ON THE SUMMABILITY OF A CLASS OF THE DERIVED CONJUGATE SERIES OF A FOURIER SERIES

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1. Let $f(t)$ be integrable $L(-\pi, \pi)$ and periodic with period 2π, and let

$$(1.1) \qquad \frac{1}{2} a_o + \sum_{1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

be its Fourier series. The series

$$(1.2) \qquad \sum_{1}^{\infty} n(b_n \cos nt - a_n \sin nt) ,$$

obtained by the term by term differentiation of the series (1.1), is called the Derived Fourier series of (1.1). The series conjugate to (1.2),

$$(1.3) \qquad - \sum_{1}^{\infty} n(a_n \cos nt + b_n \sin nt) = - \sum_{1}^{\infty} L_n ,$$

is called the Derived conjugate series of the Fourier series of f.

Suppose that $(\Lambda) = (\lambda_{n,k})$ is a triangular matrix (i.e., $\lambda_{n,k} = 0$ for $k \geq n + 1$) which is regular [cf. 1, page 43, th. 2]. If $\{s_n\}$ denotes the partial sum of the series (1.3), then the (Λ) transforms $\{t_n\}$ are given by

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$$t_n = \sum_{k=1}^{n} \lambda_{n,k} s_k ,$$

and the sequence $\{s_n\}$ will be said to be summable (Λ) to a sum s, if $t_n \to s$ as $n \to \infty$.

The summability (Λ) for the series (1.1) has been considered by Peterson [2]. In this paper, we consider the summability (Λ) for the series (1.3).

2. We write

$$\phi_x(t) = f(x+t) + f(x-t) - 2f(x) , \qquad h(t) = \frac{\phi_x(t)}{t} - d ,$$

where $d \equiv d(x)$ denotes the jump of f at $t = x$. We prove the following

THEOREM: If

(2.1) (i) $\displaystyle\int_t^{\pi} \frac{|h(u)|}{u} du = o(\log 1/t)$ as $t \to 0$;

(ii) (Λ) is a regular triangular matrix such that ;

(a) $\displaystyle\sum_{k=2}^{n} k \log k \, |\lambda_{n,k} - \lambda_{n,k+1}| = O(\log n)$;

(b) $\displaystyle\sum_{k=2}^{n} \lambda_{n,k} \log k \sim \log n$;

then $\dfrac{t_n}{\log n} \to -\dfrac{d}{\pi}$.

The relation between the conditions (2.1) and the condition

(2.2) $$\int_0^t |h(u)| \, du = o(t)$$

is brought out by the following

LEMMA: If $\int_0^t |h(u)|du = o(t)$ as $t \to 0$, then

(2.3) $$\int_t^\pi \frac{|h(u)|}{u} du = o(\log 1/t) \text{ as } t \to 0 .$$

On the other hand, if $\int_t^\pi \frac{|h(u)|}{u} du = o(\log 1/t)$ as $t \to 0$ then

(2.4) $$\int_0^t |h(u)|du = o(t \log 1/t) \text{ as } t \to 0 .$$

This lemma is known [3] with h replaced by ϕ.

3. Proof of the theorem. It is easy to see that

$$L_k = \frac{k}{\pi} \int_0^\pi \phi_x(t) \cos kt \, dt$$

$$= \frac{k}{\pi} \int_0^\pi \frac{\phi_x(t)}{t} \cdot t \cos kt \, dt$$

$$= \frac{k}{\pi} \int_0^\pi \{h(t) + d\} t \cos kt \, dt$$

$$= \frac{k}{\pi} \int_0^\pi th(t) \cos kt \, dt + \frac{d}{\pi} \int_0^\pi tk \cos kt \, dt$$

$$= \beta_k + \gamma_k , \text{ say .}$$

Integrating by parts, we get

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$$\gamma_k = -\frac{d}{\pi}\,\frac{1-(-1)^k}{k} = -\frac{d}{\pi}\,\omega_k, \quad \text{say}\,.$$

Now

$$s_n = \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \beta_k + \sum_{k=1}^{n} \gamma_k$$

$$= \frac{1}{\pi}\int_0^{\pi} th(t) \sum_{k=1}^{n} k\cos kt\, dt - \frac{d}{\pi}\sum_{k=1}^{n} \omega_k$$

$$= \frac{1}{\pi}\int_0^{\pi} th(t)\frac{d}{dt}\Big\{\sum_{k=1}^{n} \sin kt\Big\}\, dt - \frac{d}{\pi}(\log n + C + E_n)$$

