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# BOX DIMENSION FOR GRAPHS OF FRACTAL FUNCTIONS

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We calculate the box-dimension for a class of nowhere differentiable curves defined by Markov attractors of certain iterated function systems of affine maps.

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## 1. Introduction

Box dimension is one of the widely used fractal dimensions. Bedford [1] calculated the box dimension of a class of self-affine curves. These curves appear as attractors of hyperbolic iterated function systems (HIFS) of affine maps. In this paper we calculate the box dimension of curves which can be considered as Markov attractors of HIFS of affine maps.

A hyperbolic iterated function system $(X; T_1, \ldots, T_n)$ is a compact metric space together with contractive maps $T_i : X \mapsto X$. There exists a non-empty compact subset $A$ of $X$ such that

$$A = \bigcup_{i=1}^{n} T_i(A).$$

$A$ is called the attractor of the HIFS. A Markov transition matrix $M$ is an $n \times n$ irreducible 0-1 matrix. Then there exist non-empty subsets $A_1, A_2, \ldots, A_n$ of $A$ such that

$$A_i = \bigcup_{M_{ij}=1} T_i(A_j)$$

The set $A_M = \bigcup_{i=1}^{n} A_i$ is called the Markov attractor of the HIFS associated with $M$. Ellis and Branton [3] and the second named author [6] estimated the Hausdorff dimension for Markov attractors. Gibert and Massopust [5] gave the Hausdorff dimension of a certain class of fractal curves which appear as attractors of HIFS of affine maps.

In this paper, $X$ will be the unit square $[0, 1] \times [0, 1]$ and $T_i$ will have the form

$$T_i\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}a_i & 0\\ b_i & c_i\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}x_i\\y_i\end{pmatrix}$$

where $0<|a_i|<|c_i|<1$. Our main result is that under certain restrictions we have

$$\dim_B(A_i)=s$$

where $s$ is determined by

$$\left\|M\begin{pmatrix}|c_1\|a_1|^{s-1} & & 0\\ & \ddots & \\ 0 & & |c_n\|a_n|^{s-1}\end{pmatrix}\right\|=1,$$

and where $\|\cdot\|$ denotes the Perron–Frobenius eigenvalue of the matrix.

## 2. The construction of curves

The metric space we employ is a rectangular subset $I_1\times I_2$ of $\mathbf{R}^2$. Without loss of generality, we let $I_1=I_2=[0,1]$. Use $J$ to denote $[0,1]\times[0,1]$. For $i=1,2,\ldots,k$, define $T_{ij}:J\mapsto J$ by

$$T_{ij}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}a_i & 0\\ b_{ij} & c_{ij}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}x_i\\y_{ij}\end{pmatrix}, j=1,2,\ldots,l_i.$$

For $n=l_1+l_2+\cdots+l_k$, define an $n\times n$ matrix $M$ in the following way: We use $M_{(ij)(uv)}$ to denote the $(l_1+\cdots+l_{i-1}+j, l_1+\cdots+l_{u-1}+v)$ element of $M$. First we let $M_{(ij)(iv)}=\delta_{jv}$. Furthermore, for each $(ij)$ and each $u$ we define $M_{(ij)(uv)}=1$ for exactly one $v\in\{1,2,\ldots,l_u\}$, and $M_{(ij)(uv)}=0$ for all other cases. Assume that $M$ is irreducible. Suppose $T_{ij}$, $i=1,2,\ldots,k, j=1,2,\ldots,l_i$ satisfying the following conditions:

1. $a_i>0$ and $a_1+a_2+\cdots+a_k=1$, $x_1=0$ and $x_{i+1}=a_1+\cdots+a_i$, $i=1,2,\ldots,k-1$;

2. let $(0,y_j)$ be the fixed point of $T_{1j}, j=1,2,\ldots,l_1$ and $(1,y_j')$ be the fixed point of $T_{kj}, j=1,2,\ldots,l_k$. We assume that there exists a $y_0\in[0,1]$ such that $P_2T_{uv}(0,y_j)^T=y_0$ if $M_{(uv)(1j)}=1, u\neq 1$ and $P_2T_{uv}(1,y_j')^T=y_0$ if $M_{(uv)(kj)}=1, u\neq k$, where $P_2$ is the projective map to the second coordinate.

