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# INFINITE FAMILIES OF ARITHMETIC IDENTITIES FOR 4-CORES

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**Abstract**

Let $u(n)$ and $v(n)$ be the number of representations of a nonnegative integer $n$ in the forms $x^2 + 4y^2 + 4z^2$ and $x^2 + 2y^2 + 2z^2$, respectively, with $x, y, z \in \mathbb{Z}$, and let $a_4(n)$ and $r_3(n)$ be the number of 4-cores of $n$ and the number of representations of $n$ as a sum of three squares, respectively. By employing simple theta-function identities of Ramanujan, we prove that $u(8n + 5) = 8a_4(n) = v(8n + 5) = \frac{1}{3}r_3(8n + 5)$. With the help of this and a classical result of Gauss, we find a simple proof of a result on $a_4(n)$ proved earlier by K. Ono and L. Sze ['4-core partitions and class numbers', *Acta Arith.* **80** (1997), 249–272]. We also find some new infinite families of arithmetic relations involving $a_4(n)$.

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## 1. Introduction

A partition $\lambda$ is said to be a $t$-core if and only if it has no hook numbers that are multiples of $t$; or if and only if $\lambda$ has no rim hooks that are multiples of $t$. If $a_t(n)$ denotes the number of partitions of $n$ that are $t$-cores, then the generating function for $a_t(n)$ is given by [3, Equation (2.1)]

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{(q^t; q^t)_\infty^t}{(q; q)_\infty};$$

here and throughout the paper, we assume that $|q| < 1$ and use the standard notation

$$(a; q)_\infty := \prod_{n=0}^{\infty}(1 - aq^n).$$

In particular, the generating function for $a_4(n)$ is given by

$$\sum_{n=0}^{\infty} a_4(n)q^n = \frac{(q^4; q^4)_\infty^4}{(q; q)_\infty}. \tag{1.1}$$

In [4, 5], Hirschhorn and Sellers used some elementary generating function manipulations to find congruences and the following infinite families of arithmetic relations involving 4-cores: for $k \geq 1$,

$$3^k a_4(3n) = a_4\Big(3^{2k+1}n + \frac{5 \times 3^{2k} - 5}{8}\Big), \tag{1.2}$$

$$(2 \times 3^k - 1)a_4(3n+1) = a_4\Big(3^{2k+1}n + \frac{13 \times 3^{2k} - 5}{8}\Big), \tag{1.3}$$

$$\Big(\frac{3^{k+1}-1}{2}\Big)a_4(9n+2) = a_4\Big(3^{2k+2}n + \frac{7 \times 3^{2k+1} - 5}{8}\Big), \tag{1.4}$$

$$\Big(\frac{3^{k+1}-1}{2}\Big)a_4(9n+8) = a_4\Big(3^{2k+2}n + \frac{23 \times 3^{2k+1} - 5}{8}\Big). \tag{1.5}$$

Again, if $h(-D)$ denotes the class number of primitive binary quadratic forms with discriminant $-D$ and $a_4(n)$ denotes the number of 4-cores of $n$, then, for a square-free integer $8n+5$, Ono and Sze [7] proved that

$$a_4(n) = \tfrac{1}{2}h(-32n-20). \tag{1.6}$$

Employing (1.6) and the index formulas for class numbers, Ono and Sze [7] proved (1.2)–(1.5) and some general identities conjectured by Hirschhorn and Sellers [5].

The main purpose of this paper is to use Ramanujan's simple theta-function identities to prove that $u(8n+5) = 8a_4(n) = v(8n+5) = \frac{1}{3}r_3(8n+5)$, where $u(n)$ and $v(n)$ denote the number of representations of a nonnegative integer $n$ in the form $x^2 + 4y^2 + 4z^2$ and $x^2 + 2y^2 + 2z^2$, respectively, with $x, y, z \in \mathbb{Z}$, and $r_3(n)$ denotes the number of representations of $n$ as a sum of three squares. With the aid of this, we find new proofs of (1.2)–(1.5) as well as some analogous new infinite families of identities for $a_4(n)$. We note that Hirschhorn and Sellers [6] also proved the identity $8a_4(n) = \frac{1}{3}r_3(8n+5)$ from which (1.2)–(1.5) can easily be deduced with the help of the other results in [6].