(where C is a constant and $E_n \to 0$ as $n \to \infty$)

$$= \frac{1}{\pi}\int_0^{\pi} th(t)\frac{d}{dt}\Big\{\frac{\cos t/2 - \cos(n+1/2)t}{2\sin t/2}\Big\}\, dt$$

$$- \frac{d}{\pi}(\log n + C + E_n)\,.$$

Therefore

$$\frac{t_n}{\log n} = \frac{1}{\log n}\sum_{k=1}^{n} \lambda_{n,k}\frac{1}{\pi}\int_0^{\pi} th(t)\frac{d}{dt}\Big\{\frac{\cos t/2 - \cos(k+1/2)t}{2\sin t/2}\Big\}\, dt$$

$$- \frac{d}{\pi}\,\frac{1}{\log n}\sum_{k=1}^{n} \lambda_{n,k}\log k - \frac{C}{\log n}\,\frac{d}{\pi}\sum_{k=1}^{n}\lambda_{n,k}$$

$$- \frac{d}{\pi}\,\frac{C}{\log n}\sum_{k=1}^{n}\lambda_{n,k}E_k$$

(3.1) $$= J_1 + J_2 + J_3 + J_4, \quad \text{say}.$$

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E_k being a null sequence,

(3.2) $$|J_4| = o(1) .$$

Next,

$$|J_3| = \left|\frac{C}{\log n} \frac{d}{\pi} \sum_{k=1}^{n} \lambda_{n,k}\right|$$

$$= O\left(\frac{1}{\log n}\right)$$

(3.3) $$= o(1) .$$

Since by hypothesis $\sum_{k=1}^{n} \lambda_{n,k} \log k \sim \log n$,

(3.4) $$J_2 \to -\frac{d}{\pi} .$$

Now we consider

$$|J_1| = \left|\frac{1}{\log n} \sum_{k=1}^{n} \lambda_{n,k} \frac{1}{\pi} \int_0^{\pi} t\, h(t) \frac{d}{dt} \left\{\frac{\cos t/2 - \cos(k + 1/2)t}{2 \sin t/2}\right\} dt\right|$$

$$\leq \left|\frac{1}{\log n} \sum_{k=1}^{n} \lambda_{n,k} \frac{1}{\pi} \int_0^{1/k} t\, h(t) \frac{d}{dt} \left\{\frac{\cos t/2 - \cos(k+1/2)t}{2 \sin t/2}\right\} dt\right|$$

$$+ \left|\frac{1}{\log n} \sum_{k=1}^{n} \lambda_{n,k} \frac{1}{\pi} \int_{1/k}^{\pi} t\, h(t) \frac{d}{dt} \left\{\frac{\cos t/2 - \cos(k + 1/2)t}{2 \sin t/2}\right\} dt\right|$$

$$= \left|\frac{1}{\log n} \sum_{k=1}^{n} \lambda_{n,k} P_k\right| + \left|\frac{1}{\log n} \sum_{k=1}^{n} \lambda_{n,k} Q_k\right| , \text{ say}$$

$$= |I_1| + |I_2| .$$

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On integrating by parts and applying the condition (2.4), which is implied by (2.1), it is easy to see that

$$|P_k| = o(\log k) .$$

Therefore

$$|I_1| = o(\frac{1}{\log n} \sum_{k=1}^{n} |\lambda_{n,k}| \log k)$$

$$= o(1) .$$

Next

$$|I_2| = |\frac{1}{\log n} \sum_{k=1}^{n} \lambda_{n,k} \frac{1}{\pi} \int_{1/k}^{\pi} t\, h(t) \frac{d}{dt} \{\frac{\cos t/2 - \cos(k+1/2)t}{2 \sin t/2}\} dt|$$

$$\leq |\frac{1}{\log n} \sum_{k=1}^{n} \lambda_{n,k} \frac{1}{\pi} \int_{1/k}^{\pi} t\, h(t) \cos t/2 \{\frac{\cos t/2 - \cos(k+1/2)t}{(2 \sin t/2)^2}\} dt|$$

$$+ |\frac{1}{\log n} \sum_{k=1}^{n} \lambda_{n,k} \frac{1}{\pi} \int_{1/k}^{\pi} t\, h(t)\, 2 \sin t/2 \frac{1/2 \sin t/2 - (k+1/2) \sin(k+1/2 ...}{(2 \sin t/2)^2}$$