For each $k$-tuple $(j(1),j(2),\ldots,j(k))$, where $1\le j(i)\le l_i$, let $\Gamma=\bigcup_{i=1}^k A_{ij(i)}$. Then we have

**Theorem 1.** *$\Gamma$ is the graph of a continuous function $\varphi:[0,1]\mapsto\mathbf{R}$.*

**Proof.** For each sequence $i_1,i_2,\ldots$, by the definition of $M$, there exists exactly one sequence $(i_1j(i_1)),(i_2j_2),\ldots$ such that $M_{(i_tj_t)(i_{t+1}j_{t+1})}=1$. If the elements of the sequence $i_1,i_2,\ldots$ are not all 1 or $k$ except finite many, then there exists exactly one point $(x,y)$

such that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \lim_{m\to\infty} T_{i_1 j(i_1)} T_{i_2 j_2} \dots T_{i_m j_m} \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Define $\varphi(x) = y$.

For the sequence $i_1 i_2 \dots i_m 111\dots$ and $i_1 i_2 \dots i_m - 1kkk\dots$ let

$$\begin{pmatrix} x \\ y \end{pmatrix} = T_{i_1 j(i_1)} \dots T_{i_m j_m} \begin{pmatrix} 0 \\ y_j \end{pmatrix}, \quad \text{where } M_{(i_m j_m)(1_j)} = 1$$

and

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = T_{i_1 j(i_1)} \dots T_{i_m - 1 j'_m} \begin{pmatrix} 1 \\ y'_{j'} \end{pmatrix}, \quad \text{where } M_{(i_m - 1 j'_m)(k j')} = 1.$$

By conditions 1 and 2, we can easily see that $T_{i_m j_m}(0, y_j)^T = T_{i_m - 1 j'_m}(1, y'_{j'})^T$. Hence $(x, y) = (x', y')$. Again, we define $\varphi(x) = y$.

On the other hand, it is easy to see that for each $x \in [0, 1]$ there exists one (or two for countably many) sequence $i_1, i_2, \dots$ such that

$$x = P_1 \lim_{m\to\infty} T_{i_1 j(i_1)} T_{i_2 j_2} \dots T_{i_m j_m} \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $P_1$ is the projection to the first coordinate. Hence we have defined a function $\varphi : [0, 1] \mapsto \mathbf{R}$.

Next we show that $\varphi$ is continuous by showing that $\Gamma$ is a continuous image of [0,1].

For $x = \sum_{m=1}^{\infty} (i_m - 1)/k^m$, $i_m \in \{1, 2, \dots, k\}$ define $\psi : [0, 1] \mapsto \Gamma$ by letting

$$\psi(x) = \lim_{m\to\infty} T_{i_1 j(i_1)} T_{i_2 j_2} \dots T_{i_m j_m} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where $j_l$ determined by $M_{(i_{l-1} j_{l-1})(i_l j_l)} = 1$. We show that $\psi$ is continuous. Let $\alpha = \max_{i,j}\{\mathrm{Lip}(T_{ij})\}$. Given $\varepsilon > 0$ choose $N$ large enough such that $\alpha^N < \varepsilon/(2\sqrt{2})$. Let $\delta = k^{-N+1}$. For $x = \sum_{m=1}^{\infty}(i_m - 1)/k^m$ and $x' = \sum_{m=1}^{\infty}(u_m - 1)/k^m$, if $|x - x'| < \delta$ we must have $i_1 = u_1, i_2 = u_2, \dots, i_N = u_N$ or $i_1 = u_1, i_2 = u_2, \dots, i_{l-1} = u_{l-1}, i_l = u_l + 1$ and $i_{l+1} = \dots = i_N = 1$, $u_{l+1} = \dots = u_N = k$. In the first case, it is easy to see $|\psi(x) - \psi(x') < \varepsilon$. In the second case, we have

$$\psi(x) = \lim_{m\to\infty} T_{i_1 j(i_1)} T_{i_2 j_2} \dots T_{i_l j_l} (T_{1_j})^{N-l} T_{i_{N+1} j_{N+1}} \dots T_{i_m j_m} \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and

$$\psi(x') = \lim_{m\to\infty} T_{i_1 j(i_1)} T_{i_2 j_2} \dots T_{i_l-1,v_l} (T_{kv})^{N-l} T_{u_{N+1} v_{N+1}} \dots T_{u_m v_m} \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

As in the above, let $(0, y_j)$ be the fixed point of $T_{1j}$ and $(1, y'_v)$ the fixed point of $T_{kv}$. By the second assumption we know that

$$T_{i_l j_l} \begin{pmatrix} 0 \\ y_j \end{pmatrix} = T_{i_l-1 v_l} \begin{pmatrix} 1 \\ y'_v \end{pmatrix}.$$

