ON THE CLIFFORD COLLINEATION, TRANSFORM AND SIMILARITY GROUPS

(III) GENERATORS AND INVOLUTIONS

BEVERLEY BOLT

(received 26 October, 1960)

1. Introduction

In this paper the Clifford groups $PCT(p^m), p > 2$, $PCG(p^m)$ and $CS'(p^m)$, and the factor groups $\frac{1}{2}CS'(p^m)$, which were defined in Paper I of this series (Bolt, Room and Wall [1])[1], are considered as transformations of projective $[p^m - 1]$ over the complex field, C. We note that the geometrical results are the same if any of the corresponding groups CT, CG or $\mathscr{C}\mathscr{G}$, and CS, respectively, are considered instead.

In § 2 explicit generators of PCT, CS' and PCG are given and the involutions of PCT are determined. In § 3 the configurations of linear spaces in $[p^m - 1]$ formed by the invariant spaces of the involutions of PCT are examined. Familiarity with the results and notation of Bolt, Room and Wall ([1], [2]) is assumed throughout but, for convenience, the following results are quoted.

1.1 $PCT(p^m) \cong ASp(2m, p), p > 2$; ASp is the group of symplectic affine transformations $(T, \boldsymbol{t}): \boldsymbol{\alpha} \to \boldsymbol{\alpha}T' + \boldsymbol{t}$, where $T \in Sp(2m, p), \boldsymbol{t}, \boldsymbol{\alpha} \in \mathscr{V}_{2m}$, and $\mathscr{V}_k = \mathscr{V}_k(p)$ is the k-dimensional space of all row vectors $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_k)$ over $GF(p)$.

1.2 $(S, \boldsymbol{s})(T, \boldsymbol{t}) = (ST, \boldsymbol{t}S' + \boldsymbol{s})$.

1.3 $PCG(p^m) \cong$ the normal subgroup of ASp formed by the "translations" $(I, \boldsymbol{t})$, where I is the identity $2m \times 2m$ matrix.

If $W^{\boldsymbol{\alpha}} \in PCG$ and $W^{\boldsymbol{\alpha}} \leftrightarrow (I, \boldsymbol{\alpha})$, then $W^{\boldsymbol{\alpha}}$ is a matrix with one non-zero element in each row and column given by

1.4 $$(W^{\boldsymbol{\alpha}})_{\boldsymbol{\lambda}}^{\boldsymbol{\lambda}+\boldsymbol{a}_2} = \omega^{\boldsymbol{a}_1 \cdot (\boldsymbol{\lambda}+\boldsymbol{a}_2)}$$

where $\boldsymbol{\alpha} = (\boldsymbol{a}_1, \boldsymbol{a}_2)$, $\boldsymbol{\lambda}, \boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathscr{V}_m$, $\omega = \exp(2\pi i/p)$; and the rows and columns of the $p^m \times p^m$ matrices are numbered in the reversed scale of p, so that $\boldsymbol{\lambda}$ is the index of the row or column with position number $\lambda_1 + p\lambda_2 + \cdots + p^{m-1}\lambda_m$.

[1] Hereafter referred to as Paper I.

1.5 $$CT(p^m)/CG(p^m) \cong Sp(2m, p);$$

and moreover $CT(p^m)$ contains a subgroup $CS(p^m)$ such that[2]

1.6 $$PCS(p^m) \cong CS'(p^m) \cong Sp(2m, p), \quad p > 2.$$

$CS'(p^m)$ corresponds to the transformations $(T, \mathbf{0})$ of ASp and contains a single self-conjugate element J corresponding to $(-I, \mathbf{0})$; $CS(p^m)$ is the centralizer of J in $CT(p^m)$.

Let $X_T \in CS'$ where $X_T \leftrightarrow (T, \mathbf{0})$, then $W^t X_T \in PCT$ and $W^t X_T \leftrightarrow (T, \boldsymbol{t})$. Write

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A, B, C, D are $m \times m$ matrices. Let $d_T =$ rank of C.

Define M to be an $m \times m$ matrix composed of d_T columns of C and $m - d_T$ columns of D such that the columns of D, $i_{d_T+1}, \cdots, i_m$, together with the columns of C span $\mathscr{V}_m$. The i-th column of M is the i-th column of D where i is one of $i_{d_T+1}, \cdots, i_m$ and the i-th column of C otherwise.

Define $\mathscr{V}_T$ as the supspace of $\mathscr{V}_{2m}$ formed by all vectors $(\boldsymbol{a}_2, \boldsymbol{A}_2)$ where $(\boldsymbol{a}_1, \boldsymbol{a}_2)$ runs over $\mathscr{V}_{2m}$ and $(\boldsymbol{A}_1, \boldsymbol{A}_2) = (\boldsymbol{a}_1, \boldsymbol{a}_2)T'$.