$$= |I_{2,1}| + |I_{2,2}| , \text{ say} .$$

Applying condition (2.1) of the hypothesis, it is easy to see that

$$(3.6) \qquad |I_{2,1}| = o(1)$$

Now

$$|I_{2,2}| \leq |\frac{1}{\log n} \sum_{k=1}^{n} \lambda_{n,k} \frac{1}{\pi} \int_{1/k}^{\pi} t\, h(t)\, dt|$$

$$+ |\frac{1}{\log n} \sum_{k=1}^{n} \lambda_{n,k} \frac{1}{\pi} \int_{1/k}^{\pi} t\, h(t) \frac{(k+1/2) \sin (k+1/2)t}{2 \sin t/2} dt|$$

$$= |I'_{2,2}| + |I''_{2,2}| .$$

Using (2.1), we get

$$(3.7) \qquad |I'_{2,2}| = o(1) .$$

Finally

$$|I''_{2,2}| = \left|\frac{1}{\log n} \sum_{k=1}^{n} \lambda_{n,k} \frac{1}{\pi} \int_{1/k}^{\pi} t\, h(t) \frac{(k+1/2) \sin (k+1/2)t}{2 \sin t/2} dt\right|$$

$$= \left|\frac{1}{\log n} \sum_{k=1}^{n} \lambda_{n,k} \frac{1}{\pi} \int_{1/k}^{\pi} t\, h(t) \frac{1}{2 \sin t/2} [K_k(t) - K_{k-1}(t)] dt\right| ,$$

where

$$|K_k(t)| = \left| \sum_{v=1}^{k} (v + 1/2) \sin (v + 1/2)t \right| \leq \frac{Ak}{t} \quad \text{in } (\frac{1}{k}, \pi) .$$

Also

$$|I''_{2,2}| \leq \frac{1}{\log n} \left| \sum_{k=1}^{n} (\lambda_{n,k} - \lambda_{n,k+1}) \frac{1}{\pi} \int_{1/k}^{\pi} \frac{t\, h(t)}{2 \sin t/2} K_k(t)\, dt \right|$$

$$+ \frac{1}{\log n} \left| \sum_{k=1}^{n} \lambda_{n,k} \frac{1}{\pi} \int_{1/k}^{1/(k-1)} \frac{t\, h(t)}{2 \sin t/2} K_k(t)\, dt \right|$$

$$= |I''_{2,2,1}| + |I''_{2,2,2}|, \quad \text{say} .$$

But

$$|I''_{2,2,1}| \leq \frac{A}{\log n} \sum_{k=1}^{n} |\lambda_{n,k} - \lambda_{n,k+1}| k \int_{1/k}^{\pi} \frac{|h(t)|}{t} dt$$

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$$= o\left(\frac{1}{\log n} \sum_{k=1}^{n} |\lambda_{n,k} - \lambda_{n,k+1}| k \log k\right)$$

(3.8) $$= o(1) ,$$

and

$$|I''_{2,2,2}| = O\left[\frac{1}{\log n} \sum_{k=1}^{n} |\lambda_{n,k}| \frac{k}{\pi} \int_{1/k}^{1/(k-1)} \frac{|h(t)|}{t} dt\right]$$

$$= O\left[\frac{1}{\log n} \sum_{k=1}^{n} |\lambda_{n,k}| T_k\right] , \quad \text{say,}$$

where

$$T_k = \frac{k}{\pi} \int_{1/k}^{1/(k-1)} \frac{|h(t)|}{t} dt .$$

By using (2. 1) we get

$$|T_k| = o(1) .$$

Therefore

$$|I''_{2,2,2}| = o\left(\frac{1}{\log n}\right)$$

(3. 9) $$= o(1) .$$

Combining (3. 8) and (3. 9) we get

(3. 10) $$|I''_{2,2}| = o(1) ,$$

and combining (3. 7) and (3. 10) we get

(3. 11) $$|I_{2,2}| = o(1) ,$$

which on combination with (3. 6) gives

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(3.12) $$|I_2| = o(1) .$$

Combining (3.2), (3.3), (3.4), (3.5) and (3.12) completes the proof.

In particular if we choose $\lambda_{n,k} = \frac{1}{n+1}$ for $k \leq n$ and zero for $k > n$, the (Λ) method of summability reduces to $(C, 1)$ method of summability. Also, this choice of $(\lambda_{n,k})$ satisfies all the conditions imposed on the matrix in our theorem and with this choice our theorem reduces to a theorem due to Mohanty and Nanda [4].

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