BULL. AUSTRAL. MATH. SOC. 20D10
VOL. 21 (1980), 229-236.

NORMAL FITTING CLASSES AND HALL SUBGROUPS

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It was shown by Bryce and Cossey that each Hall π-subgroup of a group in the smallest normal Fitting class S_* necessarily lies in S_*, for each set of primes π. We prove here that for each set of primes π such that $|\pi| \geq 2$ and π' is not empty, there exists a normal Fitting class without this closure property. A characterisation is obtained of all normal Fitting classes which do have this property.

Let F be a normal Fitting class closed under taking Hall π-subgroups, in the sense of the paragraph above, and let S_π denote the Fitting class of all finite soluble π-groups, for some set of primes π. The second main theorem is a characterisation of the groups in the smallest Fitting class containing F and S_π in terms of their Hall π-subgroups.

1. Introduction

Let F be a normal Fitting class of finite soluble groups and π a set of primes. F is said to be *closed under taking Hall π-subgroups* if each group in F possesses a Hall π-subgroup which lies in F. Since every normal Fitting class contains all finite nilpotent groups [3, Theorem 5.1], we avoid triviality by assuming that $|\pi| \geq 2$ and that π' is not

Received 24 September 1979. This paper is based, in part, on a section of the author's PhD thesis, which was submitted at the University of East Anglia in April 1979. The financial support of the Science Research Council is gratefully acknowledged.

https://doi.org/10.1017/S0004972700006055 Published online by Cambridge University Press

empty. Bryce and Cossey showed that the smallest normal Fitting class is closed under taking Hall π-subgroups, for each set of primes π [*6*, 4.15]. This fact can be more easily deduced from a result of Hauck [*8*, Chapter 6]. In Section 3 of this paper, we prove the following result.

THEOREM 1. *Let* π *be a set of primes such that* $|\pi| \geq 2$ *and* π' *is not empty. Then there exists a normal Fitting class which is not closed under taking Hall* π*-subgroups.*

The concept of the join of two Fitting classes was introduced in [*7*]. The join of Fitting classes X and Y is defined to be the smallest Fitting class containing their union. For each set of primes π, let S_π denote the Fitting class of all finite soluble π-groups, and recall that a subgroup N of the direct product $G \times H$ of groups G and H is said to be *subdirect* in $G \times H$ if $N(1 \times H) = G \times H = (G \times 1)N$. Our second main result is proved in Section 4.

THEOREM 2. *Let* π *be a set of primes and* F *a normal Fitting class closed under taking Hall* π*-subgroups. Let* H *be a Hall* π*-subgroup of a group* G. *Then* G *lies in* $S_\pi \vee F$ *if and only if* $(G \times H)_F$ *is subdirect in* $G \times H$.

That many normal Fitting classes are closed under taking Hall π-subgroups for a given set of primes π is ensured by the characterisation of these Fitting classes obtained in Theorem 5 of Section 3.

2. Preliminaries

All groups mentioned are finite and soluble. Basic definitions and facts concerning Fitting classes and the $*$-operation may be found in [*3*] and [*10*]. The notation is standard and is described in [*7*]. We point out that as a consequence of [*10*, Theorem 2.2c)], the normal Fitting class S_* is contained in every normal Fitting class. We list the following results for the reader's convenience.

I [*7*, Corollary 2.6]. *Let* X *and* Y *be Fitting classes such that* $X \subseteq Y^*$. *Then a group* G *lies in* $X \vee Y$ *if and only if there exists a group* K *in* X *such that* $(G \times K)_Y$ *is subdirect in* $G \times K$.

When $X = S_\pi$, for a set of primes π, and Y is a normal Fitting

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class closed under taking Hall π-subgroups, Theorem 2 will allow us to dispense with the arbitrary choice of the group K in I. The next result can be deduced from I and Theorem 2.9 of [7].

II. *Let* X, Y *and* Z *be Fitting classes such that* $X \subseteq Y^*$.

1. *If* $X \subseteq Z$, *then* $(X \vee Y) \cap Z = X \vee (Y \cap Z)$.

2. *If* $Y \subseteq Z$, *then* $(X \vee Y) \cap Z = (X \cap Z) \vee Y$.