Let $E = T_{i_l j_l}(T_{1j})^{N-l} J \bigcup T_{i_l-1 v_l} T_{i_l-1,v_l}(T_{kv})^{N-l} J$. Then $\operatorname{diam}(E) \le 2\alpha^{N-l+1}\sqrt{2}$, since the two parts of the union have a common point. Therefore

$$|\psi(x) - \psi(x')| \le \operatorname{diam}(T_{i_1 j(i_1)} T_{i_2 j_2} \dots T_{i_{l-1} j_{l-1}} E) \le \alpha^{l-1} \operatorname{diam}(E) < \varepsilon. \qquad \square$$

**Example 1.** Let $k > 2$. Define $T_{ij} : J \mapsto J$ $(i = 1, 2, \dots, k; j = 1, 2)$ as follows:

$$T_{i1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{k} & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{i-1}{k} \\ 1-\alpha \end{pmatrix},$$

$$T_{i2} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{k} & 0 \\ 0 & 1-\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{i-1}{k} \\ 0 \end{pmatrix},$$

where $\min\{\alpha, 1-\alpha\} > \frac{1}{k}$. Let $M$ be defined by

$$M_{(ij)(uv)} = \begin{cases} 1 & \text{if } (ij) = (uv) \text{ or } i \ne u, j \ne v \\ 0 & \text{otherwise.} \end{cases}$$

When $k = 3$

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to check that (1) and (2) are satisfied. We let $j(i) = 1, i = 1, 2, 3$. The continuous function $f_{k,\alpha}$, can be defined in the following way: for $x = \sum_{m=1}^{\infty} x_m/k^m$, $x_m \in \{0, 1, \dots, k-1\}$, let

$$f_{k,\alpha}(x)=\sum_{m=1}^{\infty}\alpha^{l_m}(1-\alpha)^{m-1_m}u_m$$

where $u_1=1$ and

$$u_{m+1}=\begin{cases}u_m & \text{if } x_{m+1}=x_m\\ 1-u_m & \text{otherwise}\end{cases}$$

and $l_m=u_1+u_2+\cdots+u_m-1$. Figures 1 to 4 show the first four steps of iteration, where $\alpha=1/2$. When $\alpha=1/2$ we write $f_{k,\alpha}$ as $B_k$ and call it a Bush function. Functions of this kind were first considered by K. A. Bush [2] as an example of continuous nowhere differentiable functions.

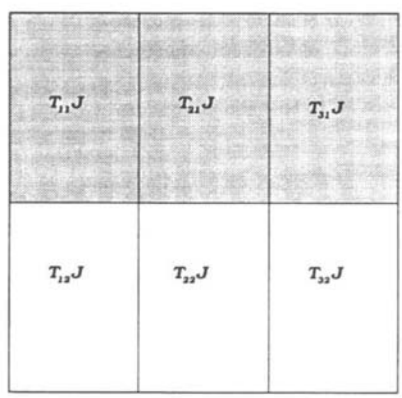


FIGURE 1 Step 1($k=3,\alpha=1/2$).

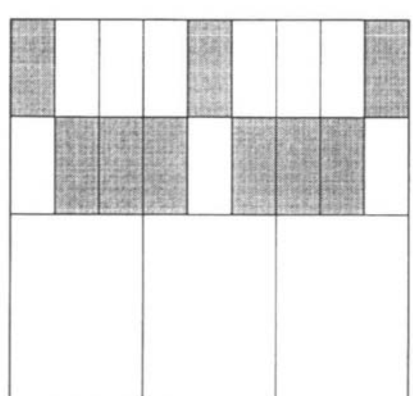

FIGURE 2 Step 2 ($k=3,\alpha=1/2$).

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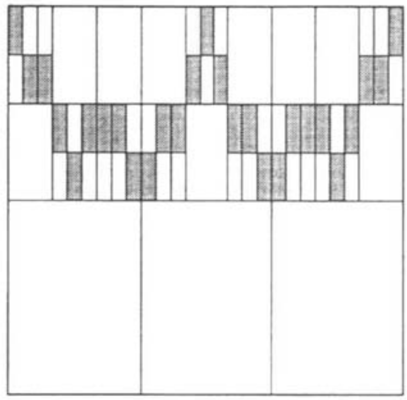

FIGURE 3 Step 3 ($k = 3, \alpha = 1/2$)

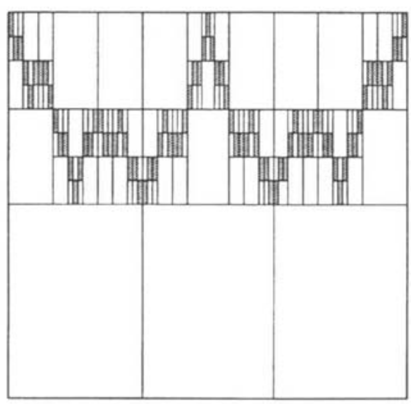

FIGURE 4 Step 4 ($k = 3, \alpha = 1/2$)

The function $\varphi$ in Theorem 1 is usually nowhere differentiable. In some cases it may be differentiable almost everywhere. We call this the degenerate case. In fact if we let $b_i = 0$ for all $i$ and $y_{ij} = 0$ for all $(i, j)$ then we have $\varphi \equiv 0$. In the next section, we calculate the box dimension of the graph of $\varphi$ in non-degenerate cases.