THEOREM 1.

$$(X_T)^{\boldsymbol{\mu}}_{\boldsymbol{\lambda}} = \left(\frac{\det M}{p}\right) \left(\left(\frac{2}{p}\right) \theta\right)^{-d_T} \eta_{\boldsymbol{\lambda}\boldsymbol{\mu}}$$

where

$$\eta_{\boldsymbol{a}_2 \boldsymbol{A}_2} = \omega^{\frac{1}{2}(\boldsymbol{A}_1 \cdot \boldsymbol{A}_2 - \boldsymbol{a}_1 \cdot \boldsymbol{a}_2)},$$

and

$$\eta_{\boldsymbol{\lambda}\boldsymbol{\mu}} = 0 \quad \text{if} \quad (\boldsymbol{\lambda}, \boldsymbol{\mu}) \notin \mathscr{V}_T;$$

$$\left(\frac{s}{p}\right) \text{is Legendre's symbol,}$$

and

$$\theta = \begin{cases} p^{\frac{1}{2}} & p \equiv 1 \pmod 4 \\ ip^{\frac{1}{2}} & p \equiv 3 \pmod 4). \end{cases}$$

We note that X_T has p^{d_T} non-zero elements in each row and column, and in particular

(i) if C is the zero matrix, X_T has one non-zero element in each row and column given by

$$(X_T)^{\boldsymbol{\mu} D'}_{\boldsymbol{\mu}} = \left(\frac{\det D}{p}\right) \omega^{\frac{1}{2}(\boldsymbol{\mu} B'D) \cdot \boldsymbol{\mu}};$$

[2] The isomorphism $CS' \cong Sp$ does not hold for $p^m = 3$; see Paper I, Appendix.

https://doi.org/10.1017/S1446788700026926 Published online by Cambridge University Press

(ii) if $T = -I, X_T = J$

$$(J)_{\bar{\mu}}^{\mu} = \left(\frac{(-1)^m}{p}\right) = \begin{cases} -1 & m \text{ odd}, p \equiv 3 \pmod 4, \\ 1 & \text{otherwise}; \end{cases}$$

(iii) if C is non-singular

$$(X_T)_{\mu}^{\lambda} = \left(\frac{\det C}{p}\right)\left(\left(\frac{2}{p}\right)\theta\right)^{-m} \omega^{\frac{1}{2}\rho}$$

where

$$\rho = (\lambda AC^{-1}) \cdot \lambda - 2(\mu C^{-1}) \cdot \lambda + (\mu C^{-1} D) \cdot \mu.$$

2.1 Generators of *PCG*, *CS′* and *PCT*

From Paper I (3.1.2) any $2m$ matrices W^{α_i}, $i = 1, \cdots, 2m$, such that α_i form a basis of $\mathscr{V}_{2m}$, generate *PCG*. Hence we may take as generators W^{σ_i}, where

$$\sigma_{2i-1} = (e_i, -\nu_{i-1}), \quad \sigma_{2i} = (e_i, -\nu_i), \quad i = 1, \cdots, m,$$

with e_i the i-th unit vector in $\mathscr{V}_m$, and $\nu_i = \sum_{k=1}^{i} e_k$.

Room and Smith ([7]) showed that $Sp(2m, p)$ is generated by two elements D and Q of periods p and $4m + 2$ respectively, and $Q^{2m+1} = -I$.

2.1.1 $$D = \begin{bmatrix} I & W \\ 0 & I \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix},$$

where

I is the $m \times m$ identity matrix,
0 is the $m \times m$ zero matrix,

$$W = \begin{bmatrix} -e_1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad X = \begin{bmatrix} \nu_m \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix},$$

$$Y = \begin{bmatrix} e_1 \\ -e_1 + e_2 \\ -e_2 + e_3 \\ \cdot \\ \cdot \\ \cdot \\ -e_{m-1} + e_m \end{bmatrix}, \quad Z = \begin{bmatrix} -\nu_m \\ -\nu_m + \nu_1 \\ \cdot \\ \cdot \\ \cdot \\ -\nu_m + \nu_{m-1} \end{bmatrix}.$$

Thus from Theorem 1 we have

https://doi.org/10.1017/S1446788700026926 Published online by Cambridge University Press

THEOREM 2. *CS' is generated by the two matrices X_Q and X_D of periods $4m+2$ and p, respectively, where*

$$(X_Q)^{\lambda}_{\mu} = \left(\frac{(-1)^m}{p}\right)\left(\left(\frac{2}{p}\right)\theta\right)^{-m}\omega^{\rho}$$

$$2\rho = -\lambda_1^2 + 2\sum_{i=1}^{m-1}(\lambda_i - \lambda_{i+1})\mu_i + 2\lambda_m\mu_m,$$

and X_D is the diagonal matrix $(X_D)^{\lambda}_{\lambda} = \omega^{-\frac{1}{2}\lambda_1^2}$.