## 2. Preliminary results

For $|ab| < 1$, Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.$$

Jacobi's famous triple product identity [1, p. 35, Entry 19] takes the form

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.1}$$

The three most important special cases of $f(a, b)$ are

$$\varphi(q) := f(q, q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = (-q; q^2)_\infty^2 (q^2; q^2)_\infty, \tag{2.2}$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}, \tag{2.3}$$

and

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_\infty, \tag{2.4}$$

where the product representation in (2.2)–(2.4) arises from (2.1).

We note that, by (2.3) and manipulation of the $q$-products, (1.1) reduces to

$$\sum_{n=0}^{\infty} a_4(n) q^n = \psi(q)\psi^2(q^2). \tag{2.5}$$

In the following lemmas, we state some theta-function identities of Ramanujan which will be used in our subsequent sections. (The first three are trivial.)

Lemma 2.1 [1, Entry 25 (i) and (ii), p. 40]. *We have*

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8). \tag{2.6}$$

Lemma 2.2 [1, Entry 25 (v) and (vi), p. 40]. *We have*

$$\varphi^2(q) = \varphi^2(q^2) + 4q\psi^2(q^4). \tag{2.7}$$

Lemma 2.3 [1, p. 49, Corollary (i)]. *We have*

$$\varphi(q) = \varphi(q^{25}) + 2qf(q^{15}, q^{35}) + 2q^4 f(q^5, q^{45}). \tag{2.8}$$

Lemma 2.4 [1, p. 262, Entry 10(iv)]. *We have*

$$\varphi^2(q) - \varphi^2(q^5) = 4qf(q, q^9)f(q^3, q^7). \tag{2.9}$$

## 3. Main theorems

In this section, we present the relations among $u(n)$, $v(n)$, $r_3(n)$ with $a_4(n)$.

Theorem 3.1. *If $u(n)$ and $v(n)$ denote the number of representations of a nonnegative integer $n$ in the forms $x^2 + 4y^2 + 4z^2$ and $x^2 + 2y^2 + 2z^2$, respectively, where $x, y, z \in \mathbb{Z}$, and $a_4(n)$ is the number of 4-cores of $n$, then*

$$u(8n + 5) = 8a_4(n) = v(8n + 5). \tag{3.1}$$

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In the following process of proving (3.1), we also find some other results involving $u(n)$ and $v(n)$.

PROOF. First we prove the first equality in (3.1). Clearly, the generating function for $u(n)$ is given by

$$\sum_{n=0}^{\infty} u(n)q^n = \varphi(q)\varphi^2(q^4).$$

With the aid of (2.6), we rewrite the above as

$$\begin{aligned}\sum_{n=0}^{\infty} u(n)q^n &= \varphi^2(q^4)(\varphi(q^4) + 2q\psi(q^8)) \\ &= \varphi^3(q^4) + 2q\varphi^2(q^4)\psi(q^8).\end{aligned} \tag{3.2}$$

Extracting the terms involving $q^{4n}$, $q^{4n+1}$, $q^{4n+2}$ and $q^{4n+3}$ respectively in (3.2),

$$\sum_{n=0}^{\infty} u(4n)q^n = \varphi^3(q), \tag{3.3}$$

$$\sum_{n=0}^{\infty} u(4n+1)q^n = 2\varphi^2(q)\psi(q^2), \tag{3.4}$$

$$u(4n+2) = 0,$$
$$u(4n+3) = 0.$$

Now, with the help of (2.6), we can rewrite (3.3) in the form

$$\sum_{n=0}^{\infty} u(4n)q^n = \varphi^3(q^4) + 6q\varphi^2(q^4)\psi(q^8) + 12q^2\varphi(q^4)\psi^2(q^8) + 8q^3\psi^3(q^8). \tag{3.5}$$