## 3. Main results

In this section we calculate the box dimension of $\Gamma$ under certain conditions. We first establish a more general result.

Let $(J; T_1, T_2, \ldots, T_n)$ be a HIFS where $T_i : J \mapsto J$ is defined by

$$T_i\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_i & 0 \\ b_i & c_i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

with $0 < |a_i| \le |c_i| < 1$. Let $M$ be an $n \times n$ Markov transition matrix. Let $A_M$ be the Markov attractor of the HIFS associated with $M$. Let $s$ be the number such that

$$\left\| M \begin{pmatrix} |c_1||a_1|^{s-1} & & 0 \\ & \ddots & \\ 0 & & |c_n||a_n|^{s-1} \end{pmatrix} \right\| = 1. \tag{1}$$

Then we have

**Proposition 2.** $\dim_B(A_M) \le s$.

**Proof.** Let $\sum_n = \{1, 2, \ldots, n\}^N$. Let $\sum_M$ be a subset of $\sum_n$ which consists of all the elements $(i_1 i_2 \ldots)$ such that $M_{i_j i_{j+1}} = 1$. Denote

$$M(s) = M \begin{pmatrix} |c_1||a_1|^{s-1} & & 0 \\ & \ddots & \\ 0 & & |c_n||a_n|^{s-1} \end{pmatrix}.$$

By the Perron–Frobenius Theorem, there exists a vector $\mathbf{p} = (p_1, p_2, \ldots, p_n)^T$ with $p_i > 0$ such that

$$M(s)\mathbf{p} = \mathbf{p}.$$

We assume that $\sum_{i=1}^n p_i = 1$. Define a probability measure on $\sum_n$ by letting

$$\begin{aligned} &\mu([i]) = p_i, \\ &\mu([ij]) = M(s)_{ij} p_j \\ &\ldots \ldots \\ &\mu([i_1 i_2 \ldots i_k]) = M(s)_{i_1 i_2} M(s)_{i_2 i_3} \ldots M(s)_{i_{k-1} i_k} p_{i_k}, \end{aligned}$$

where $[i_1 i_2 \ldots i_k]$ is the cylinder set which contains all elements which begin with $i_1 i_2 \ldots i_k$. Clearly, the support of $\mu$ is $\sum_M$. Let $a = \min\{|a_1|, |a_2|, \ldots, |a_n|\}$. Given $\delta > 0$, suppose $a^m \ge \delta > a^{m+1}$. For each $x \in A_M$, there exist $i_1, i_2, \ldots, i_l$ with $\delta > |a_{i_1} a_{i_2} \ldots a_{i_l}| > a^{m+2}$ and $M_{i_j i_{j+1}} = 1$ such that $x \in T_{i_1} \ldots T_{i_l} J = J_{i_1 \ldots i_l}$. Let

$$\mathcal{C} = \{[i_1 \ldots i_l];\ l \text{ is the first number such that } \delta > |a_{i_1} a_{i_2} \ldots a_{i_l}| > a^{m+2}, M_{i_j i_{j+1}} = 1\}.$$

It is easy to see that if $[i_1 \dots i_l], [j_1 \dots j_t] \in \mathcal{C}$ and $[i_1 \dots i_l] \neq [j_1 \dots j_t]$, then $[i_1 \dots i_l] \cap [j_1 \dots j_t] = \emptyset$. Therefore $\mathcal{C}$ is a disjoint cover of $\sum_M$.

Now we calculate how many $\delta$-squares (square of side length $\delta$) are needed to cover $A_M$. The height and width of $J_{i_1 \dots i_l}$ are $|c_{i_1} \dots c_{i_l}|$ and $|a_{i_1} \dots a_{i_l}| < \delta$ respectively. Hence, at most $[|c_{i_1} \dots c_{i_l}/a_{i_1} \dots a_{i_l}|] + 1$ $\delta$-squares are needed to cover $J_{i_1 \dots i_l} \cap A_M$.

