary to discuss hypothetical zeros off the critical line. None the less Vinogradov's zero-free region for $\zeta(s)$ plays a crucial role just as before.

Although it is not strictly relevant to the present paper, it may be of interest to observe at this point that, contrary to the impression gained from following the historical development of the subject, it is possible to prove Huxley's theorem without recourse to Vinogradov's zero-free region. Littlewood's estimate

$$1 - \beta \gg \frac{\log\log(3 + |\gamma|)}{\log(3 + |\gamma|)} \quad (\zeta(\beta + i\gamma) = 0)$$

Received November 28, 1981 and in revised form March 25, 1982.

https://doi.org/10.4153/CJM-1982-095-9 Published online by Cambridge University Press

suffices, providing that one uses the "log-free" zero-density theorem

$$N(\sigma, T) \ll T^{(12/5+\epsilon)(1-\sigma)},$$

(for any $\epsilon > 0$), which follows from [7]. However this last bound has no counterpart in the approach we shall consider in the present paper.

It is of interest to compare our method with that used by Hoheisel [3]. There too an identity (similar to (2) in fact) was employed, and mean-value estimates were applied directly to the resulting Dirichlet polynomials, without introducing $N(\sigma, T)$.

2. Vaughan's identity and its generalization. Let $\mu(n)$ be the Möbius function and define

$$M_X(s) = M(s) = \sum_{n \leqq X} \mu(n) n^{-s},$$
$$N(s) = \sum_{n \leqq X} \Lambda(n) n^{-s}.$$

We then have

$$\zeta'(s)/\zeta(s) = \zeta'(s)M(s) + \zeta(s)M(s)N(s) + (\zeta'(s)/\zeta(s) + N(s))(1 - \zeta(s)M(s)) - N(s). \tag{2}$$

On picking out the coefficient of n^{-s} on each side one obtains Vaughan's identity, namely

$$\sum_{X < n \leqq x} \Lambda(n) f(n) = S_1 - S_2 - S_3,$$

where

$$S_1 = \sum_{m \leqq X} \mu(m) \sum_{X < mn \leqq x} (\log n) f(mn),$$
$$S_2 = \sum_{m,n \leqq X} \mu(m)\Lambda(n) \sum_{X < mnr \leqq x} f(mnr),$$
$$S_3 = \sum_{\substack{m,n > X \\ X < mn \leqq x}} \Lambda(m) c(n) f(mn),$$

with

$$c(n) = \sum_{d \mid n, d \leqq X} \mu(d). \tag{3}$$

The sums S_1 and S_2 may be estimated by considering expressions of the shape

$$\sum_m a_m \sum_n f(mn), \tag{4}$$

in which the range for n is 'long'. On the other hand the sum S_3 is a bilinear form of the shape

$$\sum_m \sum_n a_m b_n f(mn), \tag{5}$$

in which neither m nor n can be 'too big'. Various methods may be applied to bound the S_i. The quality of the estimates depends on the function $f(\cdot)$, of course, but also on the ranges of the variables m, n above, and it is this latter point with which we are concerned here. There is a considerable amount of flexibility in applying Vaughan's identity. For example, the three variables occurring in S_2 can be re-grouped for part of their ranges so as to produce a bilinear form of the type (5). Moreover the identity may be iterated so as to apply it to the $\Lambda(\cdot)$ functions occurring in S_2 or S_3. Similarly a modified identity may be applied to the $\mu(\cdot)$ function. One may then try to regroup the variables of summation so as to produce expressions of types (4) and (5) with the most efficient ranges possible. The net result of all these manipulations (see [**2**, Lemmas 2 and 3] for example) is a mess.

We shall therefore use the following trivial identity (which is not, strictly speaking, a generalization of Vaughan's).