It is easily verified that $\boldsymbol{\sigma}_i Q' = \boldsymbol{\sigma}_{i+1}\ i = 1, \cdots, 2m-1$. Hence as a subgroup of PCT, PCG is generated by X_Q and $W^{\boldsymbol{\sigma}_1}$. Further

2.1.2 $$\boldsymbol{\sigma}_1 D' = \boldsymbol{\sigma}_1.$$

We next prove:

THEOREM 3. *$PCT(p^m)$ is generated by the two matrices X_Q and $W^{\boldsymbol{\sigma}_1}X_D$ of periods $4m+2$ and p, respectively.*

We need only show that $(Q, \mathbf{0})$ and $(D, \boldsymbol{\sigma}_1)$ generate $(D, \mathbf{0})$ and $(I, \boldsymbol{\sigma}_1)$. Now

$$\begin{aligned}(Q, \mathbf{0})^{2m+1} &= (-I, \mathbf{0}).\\ (-I, \mathbf{0})(D, \boldsymbol{\sigma}_1) &= (-D, -\boldsymbol{\sigma}_1).\\ (-D, -\boldsymbol{\sigma}_1)^2 &= (D^2, \boldsymbol{\sigma}_1 D' - \boldsymbol{\sigma}_1)\\ &= (D^2, \mathbf{0}) \text{ from 2.1.2,}\end{aligned}$$

and

$$(D^2, \mathbf{0})^{\frac{1}{2}(p+1)} = (D, \mathbf{0}).$$

Further

$$(D, \boldsymbol{\sigma}_1)(D^{-1}, \mathbf{0}) = (I, \boldsymbol{\sigma}_1).$$

It is easily verified, using 2.1.2, that $(D, \boldsymbol{\sigma}_1)$ has period p. Q.E.D.

2.2. Involutions of *PCT*

Dickson ([3]) showed that $Sp(2m, p)$ contains exactly m sets of conjugate, involutory substitutions. The r-th set includes

$$E_{m,r} = \frac{(p^{2m}-1)(p^{2m-2}-1)\cdots(p^{2m-2r+2}-1)}{(p^{2r}-1)(p^{2r-2}-1)\cdots(p^2-1)}p^{2r(m-r)}$$

substitutions each conjugate with the diagonal matrix

$$T_r = [-I_r, I_{m-r}, -I_r, I_{m-r}], \quad 0 < r \leqq m,$$

where I_s is the $s \times s$ identity matrix. Furthermore, every substitution S of $Sp(2m, p)$, such that $S^2 = -I$, is conjugate with $G = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$. In $\frac{1}{2}Sp(2m, p)$, T_r and T_{m-r} coincide and a new set of involutions is given by the matrices for which $S^2 = -I$.

Consider next an involutory substitution (T, t) of $ASp(2m, p)$. Then

$$(T, t)^2 = (T^2, tT' + t) = (I, 0).$$

So

$$T^2 = I \quad \text{and} \quad tT' = -t.$$

Thus every involution T of $Sp(2m, p)$ determines p^{2r} involutions (T, t) in $ASp(2m, p)$ where t belongs to the minus-subspace $\mathscr{V}_{2r}$ of T.

Now $(I, -s)(T, 0)(I, s) = (T, s(T' - I))$ from 1.2 and $s(T' - I)$ runs over $\mathscr{V}_{2r}$ as s runs over $\mathscr{V}_{2m}$. So $(T, 0)$ is conjugate to (T, t) where $tT' = -t$.

THEOREM [3] 4. *In PCT there are m classes of conjugate involutions. The r-th class corresponds to* $(T_r, 0)$ *in* $ASp(2m, p)$ *and contains* $p^{2r} E_{m,r}$ *involutions.*

In particular, $-I$ determines p^{2m} conjugate involutions $W^{\alpha} J$ in PCT corresponding to $(-I, \alpha)$, each of which determines a "symplectic" subgroup of PCT consisting of elements corresponding to $(T, \frac{1}{2}\alpha(I - T'))$, $T \in Sp(2m, p)$ (cf. 1.6). This subgroup of $ASp(2m, p)$ leaves invariant the point $\frac{1}{2}\alpha$ of $\mathscr{V}_{2m}$, so there is a $1 - 1$ correspondence between the involutions $W^{\alpha} J$ and the points $\frac{1}{2}\alpha$ of $\mathscr{V}_{2m}$.

As a substitution on projective $[p^m - 1]$, J leaves invariant a pair of dual spaces, its eigenspaces corresponding to the eigenvalues $1, -1$; and since J is self-conjugate in CS' its eigenspaces are invariant under CS'. From Theorem 1, (ii) the trace of J is 1 or -1 so the dimensions of its eigenspaces differ by 1 and are therefore $[\frac{1}{2}(p^m - 1)]$ and $[\frac{1}{2}(p^m - 3)]$.

We turn next to the configuration in $[p^m - 1]$ determined by the eigenspaces of the p^{2m} involutions $W^{\alpha} J$, which is invariant under PCT and occupies a fundamental position in the geometry of the group.