$$
\begin{aligned}
\sum_{[i_1 \dots i_l] \in C} \left( \left[ \left| \frac{c_{i_1} \dots c_{i_l}}{a_{i_1} \dots a_{i_l}} \right| \right] + 1 \right) &\leq 2 \sum_{[i_1 \dots i_l] \in C} \left| \frac{c_{i_1} \dots c_{i_l}}{a_{i_1} \dots a_{i_l}} \right| \\
&\leq 2 \sum_{[i_1 \dots i_l] \in C} \left| \frac{c_{i_1} \dots c_{i_l}}{a_{i_1} \dots a_{i_l}} \right| \left( \frac{|a_{i_1} \dots a_{i_l}|}{a^2} \right)^s \delta^{-s} \\
&= \frac{2\delta^{-s}}{a^{2s}} \sum_{[i_1 \dots i_l] \in C} |c_{i_1} \dots c_{i_l}| |a_{i_1} \dots a_{i_l}|^{s-1} \\
&\leq \frac{2\delta^{-s}}{a^{2s} \cdot \min_j \{p_j\}} \sum_{[i_1 \dots i_l] \in C} |c_{i_1} \dots c_{i_l}| |a_{i_1} \dots a_{i_l}|^{s-1} p_{i_l} \\
&= \frac{2\delta^{-s}}{a^{2s} \cdot \min_j \{p_j\}} \sum_{[i_1 \dots i_l] \in C} M(s)_{i_1 i_2} \dots M(s)_{i_{l-1} i_l} p_{i_l} \\
&= \frac{2\delta^{-3}}{a^{2s} \cdot \min_j \{p_j\}} \sum_{[i_1 \dots i_l] \in C} \mu([i_1 \dots i_l]) \\
&= \frac{2\delta^{-s}}{a^{2s} \min_j \{p_j\}}.
\end{aligned}
$$

Therefore, for any $\delta > 0$, at most $2\delta^{-s}/a^{2s} \cdot \min_j\{p_j\}$ squares are needed to cover $A_M$. Hence $\dim_B(A_M) \leq s$.

When not all $b_i = 0$, we need the following lemma. In the following we use $|J_{i_1 \dots i_m}|_H$ and $|J_{i_1 \dots i_m}|_W$ to denote the height and width of $J_{i_1 \dots i_m}$ respectively.

**Lemma 1.** *There exists $\alpha > 0$ such that*

$$|J_{i_1 \dots i_m}|_H \leq B|c_{i_1} \dots c_{i_m}|.$$

**Proof.** When $m = 1$, we have $|J_{i_1}|_H \leq |c_{i_1}| + |b_{i_1}|$. Let $c = \max\{|b_i|/|c_i|\}$. Then $|J_{i_1}|_H \leq (1 + c)|c_{i_1}|$. Assume that

$$|J_{i_2 \dots i_{m+1}}|_H \leq \alpha_m |c_{i_2} \dots c_{i-m+1}|.$$

Then

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$$\begin{aligned}|J_{i_1 i_2\ldots i_{m+1}}|_H &= |T_{i_1} J_{i_2\ldots i_{m+1}}|_H \\ &\le |c_{i_1}||J_{i_2\ldots i_{m+1}}|_H + |b_{i_1}||J_{i_2\ldots i_{m+1}}|_W \\ &= |c_{i_1}||J_{i_2\ldots i_{m+1}}|_H + |b_{i_1}||a_{i_2}\ldots a_{i_{m+1}}| \\ &\le \alpha_m |c_{i_1}\ldots c_{i_{m+1}}| + c\left|\frac{a_{i_2}\ldots a_{i_{m+1}}}{c_{i_2}\ldots c_{i_{m+1}}}\right| \\ &\le (\alpha_m + cd^m)|c_{i_2}\ldots c_{i_{m+1}}|,\end{aligned}$$

where $d = \max\{|a_i|/|c_i|\}$. Hence we can choose $\alpha_{m+1} = \alpha_m + cd^m$. Notice that $\alpha_1 = 1 + c$. Therefore,

$$\alpha_m = 1 + \sum_{i=0}^{m-1} cd^k < 1 + \frac{c}{1-d}.$$

Hence the number $1 + c/(1-d)$ can be chosen as $\alpha$. □

Now we assume that the HIFS satisfies the following conditions:

3. for any $i_1 i_2 \ldots i_m$ with $M_{i_j i_{j+1}} = 1$ let

$$y_{i_1 i_2\ldots i_m} = \inf\{y; (x, y) \in J_{i_1 i_2\ldots i_m} \cap A_M \text{ for some } x\}$$

and

$$y^*_{i_1 i_2\ldots i_m} = \sup\{y; (x, y) \in J_{i_1 i_2\ldots i_m} \cap A_M \text{ for some } x\}.$$

We assume that there exist $\beta > 0$ such that for any $\varepsilon > 0$

$$y^*_{i_1 i_2\ldots i_m} - y_{i_1 i_2\ldots i_m} \ge \beta |c_{i_1} c_{i_2} \ldots c_{i_m}|i^{1+\varepsilon}$$

4. for any $i_1 i_2 \ldots i_m$ with $M_{i_j i_{j+1}} = 1$

$$P_2(J_{i_1 i_2\ldots i_m} \cap A_M) = [y_{i_1 i_2\ldots i_m}, y^*_{i_1 i_2\ldots i_m}],$$

is an interval; and

5. open set condition. If $M_{ij} = 1$ and $i \neq j$, then $T_i J \cap T_j J = \emptyset$.