LEMMA 1. *For any integer $k \geqq 1$ we have*

$$(6) \quad \zeta'(s)/\zeta(s) = \sum_{j=1}^{k} (-1)^{j-1}\binom{k}{j}\zeta(s)^{j-1}\zeta'(s)M(s)^j + \zeta(s)^{-1}(1 - \zeta(s)M(s))^k\zeta'(s).$$

To apply Lemma 1 to the sum

$$S = \sum_{n \leqq x} \Lambda(n)f(n),$$

for example, one chooses $X^k \geqq x$ and picks out the relevant coefficients of n^{-s}. The last term on the right hand side of (6) therefore makes no contribution, since

$$(7) \quad \zeta(s)M(s) - 1 = \sum_{n>X} c(n)n^{-s},$$

with $c(n)$ as in (3). On splitting up each range of summation into intervals $N < n \leqq 2N$, one finds that S is a linear combination of $O((\log x)^{2k})$ sums of the form

$$(8) \quad \sum_{n_i \in I_i, n_1 n_2 \ldots n_{2k} \leqq x} (\log n_1)\mu(n_{k+1}) \ldots \mu(n_{2k})f(n_1 n_2 \ldots n_{2k}),$$

in which $I_i = (N_i, 2N_i]$, $\prod N_i < x$, and $2N_i \leqq X$ if $i > k$. (Some of the intervals I_i may contain only the integer 1.) By choosing k to be large, and X to be a small power of x, one may bring the part of the sum (8) involving the 'unknown' coefficients $\mu(n_i)$ as closely under control as one likes.

We illustrate this by an example. Assume that $f(n) = 0$ for $n \leqq x/2$ whence we may take $\prod(2N_i) > x/2$, if the sum (8) is to be non-empty. Let $2 \leqq u \leqq v \leqq z \leqq x$ and suppose that $u^2 \leqq z$, $128uz^2 \leqq x$ and $2^{18}x \leqq v^3$. Choose $k = 3$ and $X = x^{1/3} \leqq z$. Suppose firstly that $N_i \geqq z$

for some i (necessarily $i \leqq 3$). Then one may group the remaining variables together to produce a sum of type (4) (possibly with a weight $\log n$ included) for which $n \geqq z$. On the other hand, if $N_i < z$ for all i, we may group the variables in (8) so as to produce a bilinear form of type (5) with $u \leqq m \leqq v$. To do this we show, by contradiction, that some product P of the N_i's lies in the range $u \leqq P \leqq v/64$. Certainly we may assume for each individual N_i that either $N_i < u$ or $v/64 < N_i < z$. Suppose the latter case occurs s times. The remaining N_i, being less than u, may be formed into t (say) products P, all lying in the range $u < P < u^2$, together with a factor Q (say) for which $1 \leqq Q \leqq u$. (Q could be an empty product.) We now find that

$$uz^2 \leqq 2^{-7}x < \prod N_i \leqq Qu^{2t}z^s \leqq uz^{s+t},$$

on using the bounds $u^2 \leqq z$ and $128uz^2 \leqq x$. It follows that $s + t \geqq 3$. We then have, since $u < P \leqq v/64$ is ruled out,

$$x > \prod N_i \geqq (v/64)^{s+t} \geqq (v/64)^3 \geqq x,$$

on using the bound $v^3 \geqq 2^{18}x$. This contradiction shows that it is always possible to produce a bilinear form of the required shape. The result we have just proved is a version of [**2**, Lemma 3].

The reader should have little difficulty in proving similarly the the following alternative result; that if $f(n) = 0$ for $n \leqq x/2$, and $uv \leqq x^{1-1/k}$, then one may put (8) either into the form (4) with $n \geqq x/(uv)$, or into the form (5) with $u < m \leqq 2^{2k}x/v$. These are essentially the shapes of the sums that arise from Vaughan's identity, which is therefore included in our method.

In practice it is often the case that the ranges for (4) and (5) produced by Vaughan's identity are the most suitable, so that Lemma 1 gives no advantage. However, and this is an important point, the expression (8) may potentially be estimated by considering double sums

$$\sum_{m,n} f(mnr)$$

or trilinear forms

$$\sum_{m,n,r} a_m b_n c_r f(mnr),$$

for example.

It is easy to compare the result of Vinogradov's method with the expression (8). In place of the constraint $2N_i \leqq X$ for $i > k$, Vinogradov has

$$n_i \Big| \prod_{p \leqq X} p.$$

This condition is far more cumbersome, but nevertheless yields similar conclusions to those that follow from Lemma 1.

As far as the zero-density method is concerned, we remark only that the relation (7) plays a crucial role in the zero detection procedure (see [**8**, Chapter 12] for example) and that Jutila's device [**6**] of choosing X to be small corresponds to our taking k to be large. The connection between the two techniques, though not perhaps immediately apparent, is none the less quite strong.