We now introduce a notation of Hauck [8]. Let F be a Fitting class and π a set of primes. Then $Y(S_\pi, F)$ denotes the Fitting class of groups in which each Hall π-subgroup lies in F. The following theorem is a consequence of Hilfssatz 3 of [1].

III. $Y(S_\pi, F)$ *is a normal Fitting class, for each set of primes* π *and normal Fitting class* F.

Finally, we have a theorem collated from various sources, which will be crucial to the proof of Theorem 1.

IV. *Let* p *and* q *be distinct primes. There exists a group* $H(p, q)$ *such that* $O_p(H(p, q)) = H(p, q)_{S_*}$ *and* $|H(p, q)/H(p, q)_{S_*}| = q$.

If $q \mid p-1$, then the existence of $H(p, q)$ is established in [2]. The existence of $H(p, q)$ when $q \nmid p-1$ is a consequence of the main theorems of [5] and [9]. Details of the construction of a suitable group $H(p, q)$ may be found in [4, Chapter 3.7].

3. Normal Fitting classes closed under taking Hall π-subgroups

Let π be a non-trivial set of primes, in the sense of Theorem 1. Choose distinct primes p, q and r such that p and q are in π, and r is in π'. Set $K = H(p, q)$, $L = H(r, q)$ and denote by G the normal subgroup $(K_{S_*} \times L_{S_*})\langle(k, l)\rangle$ of $K \times L$, where k and l are elements of order q in K and L respectively. Set $F = \mathrm{Fit}\{G\} \vee S_*$. Certainly F is a normal Fitting class, since $S_* \subseteq F \subseteq S$ [10].

Proof of Theorem 1. The candidate is F. Since G lies in F and each Hall π-subgroup of G is isomorphic to K it is sufficient to prove that K is not in F. We begin by examining G.

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If G is in S_*, then $G \leq (K \times L)_{S_*}$. This implies that $K \times L = (K \times 1)(K \times L)_{S_*}$ and it follows from the definition of a Fitting class that L lies in $S_\pi \vee S_*$. Let Q be a Sylow q-subgroup of L. Certainly Q is a Hall π-subgroup of L, and so by Theorem 2, $(L \times Q)_{S_*}$ is subdirect in $L \times Q$. Since Q is nilpotent, Q lies in S_*, which leads to the contradiction that L is in S_*. We conclude that G does not lie in S_*, and consequently that $G_{S_*} = K_{S_*} \times L_{S_*} = (K \times L)_{S_*}$.

Suppose now that K lies in F. By [7, Corollary 2.5], G possesses a characteristic subgroup N such that $(K \times N)_{S_*}$ is subdirect in $K \times N$. If $N \leq G_{S_*}$, then K must lie in S_*, a contradiction. We may therefore assume that $NG_{S_*} = G$. It follows that $(K \times G)_{S_*}$ is subdirect in $K \times G$. There exists, therefore, an element x of G, of order q, such that (k, x) is an element of $(K \times G)_{S_*}$. Each element of order q in G is a conjugate of $(k, l)^n$, for some integer n lying between 1 and q. Since G/G_{S_*} is abelian, we have

$xG_{S_*} = (k, l)^n G_{S_*}$, for some integer n. The fact that G is a normal subgroup of $K \times L$ now establishes that (k, k^n, l^n) is an element of $(K \times K \times L)_{S_*}$. By definition of the $*$-operation [10],

$$(K \times K \times 1)_{S_*} = (K_{S_*} \times K_{S_*} \times 1)\langle (g^{-1}, g, 1) \mid g \in K \rangle .$$

We therefore have

$$(1, k^{n+1}, l^n) = (k^{-1}, k, 1)(k, k^n, l^n) \in (K \times K \times L)_{S_*} .$$

Certainly $(1, k^{n+1}, l^n)$ is an element of $1 \times K \times L$, and so $(k^{n+1}, l^n) \in (K \times L)_{S_*}$. Since $(K \times L)_{S_*} = K_{S_*} \times L_{S_*}$, the choice of k and l implies that q divides both n and $n+1$. This contradiction leads us to conclude that K does not lie in F.