**Theorem 3.** *Suppose the HIFS satisfies the above conditions. Then*

$$\dim_B(A_M) = s.$$

**Proof.** We only need to show that $\dim_B(A_M) \geq s$. Given $0 < \delta < 1$ assume that $a^m \geq \delta > a^{m+1}$. For any $x \in A_M$ there exist $i_1, \ldots i_l$ with $M_{i_j i_{j+1}} = 1$ and $\delta \leq |a_{i_1} \ldots a_{i_l}| < a^{m-2}$ such that $x \in J_{i_1 \ldots i_l}$. Given $\varepsilon > 0$, because of the assumptions 3 and 4, there are at least $[\beta |c_{i_1} \ldots c_{i_l}|^{1+\varepsilon}/\delta]$ $\delta$-squares which intersect with $J_{i_1 \ldots i_l} \cap A_M$. Since $|J_{i_1 \ldots i_l}|_W = |a_{i_1} \ldots a_{i_l}| > \delta$ and in view of the open set condition, each $\delta$-square intersects at most 4 such sets. We again use $C$ to denote all the cylinders $[i_1, \ldots i_l]$ mentioned above. Again $C$ is a disjoint cover of $\sum_M$. The following calculation gives us the number at least that many $\delta$-squares are needed to cover $A_M$.

$$\begin{aligned}
\sum_{[i_1,\ldots i_l]\in C} \frac{1}{4}\left[\frac{\beta|c_{i_1}\ldots c_{i_l}|^{1+\varepsilon}}{\delta}\right] &\geq \frac{1}{8}\sum_{[i_1,\ldots i_l]\in C} \frac{\beta|c_{i_1}\ldots c_{i_l}|^{1+\varepsilon}}{\delta} \\
&\geq \frac{\beta}{8}\sum_{[i_1,\ldots i_l]\in C} \frac{|c_{i_1}\ldots c_{i_l}|^{1+\varepsilon}}{a^m} \geq \frac{\beta}{8}\sum_{[i_1,\ldots i_l]\in C} \frac{|c_{i_1}\ldots c_{i_l}|^{1+\varepsilon}}{a_{i_1}\ldots a_{i_l}}\cdot\frac{1}{a^2} \\
&= \frac{\beta}{8a^2}\sum_{[i_1,\ldots i_l]\in C} |c_{i_1}\ldots c_{i_l}|^{1+\varepsilon}|a_{i_1}|^{s-1}\ldots|a_{i_l}|^{s-1}\cdot|a_{i_1}\ldots a_{i_l}|^{-s} \\
&\geq \frac{\beta}{8a^2}\sum_{[i_1,\ldots i_l]\in C} \mu([i_1\ldots i_l])\cdot\left(\frac{\delta}{a^2}\right)^{-s}|c_{i_1}\ldots c_{i_l}|^{\varepsilon} \\
&\geq \frac{\beta}{8}\cdot a^{2(s-1)}\cdot\delta^{-s+\varepsilon}.
\end{aligned}$$

Therefore, $\dim_B(A_M) - \varepsilon$ for any $\varepsilon > 0$. □

In the proof of Theorem 3 we can use $A_i$ (see Section 1) to replace $A_M$ and get the same result.

**Corollary.** *Under the same assumptions as Theorem 3, we have*

$$\dim_B(B_i) = s, \quad i = 1, \ldots, n.$$

**Remark.** The conditions 3 and 4 appear somewhat clumsy. But if a HIFS does not satisfy 3 or 4, $\dim_B(A_M) = s$ may not be true. We give two examples.