Finally in this section it should be mentioned that the identities discussed here all have their origins in the work of Hoheisel [**3**], where the first result of the type (1) was obtained. (See also [**1**].)

3. Huxley's theorem: preliminaries. Our proof of Huxley's Theorem will depend on the evaluation of the sums (8) by Perron's formula. We will then have a mean value of Dirichlet series to estimate, and this will be achieved using Montgomery's mean-value theorem and Huxley's version of Halász's lemma. To this extent our treatment has much in common with the earlier methods. However we avoid the use of zero-density theorems and the explicit formula for $\psi(x)$.

We shall assume that y and $x - \frac{1}{2}$ are integers (there is no loss in doing this) and that $x^{7/12} \leqq y \leqq x/2$. We then apply Lemma 1 with $f(n) = 1$ for $x - y < n \leqq x$ and $f(n) = 0$ otherwise, and we take $k = 5$, $X = x^{1/5}$. There are then $O((\log x)^{10})$ expressions of the shape (8). We shall prove that, for a suitable range of y, each expression (Σ say) is of the form

$$(9) \qquad \Sigma = yE(x) + O(y(\log x)^{-11})$$

for some function $E(x)$ depending on the particular Σ, but independent of y. It will follow that

$$(10) \qquad \sum_{x-y<n\leqq x} \Lambda(n) = yE_0(x) + O(y(\log x)^{-1}).$$

$E_0(x)$ may then be found by taking y "large" and applying the Prime Number Theorem.

We first consider the case in which some N_i in (8) satisfies $N_i \geqq x^{1/2}$, necessarily with $i \leqq 5$. Here elementary methods suffice to establish (9). Suppose firstly that $N_1 \geqq x^{1/2}$; then

$$(11) \quad \Sigma = \sum_{M<m\leqq cM} d_m \sum_{\substack{N<n\leqq 2N \\ x-y<mn\leqq x}} \log n,$$

where $c = 2^9$, $N = N_1$, $MN \leqq x$, and $|d_m| \leqq d_9(m)$. (Here, and later, $d_k(m)$ denotes the coefficient of m^{-s} in $(\zeta(s))^k$.) Moreover, we may suppose that $MN > 2^{-11}x$, for otherwise $\Sigma = 0$ and (9) holds trivially. We now observe that the inner sum of (11) is 0 for $m \leqq (x - y)/(2N)$ or $m \geqq x/N$; it is

$$\frac{y}{m} \log \left(\frac{x}{m}\right) + O\left(\frac{y^2}{xm}\right) + O(\log x)$$

in the range

$$\frac{x}{2N} < m \leqq \frac{x-y}{N};$$

and otherwise it is $\ll (1 + y/m) \log x$. Thus

$$(12) \quad \Sigma = y \sum_{M_1 < m \leqq M_2} d_m m^{-1} \log (x/m) + O\Big(y^2 x^{-1} \sum_{M < m \leqq cM} d_9(m) m^{-1}\Big)$$
$$+ O\Big((\log x) \sum_{M < m \leqq cM} d_9(m)\Big) + O\big(y(\log x) {\textstyle\sum}^{(1)} d_9(m) m^{-1}\big),$$

where

$$M_1 = \operatorname{Max}(M, x/(2N)),\ M_2 = \operatorname{Min}(cM, x/N),$$

and $\sum^{(1)}$ runs over the two intervals

$$(13) \quad (x-y)/N < m \leqq x/N$$

and

$$(x-y)/(2N) < m \leqq x/(2N).$$

Clearly the first term on the right hand side of (12) may serve as $yE(x)$ in (9), so it remains to bound the error terms. Here the following lemma will be useful.

LEMMA 2. *For any fixed integer $k \geqq 1$ and any fixed $\epsilon > 0$, we have $d_k(m) \ll m^\epsilon$ and*

$$\sum_{M - M^\epsilon < m \leqq M} d_k(m) \ll M^\epsilon (\log M)^{k-1}.$$

The second statement of the lemma may be found in [**10**] for example, and the first statement follows from the second. Now, since $M \ll x^{1/2}$ and $y \geqq x^{7/12}$, the first two error terms in (12) will each be $O(y(\log x)^{-11})$, providing that $y \leqq x(\log x)^{-48}$, as we now assume. Moreover the same will be true of the third error term, if $N \leqq y^{3/2}x^{-1/2}$, since then $y/N \geqq (x/N)^{1/3}$. However if N is larger, of order x for example, it might seem that a single term $m = 1$ could yield a contribution $\gg 1$ to $\sum^{(1)}$.