The characterisation of those normal Fitting classes closed under taking Hall π-subgroups depends on the following two results.

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LEMMA 3. *Let* π *be a set of primes and* F *a normal Fitting class which is closed under taking Hall* π*-subgroups. Let* H *be a Hall* π*-subgroup of a group* G *in* FS_π *. Then* G *lies in* $Y(S_\pi, F)$ *if and only if* $G = H_F G_F$ *.*

Proof. Suppose that G is in $Y(S_\pi, F)$. Certainly $G = HG_F$, and $H = H_F$. It is immediate that $G = H_F G_F$. Conversely, suppose that $G = H_F G_F$. Then $H = H_F(H \cap G_F)$, and by hypothesis $H \cap G_F$ lies in F . Thus $H = H_F$, ensuring that G is in $Y(S_\pi, F)$.

THEOREM 4. $S_\pi \vee Y(S_\pi, S_*) = S$, *for each set of primes* π .

Proof. Let H be a Hall π-subgroup and K a Hall π'-subgroup of a group G in $S_* S_{\pi'}$. Then $G = KG_{S_*}$, and so $H \leq G_{S_*}$. Since S_* is closed under taking Hall π-subgroups, this ensures that H lies in S_* . Thus $S_* S_{\pi'}$ is contained in $Y(S_\pi, S_*)$.

Suppose now that H is a Hall π-subgroup of a group G in $S_* S_\pi$. Then $G \times H$ is in $S_* S_\pi$, and it follows from Lemma 3 that $(H \times H)_{S_*}(G \times H)_{S_*}$ is the $Y(S_\pi, S_*)$-radical of $G \times H$. Since $(H \times H)_{S_*}(G \times H)_{S_*}$ contains $\left(H_{S_*} G_{S_*} \times H_{S_*}\right)\langle (h^{-1}, h) \mid h \in H\rangle$, and $G = HG_{S_*}$, the $Y(S_\pi, S_*)$-radical of $G \times H$ is subdirect in $G \times H$. We conclude from I that $S_* S_\pi$ is contained in $S_\pi \vee Y(S_\pi, S_*)$. It follows from [7, Theorem 2.1] that $S_* S_\pi \vee S_* S_{\pi'} = S$, and consequently $S_\pi \vee Y(S_\pi, S_*) = S$.

THEOREM 5. *Let* F *be a normal Fitting class and* π *a set of primes. Then* F *is closed under taking Hall* π*-subgroups if and only if* $F = (S_\pi \cap F) \vee (Y(S_\pi, S_*) \cap F)$.

Proof. IF. Certainly $S_\pi \cap F$ and $Y(S_\pi, S_*)$ are contained in $Y(S_\pi, F)$. Thus F is contained in $Y(S_\pi, F)$, and so is closed under taking Hall π-subgroups.

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ONLY IF. Suppose that F is closed under taking Hall π-subgroups. In other words, $F \subseteq Y(S_\pi, F)$. Since $Y(S_\pi, S_*) \subseteq Y(S_\pi, F)$, it follows from II and Theorem 4 that

$$Y(S_\pi, F) = (S_\pi \vee Y(S_\pi, S_*)) \cap Y(S_\pi, F)$$
$$= (S_\pi \cap Y(S_\pi, F)) \vee Y(S_\pi, S_*) = (S_\pi \cap F) \vee Y(S_\pi, S_*) .$$

A further application of II yields that

$$F = F \cap Y(S_\pi, F) = F \cap ((S_\pi \cap F) \vee Y(S_\pi, S_*))$$
$$= (S_\pi \cap F) \vee (Y(S_\pi, S_*) \cap F) .$$

4. The proof of Theorem 2

LEMMA 6. *Let π be a set of primes and F a normal Fitting class closed under taking Hall π-subgroups. Let H be a Hall π-subgroup of a group G in $S_\pi \vee F$. Then G_F contains H_F .*

Proof. Certainly $S_\pi \vee F \subseteq FS_\pi$, and so Lemma 3 implies that $H_F G_F$ is the $Y(S_\pi, F)$-radical of G . Since $F \subseteq Y(S_\pi, F)$, we may apply II to obtain $(S_\pi \vee F) \cap Y(S_\pi, F) = (S_\pi \cap Y(S_\pi, F)) \vee F = F$. Thus $H_F G_F$ lies in F , establishing the result.