3.1. Invariant spaces of the involutions $W^{\alpha} J$

A point in projective space of dimension $(p^m - 1)$ is determined by an ordered set of p^m homogeneous coordinates x_{λ}, where λ is the index of the coordinate with position number $\lambda_1 + p\lambda_2 + \cdots + p^{m-1}\lambda_m$. Vertices of the simplex of reference are X_{λ} opposite the respective prime faces $x_{\lambda} = 0$.

We note that $Sp(2m, p)$ contains a subgroup isomorphic to $Sp(2m - 2, p)$ containing the matrices

$$\begin{bmatrix} A & 0' & B & 0' \\ 0 & 1 & 0 & 1 \\ C & 0' & D & 0' \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

where A, B, C, D are matrices of $m-1$ rows and columns and $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$

[3] This result differs from Horadam's ([5]) for $PCT(3^2)$. It seems that the difference arises because he does not take into account that each involution commutes with 9 of the $W^{\alpha} J$.

https://doi.org/10.1017/S1446788700026926 Published online by Cambridge University Press

belongs to $Sp(2m-2, p)$. The corresponding matrices of $CS'(p^m)$ transform the $(p^{m-1}-1)$-dimensional subspace of $[p^m-1]$ determined by the points $X_{\lambda_1\cdots\lambda_{m-1}0}$ in the manner of $CS'(p^{m-1})$. Thus the geometry of $CS'(p^{m-1})$ is repeated in the $[p^{m-1}-1]$, and we show that the intersections of the eigenspaces of the $W^{\alpha}J$ are the invariant configurations of the involutions for $PCT(p^k)$, $k=1,\cdots,m-1$.

Denote by Π^{α} and Σ^{α} the invariant spaces of $W^{\alpha}J$ of dimensions $\frac{1}{2}(p^m-1)$ and $\frac{1}{2}(p^m-3)$, respectively. Then Π^{α} is the common space of the primes

3.1.1 $$x_{\lambda}-\omega^{\lambda\cdot a_1-\frac{1}{2}a_1\cdot a_2}x_{-\lambda+a_2}=0,\quad \alpha=(a_1, a_2),$$

and Σ^{α} is the common space of the primes

3.1.2 $$x_{\lambda}+\omega^{\lambda\cdot a_1-\frac{1}{2}a_1\cdot a_2}x_{-\lambda+a_2}=0.$$

Π^{α} is the minus-subspace if m is odd and $p\equiv 3 \pmod 4$, and is the plus-subspace otherwise, (from Theorem 1, ii). Since the involutions are conjugate we take Π^{0} and Σ^{0} as typical spaces.

Consider the intersection, S_1, of the p spaces $\Pi^{\rho\varepsilon_m}$, $\rho=0,\cdots,p-1$, where ε_i is the i-th unit vector of $\mathscr{V}_{2m}$. Then S_1 is the intersection of the primes

$$x_{\lambda}=\omega^{\rho\lambda_m}x_{-\lambda}=\omega^{\sigma\lambda_m}x_{-\lambda}\quad \text{all}\quad \rho, \sigma.$$

The $\frac{1}{2}(p^{m-1}+1)$ points $(X_{\lambda_1\cdots\lambda_{m-1}0}+X_{-\lambda_1\cdots-\lambda_{m-1}0})$ span S_1 which therefore has dimension $\frac{1}{2}(p^{m-1}-1)$. It is easily verified that S_1 is the complete intersection of any pair of the $\Pi^{\rho\varepsilon_m}$ and lies in no other Π^{α}. Thus, since there exists a matrix of CS' transforming any Π^{α} into Π^{ε_m}, Π^{0} meets every Π^{α} in a $[\frac{1}{2}(p^{m-1}-1)]$, and every pair of Π^{α} intersects in a $[\frac{1}{2}(p^{m-1}-1)]$ through which pass p of the Π^{α}. Thus S_1 belongs to a set of $C_2^{p^{2m}}\div C_2^{p}$ conjugate spaces in $[p^m-1]$, which lie in sets of $(p^{2m}-1)/(p-1)$ in the p^{2m} Π^{α}. Dually, the p $\Sigma^{\rho\varepsilon_m}$ lie in the intersection of the primes

$$x_{\lambda_1\cdots\lambda_{m-1}0}+x_{-\lambda_1\cdots-\lambda_{m-1}0}=0,\quad \text{i.e., in a } [p^m-\tfrac{1}{2}(p^m-3)].$$

Each pair of Σ^{α} lies in a $[p^m-\frac{1}{2}(p^m-3)]$ which contains p of the Σ^{α}.