We avoid this difficulty by the following trickery. In Section 2 it was explained that the various ranges of summation were to be split up into intervals $N < n \leqq 2N$. In the case of a variable n_i with $k < i \leqq 2k$ we necessarily take $N = X2^{-j}$ for some integer j. However for $i \leqq k$ we are free to choose $N = x2^{1/2-j}$. Then, if the range (13), say, is non-empty, we must have $\sqrt{2} = 2^j m^{-1} + O(yx^{-1})$ for some integer m. However it is an elementary fact of Diophantine approximation that

$$\left| \sqrt{2} - \frac{p}{q} \right| \gg q^{-2}$$

for any integers $p, q \neq 0$. Consequently we see that $N^2 \ll yx$ if $\sum^{(1)}$ is non-empty. Then, since Lemma 2 yields $d_9(m) \ll m^{1/2}$, we have

$$\sum{}^{(1)} \ll (1 + y/N)(x/N)^{-1/2} \ll y^{1/4}x^{-1/4} + yx^{-1/2}N^{-1/2}$$
$$\ll y^{1/4}x^{-1/4} \ll (\log x)^{-12}$$

as required. (Recall that we may assume $y \leqq x(\log x)^{-48}$ and $N \geqq y^{3/2}x^{-1/2}$ here.) We have now established (9) in the case $N_1 \geqq x^{1/2}$. If $N_i \geqq x^{1/2}$ with $2 \leqq i \leqq 5$ the treatment is similar, except that now

$$|d_m| \leqq d_9(m) \log x,$$

and the evaluation of the inner sum in (11) is slightly different.

We turn now to the sums Σ in which $N_i \leqq x^{1/2}$ for each i. In the case just considered, we put the sum into the form (4), by uniting all the variables bar one. Such a technique is too crude for the present situation. It is not sufficient to consider Σ as a bilinear form for example. Indeed we shall make use of the fact that $N_i \leqq x^{1/5}$ for $5 < i \leqq 10$, so clearly the corresponding variables may not be put together.

In considering Σ it is natural to use the Dirichlet series

$$f_i(s) = \sum_{N_i < n \leqq 2N_i} a_i(n)n^{-s},$$

with $a_1(n) = \log n$, $a_i(n) = 1$ for $2 \leqq i \leqq 5$ and $a_i(n) = \mu(n)$ otherwise. We also write

$$F(s) = \prod f_i(s) = \sum c_n n^{-s},$$

say, where

$$(14) \quad |c_n| \leqq d_{10}(n) \log x.$$

We use Perron's formula in the following guise:

$$\frac{1}{2\pi i}\int_{1/2-iT_1}^{1/2+iT_1} u^s \frac{ds}{s} = \mathscr{E}(u) + O(u^{1/2}T_1^{-1}|\log u|^{-1}),$$

where $T_1 \geqq 2$ and $\mathscr{E}(u) = 0$ for $0 < u < 1$ and $=1$ for $u > 1$. We may assume that $\prod N_i \geqq x2^{-11}$, since otherwise $\Sigma = 0$ and (9) holds trivially. Thus $c_n \neq 0$ only for $2^{-11}x \leqq n \leqq 2^{10}x$, and we find that

$$(15) \quad \frac{1}{2\pi i}\int_{1/2-iT_1}^{1/2+iT_1} F(s)\frac{x^s - (x-y)^s}{s}\,ds$$
$$= \Sigma + O\left(T_1^{-1}\sum_n |c_n|\cdot\left|\log\frac{x}{n}\right|^{-1}\right)$$
$$+ O\left(T_1^{-1}\sum_n |c_n|\cdot\left|\log\frac{x-y}{n}\right|^{-1}\right).$$