Proof of Theorem 2. IF. This follows immediately from I.

ONLY IF. Both F and S_π are contained in FS_π , so $S_\pi \vee F$ is contained in FS_π . Let T denote the set of groups G in FS_π such that for some Hall π-subgroup H of G , $(G \times H)_F$ is subdirect in $G \times H$. Since F is closed under taking Hall π-subgroups, $F \subseteq T$, and by definition of the $*$-operation $S_\pi \subseteq T$. That $T \subseteq S_\pi \vee F$ is ensured by I, and it is thus sufficient to show that T is a Fitting class.

Let H be a Hall π-subgroup of a group G in T . Certainly $G \times H$ is in $S_\pi \vee F$, and it follows from Lemma 6 that $(G \times H)_F$ contains $(H \times H)_F$. The definition of the $*$-operation, and the fact that $G = HG_F$, allow us to write $(G \times H)_F = (G_F \times H_F)\langle (h^{-1}, h) \mid h \in H\rangle$. Suppose now that N is a normal subgroup of G . Then $N = (N \cap H)N_F$ and

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$$\left(N \times (N \cap H)\right)_F = \left(N \times (N \cap H)\right) \cap \left[\left(G_F \times H_F\right)\langle \left(h^{-1}, h\right) \mid h \in H\rangle\right]$$
$$= \left(N_F \times (N \cap H)_F\right)\langle \left(h^{-1}, h\right) \mid h \in N \cap H\rangle .$$

Thus N lies in T .

If N and M are normal subgroups, and H is a Hall π-subgroup, of a group G , such that N and M are in T and $G = NM$, then certainly $H = (H \cap N)(H \cap M)$. Let h be an element of H . Then there exist elements n of N and m of M such that $h = nm$. By hypothesis, $\left(n^{-1}, n\right) \in \left(N \times (N \cap H)\right)_F$ and $\left(m^{-1}, m\right) \in \left(M \times (M \cap H)\right)_F$. Since G/G_F is abelian, $mnm^{-1}n^{-1} \in G_F$. Thus

$$\left(h^{-1}, h\right) = \left(m^{-1}n^{-1}, nm\right) = \left(n^{-1}, n\right)\left(m^{-1}, m\right)\left(mnm^{-1}n^{-1}, 1\right) (G \times H)_F ,$$

ensuring that G lies in T . This completes the proof that T is a Fitting class.

References

[1] James C. Beidleman und Peter Hauck, "Über Fittingklassen und die Lockett-Vermutung", submitted.

[2] Thomas R. Berger, "More normal Fitting classes of finite solvable groups", *Math. Z.* 151 (1976), 1-3.

[3] Dieter Blessenohl und Wolfgang Gaschütz, "Über normale Schunk- und Fittingklassen", *Math. Z.* 118 (1970), 1-8.

[4] Owen John Brison, "On the theory of Fitting classes of finite groups" (PhD thesis, University of Warwick, Warwick, 1978).

[5] R.M. Bryant and L.G. Kovács, "Lie representations and groups of prime power order", *J. London Math. Soc.* (2) 17 (1978), 415-421.

[6] R.A. Bryce and John Cossey, "A problem in the theory of normal Fitting classes", *Math. Z.* 141 (1975), 99-110.

[7] Elspeth Cusack, "The join of two Fitting classes", *Math. Z.* 167 (1979), 37-47.

[8] Peter Hauck, "Zur Theorie der Fittingklassen endlicher auflosbarer" (PhD thesis, Johannes Gutenberg-Universität in Mainz, Mainz, Germany, 1977).

[9] Harmut Laue, Hans Lausch und Garry R. Pain, "Verlagerung und normale Fittingklassen endlichen Gruppen", *Math. Z.* **154** (1977), 257-260.

[10] F. Peter Lockett, "The Fitting class $\underline{\underline{F}}^*$ ", *Math. Z.* **137** (1974), 131-136.

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