Similar results hold for the intersection s_1 of the p $\Sigma^{\rho\varepsilon_m}$. s_1 is a $[\frac{1}{2}(p^{m-1}-3)]$ spanned by the $\frac{1}{2}(p^{m-1}-1)$ points $(X_{\lambda_1\cdots\lambda_{m-1}0}-X_{-\lambda_1\cdots-\lambda_{m-1}0})$. The join of S_1 and s_1 is a $[p^{m-1}-1]$ in which the geometry of $PCT(p^{m-1})$ occurs; S_1 and s_1 are the Π^{0} and Σ^{0}, respectively, of $PCT(p^{m-1})$ in $[p^{m-1}-1]$.

The primes

$$x_{\lambda}=x_{-\lambda}=-\omega^{\lambda_m}x_{-\lambda}$$

have no common space, so that Π^{0} and Σ^{ε_m} do not intersect. It follows that Π^{0} does not intersect any Σ^{α}.

More generally consider the intersection of the p^r Π^{α}, $\alpha=\sum_{i=m-r+1}^{m}a_i\varepsilon_i$, $0<r\leqq m$. Each of these Π^{α} contains the space S_r spanned by the $\frac{1}{2}(p^{m-r}+1)$ points $X_{\lambda_1\cdots\lambda_{m-r}0\cdots0}+X_{-\lambda_1\cdots-\lambda_{m-r}0\cdots0}$, and S_r is not contained in any

https://doi.org/10.1017/S1446788700026926 Published online by Cambridge University Press

other Π^{α}. Since $W^{\alpha}J$, and therefore Π^{α}, is represented by the point $\frac{1}{2}\alpha$ of $\mathscr{V}_{2m}$ (cf. § 2.2), we note that S_r can be represented by the r-dimensional subspace, $\sum_{i=m-r+1}^{m} a_i \varepsilon_i$, of $\mathscr{V}_{2m}$. S_r is determined by any set of r linearly independent vectors of the subspace and is the complete intersection of the corresponding $r\Pi^{\alpha}$ and Π^0.

We next show that the spaces S_r of the configuration in Π^0 are in $1-1$ correspondence with the r-dimensional subspaces of $\mathscr{V}_{2m}$ on which $f(\alpha, \beta) = \mathbf{a}_1 \cdot \mathbf{b}_2 - \mathbf{a}_2 \cdot \mathbf{b}_1$ vanishes, where $\alpha = (\mathbf{a}_1, \mathbf{a}_2)$, $\beta = (\mathbf{b}_1, \mathbf{b}_2)$.

THEOREM 5. *The $r+1$ spaces $\Pi^0, \Pi^{\alpha_1}, \cdots, \Pi^{\alpha_r}$, where α_i are linearly independent vectors of $\mathscr{V}_{2m}$, intersect in a $[\frac{1}{2}(p^{m-r}-1)]$ if, and only if,*

$$f(\alpha_i, \alpha_j) = 0 \quad \textit{all} \quad i, j.$$

The sufficiency of the condition follows from Witt's Theorem (Dieudonné [4]). Since $f(\alpha, \beta)$ vanishes on the subspace of $\mathscr{V}_{2m}$ determined by α_i, $i = 1; \cdots, r$, there exists a symplectic transformation such that $\alpha_i \to \varepsilon_{m-r+i}$, all i.Hence there exists a transformation of CS' such that $\Pi^{\alpha_i} \to \Pi^{\varepsilon_{m-r+i}}$, so the $r+1$ spaces Π^0, Π^{α_i}, $i = 1, \cdots, r$, intersect in a $[\frac{1}{2}(p^{m-r}-1)]$ conjugate to S_r.

The second part of the theorem we assume for $PCT(p^{m-1})$ and prove by induction on m. For $m = 1$, Π^0 intersects each Π^{α} in a point and the condition is trivially satisfied. Suppose now that Π^0, Π^{α_i}, $i = 1, \cdots, r$, intersect in a $[\frac{1}{2}(p^{m-r}-1)]$, say S'_r. Since the Π^{α} are conjugate there exists a transformation of CS' for which $\Pi^{\alpha_1} \to \Pi^{\varepsilon_m}$, and which therefore transforms S'_r into a subspace of the configuration in S_1; i.e. into a $[\frac{1}{2}(p^{m-r}-1)]$ of the configuration in Π^0 for $PCT(p^{m-1})$. Thus there exists a transformation of CS' such that $S'_r \to S_r$, and the corresponding symplectic transformation maps $\sum_{i=1}^{r} b_i \alpha_i$ into $\sum_{i=m-r+1}^{m} a_i \varepsilon_i$, a subspace of $\mathscr{V}_{2m}$ on which $f(\alpha, \beta)$ vanishes. $f(\alpha, \beta)$ is invariant under the symplectic group (see Paper I, § 2.2), so the proof is complete.