Now let Δ be a fixed (small) positive quantity and take $T_1 = x^{5/12-\Delta}$,

$y \geqq x^{7/12+2\Delta}$. The first error term in (15) is then

$$(16)\quad \ll T_1^{-1}x(\log x)\sum_n d_{10}(n)|x-n|^{-1} \ll T_1^{-1}x^{1+\Delta/2}(\log x)^2 \ll y(\log x)^{-11},$$

on applying Lemma 2 with $\epsilon = \Delta/2$. A similar analysis applies to the second error term of (15). It remains therefore to consider the integral on the left of (15). The main term $yE(x)$ will come from a short subrange $|\text{Im }(s)| \leqq T_0$, where T_0 is defined in (19) below. We note that, for $\text{Re }(s) = \frac{1}{2}$,

$$|F(s)| \ll \sum n^{-1/2}|c_n| \ll (\log x)\sum n^{-1/2}d_{10}(n) \ll x^{1/2}(\log x)^{10},$$

by (14) and Lemma 2. Moreover we have

$$(17)\quad \frac{x^s-(x-y)^s}{s} = \begin{cases} x^{s-1}y + O(|s|x^{-3/2}y^2), & |\text{Im }(s)| \leqq T_0, \\ O(x^{-1/2}y), & |\text{Im }(s)| \geqq T_0. \end{cases}$$

Hence

$$8)\quad \frac{1}{2\pi i}\int_{1/2-iT_0}^{1/2+iT_0} F(s)\frac{x^s-(x-y)^s}{s}ds = y\frac{1}{2\pi i}\int_{1/2-iT_0}^{1/2+iT_0} F(s)x^{s-1}ds + O(T_0^2x^{-3/2}y^2x^{1/2}(\log x)^{10}) = yE(x) + O(y(\log x)^{-11}),$$

providing that T_0 depends only on x and that

$$y \leqq xT_0^{-2}(\log x)^{-21}.$$

Finally we deal with the range $|\text{Im }(s)| \geqq T_0$ by means of the following lemma, whose proof is given in the next section.

LEMMA 3. *We have*

$$\int_T^{2T} |F(\tfrac{1}{2}+it)|dt \ll x^{1/2}(\log x)^{-12}$$

uniformly for

$$(19)\quad \exp((\log x)^{1/3}) = T_0 \leqq T \leqq T_1 = x^{5/12-\Delta},$$

providing that every factor $f_j(s)$ of $F(s)$ has $N_j \leqq x^{1/2}$.

4. Huxley's theorem: the mean-value of $F(s)$**.** In this section we shall prove Lemma 3. We shall need Montgomery's mean-value theorem, the Halász lemma and Vinogradov's zero-free region.

We begin by writing $F(s) = F_1(s)F_2(s)$, where $F_2(s)$ is the product of those factors $f_j(s)$ for which $N_j \leqq x^{\Delta/5}$. Since

$$|f_1(\tfrac{1}{2}+it)| \ll N_1^{1/2}\log x; \quad |f_j(\tfrac{1}{2}+it)| \ll N_j^{1/2},\ (j \geqq 2),$$

we have

$$|F_2(\tfrac{1}{2} + it)| \ll Z^{1/2} \log x,$$

where Z is the product of N_j for $N_j \leqq x^{\Delta/5}$. Thus

$$(20) \quad \int_T^{2T} |F(\tfrac{1}{2} + it)|dt \ll Z^{1/2}(\log x) \int_T^{2T} |F_1(\tfrac{1}{2} + it)|dt.$$

We proceed to bound the integral on the right (I, say) by a sum over $O(T)$ well spaced points t_n; that is to say, we have

$$I \ll \sum_n |F_1(\tfrac{1}{2} + it_n)|,$$

where $T \leqq t_n \leqq 2T$, and

$$(21) \quad |t_m - t_n| \geqq 1, \ (m \neq n).$$

For each factor $f_j(s)$ of $F_1(s)$ let

$$|f_j(\tfrac{1}{2} + it_n)| = N_j^{\sigma(j,n)-1/2}.$$

We proceed to show that $\sigma(j, n)$ cannot be too close to 1. We shall treat the case $j > 5$, for which

$$f_j(s) = \sum_n \mu(n)n^{-s}.$$

The case $j \leqq 5$ would be very similar. Our starting point is the formula

$$f_j(\tfrac{1}{2} + it) = \frac{1}{2\pi i} \int_{c-iT/2}^{c+iT/2} (\zeta(\tfrac{1}{2} + it + s))^{-1} N_j^{s} \frac{2^s - 1}{s} ds + O(N_j^{1/2}T^{-1}\log x) + O(1),$$

where $c = \frac{1}{2} + (\log x)^{-1}$; this follows from Perron's formula as given by Titchmarsh [**11**, Lemma 3.12]. In the region