**Example 2.** Let

$$T_i\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} i/3 \\ 0 \end{pmatrix}, \quad i = 1, 2, 3,$$

and all entries of $M$ be 1. Then $A_M$ is the unit interval on the $x$-axis. The condition 3 is not satisfied. By (1) we have

$$s = 2 - \frac{\log 2}{\log 3} > 1 = \dim_B(A_M).$$

**Example 3.** Let

$$T_{ij}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} i/4 \\ 2(j-1)/3 \end{pmatrix}, \quad i = 1, 2, 3, 4; \quad j = 1, 2.$$

Let $M$ be a $8 \times 8$ matrix whose entries are all 1. Then

$$A_M = [0, 1] \times C$$

where $C$ is the Cantor middle-third set. This time the condition 4 is not satisfied. By (1)

$$s = \frac{5}{2} - \frac{\log 3}{2\log 2} > 1 + \frac{\log 2}{\log 3} = \dim_B(A_M).$$

Now we come back to the curve $\Gamma$ defined in Section 2. Since $\Gamma$ is a curve, condition 4 is satisfied. By the definition of $M$ and the condition 1 we can see that the open set condition holds. We will see that in a non-degenerate case, i.e. when $\varphi$ is nowhere differentiable, condition 3 is satisfied.

**Theorem 4.** *Suppose the HIFS defined in Section 2. Assume that the function $\varphi$ is nowhere differentiable. Then*

$$\dim_B(\Gamma) = s.$$

**Proof.** We need only check that condition 3 is satisfied. First we have

**Lemma 2.** *Let $C$ be a curve in $J$. Then there exists a constant $K$ such that for any $(i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n)$ we have*

$$|T_{i_1 j_1} T_{i_2 j_2} \ldots T_{i_n j_n} C|_H \geq |c_{i_1 j_1} c_{i_2 j_2} \ldots c_{i_n j_n}|(|C|_H - K|C|_W).$$

**Proof.** When $n = 1$ by the definition of $T_{ij}$ we get that

$$|T_{ij}C|_H \geq |c_{ij}||C|_H - |b_{ij}||C|_W.$$

Then for $n = 2$ we have

$$|T_{i_1j_1}T_{i_2j_2}C|_H \geq |c_{i_1j_1}|(|c_{i_2j_2}||C|_H - |b_{i_2j_2}||C|_W) - |b_{i_1j_1}||T_{i_2j_2}C|_W$$
$$= |c_{i_1j_1}c_{i_2j_2}|\left\{|C|_H - \left(\frac{|b_{i_2j_2}|}{|c_{i_2j_2}|} + \frac{|b_{i_1j_1}|a_{i_2}}{|c_{i_1j_1}c_{i_2j_2}|}\right)|C|_W\right\}.$$

In general we have

$$|T_{i_1j_1}T_{i_2j_2}\dots T_{i_nj_n}C|_H \geq |c_{i_1j_1}c_{i_2j_2}\dots c_{i_nj_n}|\left\{|C|_H - \left(\frac{|b_{i_nj_n}|}{|c_{i_nj_n}|} + \right.\right.$$
$$\left.\left. + \frac{|b_{i_{n-1}j_{n-1}}|a_{i_n}}{|c_{i_{n-1}j_{n-1}}c_{i_nj_n}|} + \dots + \frac{|b_{i_1j_1}|a_{i_2}\dots a_{i_n}}{|c_{i_1j_1}\dots c_{i_nj_n}|}\right)|C|_W\right\}$$

Hence we can choose $K = \max\{|b_{ij}|/|c_{ij}|\}\sum_{n=0}^{\infty}(\max\{a_i/|c_{ij}|\})^n$. □

Next we check that in non-degenerate cases, i.e. when $A_M$ consists of nowhere differentiable curves, the condition 3 is satisfied.

Since $A_M$ consists of nowhere differentiable curves, for each pair of $(ij)$ we can choose a piece of curve $C_{ij}$ from $A_{ij}$ such that $|C_{ij}|_H/|C_{ij}|_W \geq 2K$. For the sequence $(i_1j_1)(i_2j_2)\dots(i_nj_n)$ with $M_{(i_tj_t)(i_{t+1}j_{t+1})} = 1$, we have

$$|T_{i_1j_1}T_{i_2j_2}\dots T_{i_nj_n}J\cap A_M|_H \geq |T_{i_1j_1}T_{i_2j_2}\dots T_{i_nj_n}C_{i_{n+1}j_{n+1}}|_H$$
$$\geq |c_{i_1j_1}c_{i_2j_2}\dots c_{i_nj_n}|(|C_{i_{n+1}j_{n+1}}|_H - K|C_{i_{n+1}j_{n+1}}|_W)$$
$$\geq \frac{1}{2}|c_{i_1j_1}c_{i_2j_2}\dots c_{i_nj_n}||C_{i_{n+1}j_{n+1}}|_H$$

we complete the proof by letting $\beta = 1/2\max\{|C_{ij}|_H\}$. □

**Example 1** (continued). Use $\Gamma_{k,\alpha}$ to denote the graph of the function $f_{k,\alpha}$ defined in Section 2. We calculate $\dim_B(\Gamma_{k,\alpha})$. First we check that $\Gamma_{k,\alpha}$ satisfies condition 4. Given $i_1, i_2, \dots, i_m$,