THEOREM 6. *Π^0 contains m sets of conjugate spaces of dimension $\frac{1}{2}(p^{m-r}-1)$, $0 < r \leq m$, through each of which pass $p^r \Pi^{\alpha}$. Each $[\frac{1}{2}(p^{m-r}-1)]$ is the complete intersection of Π^0 and any r of the $p^r \Pi^{\alpha}$ for which the corresponding r points $\frac{1}{2}\alpha$ of $\mathscr{V}_{2m}$ are linearly independent. The spaces are in $1-1$ correspondence with the r-dimensional subspaces $\mathscr{V}_r$ of $\mathscr{V}_{2m}$ on which $f(\alpha, \beta)$ vanishes.*

More generally, the Π^{α} intersect in sets of p^r, in spaces of dimension $\frac{1}{2}(p^{m-r}-1)$. Similar results hold for the Σ^{α} which intersect in sets of p^r, in spaces of dimension $\frac{1}{2}(p^{m-r}-3)$ and lie in sets of p^r, in spaces of dimension $(p^m - \frac{1}{2}(p^{m-r}-3))$. In the $[p^{m-r}-1]$ spanned by a corresponding pair of spaces (e.g. S_r and s_r) the configuration for $PCT(p^{m-r})$ is repeated.

The number of $[\frac{1}{2}(p^{m-r}-1)]$ in Π^0 is equal to the number of r-dimensional subspaces $\mathscr{V}_r$ of $\mathscr{V}_{2m}$ on which $f(\alpha, \beta)$ vanishes. We enumerate these as follows:

Choose $v_1 \neq 0$ in $\mathscr{V}_{2m}$,
then choose v_i in $\mathscr{V}_{2m}$, $i = 2, \cdots, r$, such that

$$f(v_i, v_j) = 0, \quad j = 1, \cdots, i-1, \quad v_i \neq \sum_{j=1}^{i-1} \lambda_j v_j.$$

The possible number of choices for v_i, each i, is $(p^{2m-(i-1)} - p^{(i-1)})$. $v_1, \cdots, v_r$ determine $\mathscr{V}_r$, which is also determined by any basis $w_1, \cdots, w_r$. Choose w_i in $\mathscr{V}_r$, $i = 1, \cdots, r$, such that $w_i \neq \sum_{j=1}^{i-1} \mu_j w_j$; this may be done in $p^r - p^{i-1}$ ways. Then the number of $\mathscr{V}_r$ in $\mathscr{V}_{2m}$ on which $f(\alpha, \beta)$ vanishes is

$$\begin{aligned} F_{m,r} &= \frac{(p^{2m}-1)(p^{2m-1}-p)\cdots(p^{2m-r+1}-p^{r-1})}{(p^r-1)(p^{r-1}-1)\cdots(p-1)} \\ &= p^{\frac{1}{2}r(r-1)}\,\frac{(p^{2m}-1)(p^{2m-2}-1)\cdots(p^{2m-2r+2}-1)}{(p^r-1)(p^{r-1}-1)\cdots(p-1)}. \end{aligned}$$

The configuration is completely symmetrical in relation to the $p^{2m}\Pi^{\alpha}$, and there are $p^r\Pi^{\alpha}$ through each space, so the number of $[\frac{1}{2}(p^{m-r}-1)]$ in the configuration is $p^{2m}F_{m,r}/p^r = p^{2m-r}F_{m,r}$.

3.2. Invariant spaces of the involutions of CS' and $\frac{1}{2}CS'$.

We turn now to the remaining sets of involutions of PCT and consider the set of $p^{2r}E_{m,r}$ involutions corresponding to the symplectic matrix T_r (see § 2.2). If P_r is the corresponding matrix of CS' then, from Theorem 1, P_r has one non-zero element in each row and column given by

$$(P_r)_{\alpha_1\cdots\alpha_r\alpha_{r+1}\cdots\alpha_m}^{-\alpha_1\cdots-\alpha_r\alpha_{r+1}\cdots\alpha_m} = \left(\frac{(-1)^r}{p}\right) = (-1)^{\gamma},$$

and the trace of P_r is $(-1)^{\gamma}p^{m-r}$. Thus P_r leaves invariant a pair of dual spaces of dimension [4] $\frac{1}{2}(p^m + p^{m-r} - 2)$ and $\frac{1}{2}(p^m - p^{m-r} - 2)$ in $[p^m - 1]$.

The $E_{m,r}$ involutions of the set in CS' determine a set of $\frac{1}{2}E_{m,r}$ pairs of spaces in the plus-subspace of J. To investigate the configuration in Π^0 or Σ^0, take as base points in $[p^m-1]$ the $\frac{1}{2}(p^m+1)$ points $\frac{1}{2}(X_{\lambda} + X_{-\lambda})$ in Π^0 and the $\frac{1}{2}(p^m-1)$ points $\frac{1}{2}(X_{\lambda} - X_{-\lambda})$ in Σ^0. Then a matrix P of CS' reduces to the form $\begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}$, where R_1 and R_2 are matrices corresponding to P in the representations of CS' of degree $\frac{1}{2}(p^m+1)$ and $\frac{1}{2}(p^m-1)$ on Π^0 and Σ^0, respectively. Then

[4] The invariant solid Δ_{tuvw} discussed by Horadam ([6]) for $PCT(3^2)$, is not one of the eigenspaces in [8] of the involution $p(t)$ but is the section with Π^0 of the invariant [5] of $p(t)$. The main results of his paper are independent of this result but only those properties of Δ_{tuvw} relating to $\frac{1}{2}CS'$ on Π^0 hold.