$$1 - \eta \leqq \operatorname{Re}(s) \leqq \tfrac{1}{2} + c, \ |\operatorname{Im}(s) - t| \leqq T/2,$$

where $T \leqq T_1$, $T \leqq t \leqq 2T$, and

$$\eta = C(\log T_1)^{-2/3}(\log \log T_1)^{-1/3},$$

(here C is a numerical constant), we have

$$1/\zeta(s) \ll (\log T)^2;$$

this follows from [**11**, Theorem 3.11] in conjunction with the Vinogradov–Korobov estimate in the form given by Richert [**9**]. (This is the form in which we use Vinogradov's zero-free region.) We may now move the line of integration to $\operatorname{Re}(s) = \frac{1}{2} - \eta$, and use the above bound to obtain

$$f_j(\tfrac{1}{2} + it) \ll (\log x)^3(N_j^{1/2-\eta} + N_j^{1/2}T^{-1}), \ (j > 5).$$

https://doi.org/10.4153/CJM-1982-095-9 Published online by Cambridge University Press

(In fact precisely the same result may be obtained in the case $j \leqq 5$.) However it follows from (19) that $T \geqq T_0 \geqq x^\eta \geqq N_j^\eta$. Moreover, since $N_j \geqq x^{\Delta/5}$, we see that

$$N_j^{\eta/2} \geqq (\log x)^4,$$

whence, if x is large enough,

$$N_j^{\sigma(j,n)-1/2} = |f_j(\tfrac{1}{2} + it_n)| \leqq N_j^{1/2-\eta/2},$$

so that

(22) $\sigma(j, n) \leqq 1 - \eta/2.$

We now split the available range for $\sigma(j, n)$ into $O(\log x)$ ranges $I_0 = (-\infty, \frac{1}{2}]$ and

$$I_l = \left(\frac{1}{2} + \frac{l-1}{L}, \frac{1}{2} + \frac{l}{L}\right), \quad (1 \leqq l \leqq L = [\log x]).$$

Then we divide the points t_n into classes $C(j, l)$ (not necessarily disjoint) according to the value (or values) of j for which $\sigma(j, n)$ is maximal and the value of l for which $\sigma(j, n) \in I_l$. Since there are $O(\log x)$ classes, there must exist some class for which

$$I \ll (\log x) \sum_{t \in C(j,l)} |F_1(\tfrac{1}{2} + it)|.$$

However, for $t \in C(j, l)$ we have

$$|F_1(\tfrac{1}{2} + it)| = \prod N_i^{\sigma(i,n)-1/2} \leqq \prod N_i^{l/L} \leqq Y^{l/L},$$

where Y is the product of N_j for $N_j > x^{\Delta/5}$. Now, to simplify notation, let

$$\sigma = \frac{1}{2} + \frac{l-1}{L}, \quad f(s) = f_j(s), \quad N = N_j \quad \text{and} \quad R = \#C(j, l).$$

In case $l = 0$ we have $I \ll \log x$, so that Lemma 3 follows from (20), since $z \leqq (x^{\Delta/5})^{10}$, so that $z^{1/2}\, T \leqq x^{5/12}$. If $l \geqq 1$ we renumber the points t_n so that

$$C(j, l) = \{t_n : 1 \leqq n \leqq R\}.$$

Then

(23) $I \ll (Y^{\sigma-1/2}) R \log x,$

while

(24) $|f(\tfrac{1}{2} + it_n)| \gg N^{\sigma-1/2}$

for $1 \leqq n \leqq R$. Moreover we note that, by (22), $\sigma \leqq 1 - \eta/2$.

We are now in a position to apply the standard mean and large values techniques, using the ideas introduced by Montgomery [**8**] in connection with bounds for $N(\sigma, T)$. (Our notation, in particular the use of "$\sigma - \frac{1}{2}$", is intended to make the relationship clearer.) We apply the above mentioned methods to the Dirichlet polynomial

$$f(s)^g = \sum_{N^g < n \leqq (2N)^g} a_n n^{-s},$$

where

$$|a_n| \leqq (\log x)^g d_g(n).$$

If $N \geqq T^{3/5}$ we choose $g = 2$, and otherwise we take any integer g such that

$$T^{4/5} \leqq N^g \leqq T^{6/5}.$$

Since $N \geqq x^{\Delta/5}$, it follows that g is bounded in terms of Δ. The mean-value estimate ([**8**, Theorem 7.3] with $Q = 1$, $\chi = 1$, $\delta = 1$) then yields