$$P_1(J_{(i_11)(i_2j_2)\dots(i_mj_m)}) = [a, b]$$

where $a = \sum_{j=1}^{m}(i_j - 1)/k^j$ and $b = a + 1/k^m$. By the definition of $f_{k,\alpha}$, for any $x \in (a, b)$ we have the same $u_j$, $(j = 1, 2, \dots, m)$ in the expression of $f_{k,\alpha}(x)$. Suppose $u_m = 0$. Let $x_1 = a + (i_m - 1)\sum_{j=m+1}^{\infty} 1/k^j$ and $x_2 = a + \sum_{j=m+1}^{\infty} l/k^j$, where $0 \leq l \leq k-1$ and $l \neq i_m - 1$. Then

$$\min_{x\in[a,b]} f_{k,\alpha}(x) = f_{k,\alpha}(x_1) = \sum_{j=1}^{m}\alpha^{l_j}(1-\alpha)^{j-l_j}u_j$$

and

$$\max_{x\in[a,b]} f_{k,\alpha}(x) = f_{k,\alpha}(x_1)$$
$$= \sum_{j=1}^{m} \alpha^{l_j}(1-\alpha)^{j-l_j}u_j + \alpha^{l_m}(1-\alpha)^{m-l_m}\sum_{j=1}^{\infty}\alpha^j.$$

Hence we can choose $\beta = (1-\alpha)^{-1}$ in condition 3. If $u_m = 1$ we get the same result.

Let $\Lambda = \mathrm{diag}(a, b, a, b, \ldots, a, b)$ be the $2k \times 2k$ diagonal matrix whose diagonal elements are $a$ and $b$ alternatively with $a, b > 0$. We have

$$\|M\Lambda\| = \frac{a+b+\sqrt{(a-b)^2+4(k-1)^2ab}}{2}.$$

Choose $a = \alpha(1/k)^{s-1}, b = (1-\alpha)(i/k)^{s-1}$ and let $\|M\Lambda\| = 1$. We get

$$\dim_B(\Gamma_{k,\alpha}) = s = 1 + \frac{\log(1+\sqrt{(2\alpha-1)^2+4(k-1)^2\alpha(1-\alpha)}-\log 2}{\log k}.$$

When $\alpha = 1/2$, using $\Gamma_k$ to denote the graph of $B_k$, we have

$$\dim_B(\Gamma_k) = 2 - \frac{\log 2}{\log k}.$$

## 4. Concluding remarks

It is interesting to compare Theorem 4 with the main result of [4]. There Falconer considered the mixing repeller for a class $C^2$ mapping $f: M \mapsto M$ where $M$ is an open subset of $\mathbf{R}^d$. By extending the Bowen–Ruelle formula to the non-conformal setting, he obtained an estimation for the Hausdorff dimension and box dimension of the repeller under some conditions. When $d = 2$ and the repeller contains a non-differentiable arc, this gives an exact formula for the box dimension in terms of the singular values of the derivatives of the iterates of $f$.

By defining a Markov attractor using a set of linear transformations, we are effectively working directly with derivatives. If we make the formal comparison with [4] by considering $f$ defined on $\mathbf{R}^2$ by

$$f^{-1} = T_{ij}, \quad i = 1, 2, \ldots, k; \quad j = 1, 2, \ldots, l_i,$$

then Falconer's formula, based on consideration of pressure, gives the same value for the box dimension. However a fundamental condition in [4] is that

$$\|(D_x f)^{-1}\|^2 \|D_x f\| < 1. \tag{1}$$

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We have no such restriction and, in our setting, (1) translates to

$$\frac{a_i^2+b_{ij}^2+c_{ij}^2+r(a_i,b_{ij},c_{ij})}{2}\cdot\left(\frac{2}{a_i^2+b_{ij}^2+c_{ij}^2-r(a_i,b_{ij},c_{ij})}\right)^{1/2}<1, \qquad i=1,2,\ldots,k;\quad j=1,2,\ldots,l_i, \tag{2}$$

where $r(a,b,c)=\sqrt{(a^2+b^2+c^2)^2-4a^2c^2}$.

Taking the special case $b_{ij}=0$ for comparison purposes, we then have that (2) is equivalent to $c_{ij}^2<a_i$. This can never be satisfied in our Example 1 for $k\geq 4$.

As our work comes from generalising concrete examples piecing together linear maps and Falconer considers global functions in the light of thermodynamical systems, it appears likely that the connections might repay further study. We thank the referee for bringing [4] to our attention.

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