Equating the coefficients of $q^{4n}$, $q^{4n+1}$, $q^{4n+2}$, $q^{4n+3}$ respectively from both sides of (3.5),

$$u(16n) = u(4n),$$

$$\sum_{n=0}^{\infty} u(16n+4) = 6\varphi^2(q)\psi(q^2), \tag{3.6}$$

$$\sum_{n=0}^{\infty} u(16n+8) = 12\varphi(q)\psi^2(q^2), \tag{3.7}$$

$$\sum_{n=0}^{\infty} u(16n+12) = 8\psi^3(q^2). \tag{3.8}$$

From (3.8), it further follows that

$$\sum_{n=0}^{\infty} u(32n+12) = 8\psi^3(q), \tag{3.9}$$

$$u(32n+28) = 0.$$

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Again, from (3.4) and (3.6) it follows that

$$3u(4n+1) = u(16n+4). \tag{3.10}$$

Now, employing (2.7) in (3.4),

$$\sum_{n=0}^{\infty} u(4n+1)q^n = 2\psi(q^2)\varphi^2(q^2) + 8q\psi(q^2)\psi^2(q^4). \tag{3.11}$$

Extracting the terms involving $q^{2n}$ and $q^{2n+1}$ from both sides of (3.11), we respectively find that

$$\sum_{n=0}^{\infty} u(8n+1)q^n = 2\psi(q)\varphi^2(q), \tag{3.12}$$

$$\sum_{n=0}^{\infty} u(8n+5)q^n = 8\psi(q)\psi^2(q^2). \tag{3.13}$$

Employing (2.5) in (3.13) and then equating the coefficients of $q^n$ from both sides, we readily deduce the first equality of (3.1).

We now prove the second equality of (3.1). To this end, we note that the generating function for $v(n)$ is given by

$$\sum_{n=0}^{\infty} v(n)q^n = \varphi(q)\varphi^2(q^2). \tag{3.14}$$

With the help of (2.6), we rewrite (3.14) as

$$\begin{aligned}\sum_{n=0}^{\infty} v(n)q^n &= \varphi^2(q^2)(\varphi(q^4) + 2q\psi(q^8)) \\ &= \varphi^2(q^2)\varphi(q^4) + 2q\varphi^2(q^2)\psi(q^8).\end{aligned}$$

Extracting the even and odd terms,

$$\sum_{n=0}^{\infty} v(2n)q^n = \varphi^2(q)\varphi(q^2), \tag{3.15}$$

$$\sum_{n=0}^{\infty} v(2n+1)q^n = 2\varphi^2(q)\psi(q^4). \tag{3.16}$$

Now, applying (2.7) in (3.15), and then extracting the even and odd terms,

$$\sum_{n=0}^{\infty} v(4n)q^n = \varphi^3(q), \tag{3.17}$$

$$\sum_{n=0}^{\infty} v(4n+2)q^n = 4\varphi(q)\psi^2(q^2). \tag{3.18}$$

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Next, employing (2.6) in (3.17) and then extracting the terms involving $q^{4n}$, $q^{4n+1}$, $q^{4n+2}$ and $q^{4n+3}$, respectively,