$$RN^{g(2\sigma-1)} \ll (T + N^g)(\log x)\sum |a_n|^2 n^{-1}.$$

However

$$d_g(n)^2 \leqq d_{g^2}(n),$$

whence, from Lemma 2, we have

(25) $\quad \sum |a_n|^2 n^{-1} \ll (\log x)^{A_1}.$

Here A_1, and later A_2, A_3 etc., are constants depending at most on Δ. It follows that

$$R \ll (TN^{g(1-2\sigma)} + N^{g(2-2\sigma)})(\log x)^{1+A_1}.$$

In the range $\frac{1}{2} \leqq \sigma \leqq \frac{3}{4}$ we therefore have

(26) $$R \ll \begin{cases} T^{12(1-\sigma)/5}(\log x)^{1+A_1}, & N \leqq T^{3/5}, \\ N^{4-4\sigma}(\log x)^{1+A_1}, & N \geqq T^{3/5}. \end{cases}$$

Alternatively we may use Halász's lemma in the form given by Huxley [**4**, (2.9)]. This yields, on applying (25),

$$R \ll (TN^{g(4-6\sigma)} + N^{g(2-2\sigma)})(\log x)^{A_2}.$$

Consequently, for the range $\frac{3}{4} \leqq \sigma \leqq 1$, we also have (26).

If $N \leqq T_1^{3/5}$ then (26) becomes

$$R \ll T_1^{12(1-\sigma)/5}(\log x)^{A_3}.$$

Combining this with (20) and (23) it follows that

$$\begin{aligned} \int_T^{2T} |F(\tfrac{1}{2} + it)|dt &\ll Z^{1/2} Y^{\sigma-1/2} T_1^{12(1-\sigma)/5}(\log x)^{2+A_3} \\ &\ll Z^{1/2}(xZ^{-1})^{\sigma-1/2} T_1^{12(1-\sigma)/5}(\log x)^{2+A_3} \\ &\ll x^{1/2}(T_1^{12/5}Zx^{-1})^{1-\sigma}(\log x)^{2+A_3} \ll x^{1/2-2\Delta(1-\sigma)/5}(\log x)^{2+A_3}, \end{aligned}$$

since $T_1 = x^{5/12-\Delta}$ and $Z \leqq (x^{\Delta/5})^{10}$. On using $1 - \sigma \geqq \eta/2$ we see that

$$x^{-2\Delta(1-\sigma)/5}(\log x)^{2+A_3} \ll (\log x)^{-12},$$

and the estimate required for Lemma 3 follows.

There remains the case $N \geqq T_1^{3/5}$. Providing that $\Delta < 1/12$ this means that $N > x^{1/5}$, whence $f(\cdot) = f_j(\cdot)$ with $j \leqq 5$. Recall also that the case $N \geqq x^{1/2}$ has already been dealt with in Section 3. From the usual analysis of Perron's formula (see [**11**, Lemma 3.12] for example) we have

$$\sum_{N<n\leqq N+M} n^{-1/2-it} = \frac{1}{2\pi i}\int_{\mu-iT/2}^{\mu+iT/2} \zeta(\tfrac{1}{2}+it+s)\frac{(N+M)^s - N^s}{s}\,ds + O(N^{1/2}T^{-1}\log x) + O(1),$$

for $M \leqq N$, $\mu = \frac{1}{2} + (\log x)^{-1}$ and $T \leqq t \leqq 2T$. Moving the line of integration to $\operatorname{Re}(s) = 0$ yields

$$\left|\sum_{N<n\leqq N+M} n^{-1/2-it}\right| \ll \int_{T/2}^{5T/2} |\zeta(\tfrac{1}{2}+iu)|\frac{du}{1+|t-u|} + N^{1/2}T^{-1}\log x + 1.$$