3.2.1
$$t_1 = \text{trace } R_1 = \tfrac{1}{2}\sum_{\alpha}(P)^{\alpha}_{\alpha} + \tfrac{1}{2}\sum_{\alpha}(P)^{\alpha}_{-\alpha}$$
$$t_2 = \text{trace } R_2 = \tfrac{1}{2}\sum_{\alpha}(P)^{\alpha}_{\alpha} - \tfrac{1}{2}\sum_{\alpha}(P)^{\alpha}_{-\alpha},$$

since, from Theorem 1, $(P)^{\alpha}_{\beta} = (P)^{-\alpha}_{-\beta}$. The traces of the matrices corresponding to P_r in the representations on Π^0 and Σ^0 are therefore $(-1)^{\gamma}\tfrac{1}{2}(p^{m-r}+p^r)$ and $(-1)^{\gamma}\tfrac{1}{2}(p^{m-r}-p^r)$, respectively. Thus P_r leaves invariant pairs of spaces of dimension

$$\tfrac{1}{4}(p^m + p^{m-r} + p^r - 3) \quad \text{and} \quad \tfrac{1}{4}(p^m - p^{m-r} - p^r - 3) \quad \text{in} \quad \Pi^0,$$

and

$$\tfrac{1}{4}(p^m + p^{m-r} - p^r + 5) \quad \text{and} \quad \tfrac{1}{4}(p^m - p^{m-r} + p^r - 5) \quad \text{in} \quad \Sigma^0.$$

A new configuration is determined in the plus-subspace of J by the set of matrices of CS' whose square is J. If P_G is the matrix of CS' corresponding to $G = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$, then from Theorem 1

$$(P_G)^{\alpha}_{\beta} = \theta^{-m}\left(\frac{2}{p}\right)\omega^{-\beta\cdot\alpha}.$$

For m odd, $p \equiv 3 \pmod 4$, Σ^0 is the plus-subspace of J and from 3.2.1

$$t_2 = \tfrac{1}{2}\theta^{-m}\left(\frac{2}{p}\right)\sum_{\alpha}(\omega^{-\alpha\cdot\alpha} - \omega^{\alpha\cdot\alpha}) = -\left(\frac{2}{p}\right).$$

Then the subspaces of P in Σ^0 have dimension $\tfrac{1}{4}(p^m-3)$ and $\tfrac{1}{4}(p^m-7)$. If Π^0 is the plus-subspace of J the subspaces of P in Π^0 have dimension $\tfrac{1}{4}(p^m-1)$ and $\tfrac{1}{4}(p^m-5)$, since

$$t_1 = \tfrac{1}{2}\theta^{-m}\left(\frac{2}{p}\right)\sum_{\alpha}(\omega^{-\alpha\cdot\alpha} + \omega^{\alpha\cdot\alpha}) = \left(\frac{2}{p}\right).$$

Now suppose $m = 2k$, and put

$$\begin{aligned} T &= T_k, \quad \text{cf. § 2.2}, \\ &= [-I_k, I_k, -I_k, I_k], \end{aligned}$$

and

$$S = \begin{bmatrix} 0 & I_k & 0 & 0 \\ I_k & 0 & 0 & 0 \\ 0 & 0 & 0 & I_k \\ 0 & 0 & I_k & 0 \end{bmatrix}.$$

Then

$$S \in Sp(2m, p), \quad S^2 = I, \quad T^2 = I$$

and

$$TS = -ST = R.$$

Thus in the factor group $\frac{1}{2}Sp(2m, p)$, S and T commute and R is an "involution" product, and from § 2.2 every matrix of CS' whose square is J is the product of two involutions of CS'. If the matrices of CS' corresponding to S, T and R are P_S, P_T and P_R, respectively, their eigenspaces in Π^0 are related as follows:

the plus-subspace of P_R is spanned by the intersections of the plus- and minus-subspaces of P_S with the plus- and minus-subspaces of P_T, respectively;

the minus-subspace of P_R is spanned by the intersections of the plus- and minus-subspaces of P_S with the minus- and plus-subspaces of P_T, respectively.

It is readily verified that the involution T_r of $Sp(2m, p)$ does not anticommute with any involution of $Sp(2m, p)$ either when m is odd, or when m is even and $2r \neq m$.

Details of these invariant sets of spaces in $[p^m - 1]$ and in the plus-subspace of J are more easily considered for particular values of p and m. In the next section some detail is given of the cases $m = 1$ and $p = 3$, $m = 2$, for which some well-known configurations occur.