$$v(16n) = v(4n),$$

$$\sum_{n=0}^{\infty} v(16n+4)q^n = 6\varphi^2(q)\psi(q^2), \tag{3.19}$$

$$\sum_{n=0}^{\infty} v(16n+8)q^n = 12\varphi(q)\psi^2(q^2), \tag{3.20}$$

$$\sum_{n=0}^{\infty} v(16n+12)q^n = 8\psi^3(q^2). \tag{3.21}$$

It follows from (3.21) that

$$\sum_{n=0}^{\infty} v(32n+12)q^n = 8\psi^3(q), \tag{3.22}$$

$$v(32n+28) = 0.$$

Now, employing (2.7) in (3.16), and then extracting the even and odd terms,

$$\sum_{n=0}^{\infty} v(4n+1)q^n = 2\varphi^2(q)\psi(q^2), \tag{3.23}$$

$$\sum_{n=0}^{\infty} v(4n+3)q^n = 8\psi^3(q^2). \tag{3.24}$$

It follows from (3.24) that

$$\sum_{n=0}^{\infty} v(8n+3)q^n = 8\psi^3(q), \tag{3.25}$$

$$v(8n+7) = 0.$$

Also, from (3.19) and (3.23),

$$3v(4n+1) = v(16n+4). \tag{3.26}$$

On the other hand, employing (2.7) in (3.23) and then extracting the odd and even terms of the resulting identity, and with the aid of (2.5),

$$\sum_{n=0}^{\infty} v(8n+1)q^n = 2\psi(q)\varphi^2(q), \tag{3.27}$$

$$\sum_{n=0}^{\infty} v(8n+5)q^n = 8\psi(q)\psi^2(q^2) = 8\sum_{n=0}^{\infty} a_4(n)q^n. \tag{3.28}$$

From (3.28), we easily deduce the second equality of (3.1) to finish the proof. □

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COROLLARY 3.2. *If $r_3(n)$ denotes the number of representations of $n$ as a sum of three squares, then*

$$r_3(8n+5) = 3u(8n+5) = 3v(8n+5) = 24a_4(n). \tag{3.29}$$

PROOF. We note that

$$\sum_{n=0}^{\infty} r_3(n)q^n = \varphi^3(q). \tag{3.30}$$

From (3.3), (3.17) and (3.30), we deduce that

$$r_3(n) = u(4n) = v(4n). \tag{3.31}$$

Now, replacing $n$ by $2n+1$ in (3.10) and (3.26), and then employing (3.31),

$$3u(8n+5) = u(32n+20) = r_3(8n+5) \quad \text{and} \quad 3v(8n+5) = v(32n+20) = r_3(8n+5),$$

from which, with the help of (3.1), we easily deduce (3.29). □

Next we deduce the formula given above as (1.6) due to Ono and Sze [7, Theorem 2].

COROLLARY 3.3 (Ono and Sze [7, Theorem 2]). *Formula (1.6) holds.*

PROOF. A classical result due to Gauss states that if $n$ is square-free and $n > 4$, then

$$r_3(n) = \begin{cases} 24h(-n) & \text{for } n \equiv 3 \pmod 8; \\ 12h(-n) & \text{for } n \equiv 1, 2, 5, 6 \pmod 8; \\ 0 & \text{for } n \equiv 7 \pmod 8). \end{cases}$$

Now (1.6) readily follows from Corollary 3.2. □

We end this section by giving two more corollaries arising from the proof of the above theorem.

COROLLARY 3.4. *We have*

$$u(8n+1) = v(8n+1), \tag{3.32}$$

$$u(16n+8) = v(16n+8) = 3v(4n+2). \tag{3.33}$$

PROOF. Identity (3.32) follows from (3.12) and (3.27), and (3.33) follows from (3.7), (3.18) and (3.20). □

COROLLARY 3.5. *We have*

$$u(32n+12) = r_3(8n+3) = v(32n+12) = v(8n+3) = 8t_3(n), \tag{3.34}$$

*where $t_3(n)$ is the number of representations of $n$ as a sum of three triangular numbers.*

PROOF. We note that

$$\sum_{n=0}^{\infty} t_3(n)q^n = \psi^3(q). \tag{3.35}$$

Now (3.34) follows easily from (3.35), (3.9), (3.22), (3.25), (3.3) and (3.30). □

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## 4. Infinite families of arithmetic properties of $a_4(n)$

In this section, we prove some infinite families of arithmetic identities for $a_4(n)$ by using the results from the previous sections. First, we deduce the infinite families of arithmetic identities (1.2)–(1.5).