We may take $M = N$ to bound $f_j(\frac{1}{2}+it)$ for $2 \leqq j \leqq 5$, or use partial summation to bound $f_1(\frac{1}{2}+it)$. In either case, Hölder's inequality produces, in view of (24),

$$\begin{aligned} RN^{4\sigma-2} &\ll (\log x)^4 \sum_{n=1}^{R}\left\{\int_{T/2}^{5T/2}|\zeta(\tfrac{1}{2}+iu)|^4\frac{du}{1+|t_n-u|}\right\} \\ &\quad\times\left\{\int_{T/2}^{5T/2}\frac{du}{1+|t_n-u|}\right\}^3 + RN^2T^{-4}(\log x)^8 + R(\log x)^4 \\ &\ll (\log x)^8\left\{\int_{T/2}^{5T/2}|\zeta(\tfrac{1}{2}+iu)|^4\sum_{n=1}^{R}\frac{1}{1+|t_n-u|}\,du + RN^2T^{-4} + R\right\} \\ &\ll (\log x)^{13}\{T + RN^2T^{-4} + R\}, \end{aligned}$$

by Ingham's fourth power moment estimate for $\zeta(\frac{1}{2}+it)$, (see [**11**, (7.6.1)] for example). We now deduce that either

$$N^{4\sigma-2} \ll (\log x)^{13}(N^2T^{-4} + 1)$$

or

$$\begin{aligned} R &\ll TN^{2-4\sigma}(\log x)^{13} \\ &\ll T_1^{1+3(2-4\sigma)/5}(\log x)^{13} \\ &\ll T_1^{12(1-\sigma)/5}. \end{aligned}$$

In the latter case Lemma 3 follows as before. In the former case either $T \ll N^{1-\sigma}(\log x)^4$, whence

$$R \ll T \ll N^{1-\sigma}(\log x)^4 \ll x^{(1-\sigma)/2}(\log x)^4 \ll T_1^{12(1-\sigma)/5}(\log x)^4,$$

or $\sigma \leqq 7/12$, whence

$$R \ll T \ll T_1 \ll T_1^{12(1-\sigma)/5},$$

and in each case Lemma 3 again follows. Note that the argument here would fail if N were too close to x.

5. Conclusion. From (15), (16), (17) and Lemma 3 we have

$$\Sigma = yE(x) + O(y(\log x)^{-11})$$

as claimed, uniformly for

$$x^{7/12+2\Delta} \leqq y \leqq y_0 = xT_0^{-2}(\log x)^{-21}.$$

The formula (10) then follows, and it remains to find $E_0(x)$. By the Prime Number Theorem, in the form

$$\psi(x) = x + O(x \exp(-(\log x)^{1/2})),$$

we see that

$$\begin{aligned}\psi(x) - \psi(x - y_0) &= y_0E_0(x) + O(y_0(\log x)^{-1}) \\ &= y_0 + O(x \exp\{-(\log x)^{1/2}\}).\end{aligned}$$

Thus $E_0(x) = 1 + O((\log x)^{-1})$, and (1) follows.

REFERENCES

1. P. X. Gallagher, *Bombieri's mean value theorem*, Mathematika *15* (1968), 1–6.

2. D. R. Heath-Brown, *The Pjateckiĭ-Šapiro prime number theorem*, J. Number Theory, to appear.

3. G. Hoheisel, *Primzahlprobleme in der Analysis*, Sitz. Preuss. Akad. Wiss. *33* (1930), 3–11.

4. M. N. Huxley, *On the difference between consecutive primes*, Invent. Math. *15* (1972), 164–170.

5. ——— *The distribution of prime numbers* (Clarendon Press, Oxford, 1972).

6. M. Jutila, *On a density theorem of H. L. Montgomery for L-functions*, Ann. Acad. Sci. Fenn. Ser. A I (1972), No. 520.

7. ——— *On Linnik's constant*, Math. Scand. *41* (1977), 45–62.

8. H. L. Montgomery, *Topics in multiplicative number theory* (Springer, Berlin, 1971).

9. H.-E. Richert, *Zur Abschätzung der Riemannschen Zetafunktion in der Nähe der Vertikalen σ* = 1, Math. Ann. *169* (1967), 97–101.

10. P. Shiu, *A Brun–Titchmarsh theorem for multiplicative functions*, J. reine angew. Math. *313* (1980), 161–170.

11. E. C. Titchmarsh, *The theory of the Riemann Zeta-function* (Clarendon Press, Oxford, 1951).

12. R. C. Vaughan, *On the estimation of trigonometrical sums over primes, and related questions*, Institut Mittag-Leffler, Report No. *9* (1977).

13. I. M. Vinogradov, *The method of trigonometric sums in the theory of numbers* (Interscience, London, 1954).

Magdalen College,
Oxford, England