3.3. Configurations for particular values of m and p

For $PCT(p)$ the $p^2\ \Pi^{\alpha}$ intersect in $p(p+1)$ points arranged in $p+1$ simplexes which are permuted by CT. Each simplex has one vertex in every Π^{α} and one face through every Σ^{α}. The $p\ \Pi^{\alpha}$ having $\alpha = (a_1, 0)$, intersect in X_0 and are transformed by the generators X_Q and X_D (Theorem 2) into the $p+1$ points X_0, $\sum_{\lambda=0}^{p-1} \omega^{\mu\lambda^2} X_\lambda$, $\mu = 0, \cdots, p-1$, in Π^0. Then $W^{\sigma_1}X_D$ (Theorem 3) transforms the $p+1$ points into the $p(p+1)$ points X_λ, $\sum_{\lambda=0}^{p-1}\omega^{\mu\lambda^2+\sigma\lambda}X_\lambda$. It is easily verified that the points are permuted in $p+1$ simplexes $\mathscr{S}$, $\mathscr{S}_\mu$, $\mu = 0, \cdots, p-1$, where

$$\mathscr{S} \text{ is } \prod_{\lambda} x_\lambda = 0, \quad \mathscr{S}_\mu \text{ is } \prod_{\sigma} \Big(\sum_{\lambda=0}^{p-1} \omega^{\mu\lambda^2+\sigma\lambda} X_\lambda\Big) = 0,$$

in terms of their prime faces. The simplexes belong to an invariant ∞^p linear family of primals of degree p through the $p^2\Sigma^{\alpha}$, and of which the simplexes are the degenerate members. For $p = 3$, 5 the only primals of degree p containing the Σ^{α} have the form $A\ \mathscr{S} + \sum_\mu B_\mu \mathscr{S}_\mu = 0$, where A, $B_\mu \in C$. $PCT(p)$ contains only one set of involutions, the JW^{α}.

$PCT(3)$ is the Hessian group of self-transformations of the 9 inflexions of the plane canonical cubic curve, (Todd [8]). The Σ^{α}, in this case the plus-subspaces of the $W^{\alpha}J$, are the 9 inflexions and the Π^{α} are their corresponding harmonic polars. The intersections of the Π^{α} are the 12 points of the Jacobian or Hessian configuration arranged in 4 triangles. CS leaves invariant one inflexion and its harmonic polar.

Some aspects of the configuration for $PCT(3^2)$ are discussed by Horadam ([5], [6]). $PCT(3^2)$ permutes in [8] a set of 81 [4]'s, Π^α, and a set of 81 [3]'s, Σ^α; a set of 9×90 [5]'s and a set of 9×90[2]'s. $\frac{1}{2}CS'$ is isomorphic to the simple group of order 25,920 associated with the 27 lines of a cubic surface. In the [4] Π^0, the plus-subspace of J, $\frac{1}{2}CS'$ leaves invariant the Burkhardt primal (Todd [9]). The Π^α intersect Π^0 in the 40 polar lines of the 40 Jacobian planes of the primal. The 40 lines intersect in 40 points, the poles of the 40 Steiner solids associated with the primal. The 90 [5]'s and the 90 [2]'s in [8] determined by the 90 conjugate involutions of CS' intersect Π^0 in the 45 Jordan primes and 45 nodes of the Burkhardt primal, respectively. The 270 conjugate involutions of $\frac{1}{2}CS'$ each leave invariant an f-plane (which is the intersection of 2 Jordan primes), and an e-line (which is the join of 2 nodes of the primal).

References

[1] Bolt, Beverley, Room, T. G. and Wall, G. E., On the Clifford Collineation, Transform and Similarity Groups I, This *Journal* 2 (1961), 60-79.

[2] Bolt, Beverley, Room, T. G. and Wall, G. E., On the Clifford Collineation, Transform and Similarity Groups II, ibid., 80-96.

[3] Dickson, L. E., *Linear Groups with an Exposition of the Galois Field Theory* (Leipzig, 1901).

[4] Dieudonné, J., *La Géométrie des Groupes Classiques* (Berlin, 1955), Ergebnisse der Mathematik, (N.F.) 5.

[5] Horadam, A. F., *Quart. J. Math.* (Oxford 2nd Series) 8 (1957), 241—259.

[6] Horadam, A. F., *Canadian J. Math.* 11 (1959) 18—33.

[7] Room, T. G. and Smith, R. J., *Quart. J. Math.* (Oxford 2nd Series) 9 (1958), 177—182.

[8] Todd, J. A., *Analytical and Projective Geometry* (London, 1947) 172.

[9] Todd, J. A., *Proc. Camb. Phil. Soc.* 46 (1950), 73—90.

Columbia University,
New York,
U.S.A.

The University of Sydney,
Sydney,
Australia.

https://doi.org/10.1017/S1446788700026926 Published online by Cambridge University Press