THEOREM 4.1 (Hirschhorn and Sellers [5]). *The identities (1.2)–(1.5) hold.*

PROOF. Cooper and Hirschhorn [2] found the following arithmetic properties of $r_3(n)$. For any nonnegative integer $n$ and any integer $k \geq 1$,

$$3^k r_3(6n+5) = r_3(9^k(6n+5)), \tag{4.1}$$
$$(2 \times 3^k - 1) r_3(24n+13) = r_3(9^k(24n+13)), \tag{4.2}$$
$$\Big(\frac{3^{k+1}-1}{2}\Big) r_3(72n+21) = r_3(9^k(72n+21)), \tag{4.3}$$
$$\Big(\frac{3^{k+1}-1}{2}\Big) r_3(72n+69) = r_3(9^k(72n+69)). \tag{4.4}$$

Replacing $n$ by $4n$ in (4.1),

$$3^k r_3(8(3n)+5) = r_3\Big(8\Big(3^{2k+1}n + \frac{5 \times 3^{2k} - 5}{8}\Big) + 5\Big),$$

from which we readily deduce (1.2) by employing (3.29).

In a similar fashion, (1.3)–(1.5) follow from (4.2)–(4.4), respectively. □

In the next theorem we give some more infinite families of arithmetic identities for $a_4(n)$.

THEOREM 4.2. *If $a_4(n)$ denotes the number of 4-cores of $n$, and $k \geq 1$ then*

$$5a_4(5n+2) = a_4(125n+65), \tag{4.5}$$
$$5a_4(5n+3) = a_4(125n+90), \tag{4.6}$$
$$\Big(\frac{5^{k+1}-1}{4}\Big) a_4(25n) = a_4\Big(5^{2k+2}n + \frac{5^{2k+1}-5}{8}\Big), \tag{4.7}$$
$$\Big(\frac{5^{k+1}-1}{4}\Big) a_4(25n+5) = a_4\Big(5^{2k+2}n + \frac{9 \times 5^{2k+1}-5}{8}\Big), \tag{4.8}$$
$$\Big(\frac{5^{k+1}-1}{4}\Big) a_4(25n+10) = a_4\Big(5^{2k+2}n + \frac{17 \times 5^{2k+1}-5}{8}\Big), \tag{4.9}$$
$$\Big(\frac{5^{k+1}-1}{4}\Big) a_4(25n+20) = a_4\Big(5^{2k+2}n + \frac{33 \times 5^{2k+1}-5}{8}\Big). \tag{4.10}$$

Before proving the theorem, we prove the following lemma concerning $r_3(n)$.

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Lemma 4.3. *If $r_3(n)$ denotes the number of representations of $n$ as a sum of three squares, then*

$$5r_3(5n+1) = r_3(25(5n+1)), \tag{4.11}$$

$$5r_3(5n+4) = r_3(25(5n+4)), \tag{4.12}$$

$$\Big(\frac{5^{k+1}-1}{4}\Big)r_3(25n+5) = r_3(25^k(25n+5)), \tag{4.13}$$

$$\Big(\frac{5^{k+1}-1}{4}\Big)r_3(25n+10) = r_3(25^k(25n+10)), \tag{4.14}$$

$$\Big(\frac{5^{k+1}-1}{4}\Big)r_3(25n+15) = r_3(25^k(25n+15)), \tag{4.15}$$

$$\Big(\frac{5^{k+1}-1}{4}\Big)r_3(25n+20) = r_3(25^k(25n+20)). \tag{4.16}$$

Proof. Employing the 5-dissection of $\varphi(q)$ from (2.8) in (3.30) and then extracting the terms involving $q^{5l+r}$ for $r = 0, 1, 2, 3, 4$, respectively,

$$\sum_{n=0}^{\infty} r_3(5n)q^n = \varphi^3(q^5) + 24q\varphi(q^5)f(q,q^9)f(q^3,q^7), \tag{4.17}$$

$$\sum_{n=0}^{\infty} r_3(5n+1)q^n = 6\varphi^2(q^5)f(q^3,q^7) + 24qf^2(q^3,q^7)f(q,q^9), \tag{4.18}$$

$$\sum_{n=0}^{\infty} r_3(5n+2)q^n = 12\varphi(q^5)f^2(q^3,q^7) + 8q^2f^3(q,q^9),$$

$$\sum_{n=0}^{\infty} r_3(5n+3)q^n = 8f^3(q^3,q^7) + 12q\varphi(q^5)f^2(q,q^9),$$

$$\sum_{n=0}^{\infty} r_3(5n+4)q^n = 6\varphi^2(q^5)f(q,q^9) + 24qf^2(q,q^9)f(q^3,q^7). \tag{4.19}$$

Now, employing (2.9) in (4.17),

$$\sum_{n=0}^{\infty} r_3(5n)q^n = 6\varphi^2(q)\varphi(q^5) - 5\varphi^3(q^5), \tag{4.20}$$

which we rewrite, with the aid of (3.30), as

$$\sum_{n=0}^{\infty} r_3(5n)q^n = 6\varphi^2(q)\varphi(q^5) - 5\sum_{n=0}^{\infty} r_3(n)q^{5n}. \tag{4.21}$$

Similarly, employing (2.9) in (4.18) and (4.19),

$$\sum_{n=0}^{\infty} r_3(5n+1)q^n = 6\varphi^2(q)f(q^3,q^7) \tag{4.22}$$

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and

$$\sum_{n=0}^{\infty} r_3(5n+4)q^n = 6\varphi^2(q)f(q,q^9), \tag{4.23}$$

respectively.

Again, using (2.9) in (4.20), and then extracting the terms involving $q^{5n}$, we deduce that

$$\sum_{n=0}^{\infty} r_3(25n)q^n = 6\varphi(q)\varphi^2(q^5) + 48q\varphi(q)f(q,q^9)f(q^3,q^7) - 5\varphi^3(q). \tag{4.24}$$

Employing (2.9) once again in (4.24),

$$\sum_{n=0}^{\infty} r_3(25n)q^n = 7\varphi^3(q) - 6\varphi(q)\varphi^2(q^5),$$

which we rewrite, with the help of (2.8), as

$$\sum_{n=0}^{\infty} r_3(25n)q^n = 7\varphi^3(q) - 6\varphi^2(q^5)(\varphi(q^{25}) + 2qf(q^{15},q^{35}) + 2q^4f(q^5,q^{45})). \tag{4.25}$$

Now, employing (3.30) in (4.25), and then extracting the terms involving $q^{5n}$,

$$\sum_{n=0}^{\infty} r_3(125n)q^n = 7\sum_{n=0}^{\infty} r_3(5n)q^n - 6\varphi^2(q)\varphi(q^5). \tag{4.26}$$

Employing (4.21) in (4.26), we arrive at

$$5\sum_{n=0}^{\infty} r_3(n)q^{5n} = 6\sum_{n=0}^{\infty} r_3(5n)q^n - \sum_{n=0}^{\infty} r_3(125n)q^n. \tag{4.27}$$

We are now in a position to prove (4.11)–(4.16). First we prove (4.11) and (4.12). Equating the terms involving $q^{5n+1}$ and $q^{5n+4}$, respectively, from both sides of (4.25), we obtain

$$\sum_{n=0}^{\infty} r_3(125n+25)q^n = 7\sum_{n=0}^{\infty} r_3(5n+1)q^n - 12\varphi^2(q)f(q^3,q^7) \tag{4.28}$$

and

$$\sum_{n=0}^{\infty} r_3(125n+100)q^n = 7\sum_{n=0}^{\infty} r_3(5n+4)q^n - 12\varphi^2(q)f(q,q^9), \tag{4.29}$$

respectively. Employing (4.22) and (4.23) in (4.28) and (4.29) respectively, and then equating the coefficients of $q^n$ from both sides of the resulting identities, we readily deduce (4.11) and (4.12).

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Next, we prove (4.13). Equating the coefficients of $q^{5n+1}$ from both sides of (4.27), we deduce that

$$6r_3(25n+5) = r_3(25(25n+5)), \tag{4.30}$$

which is (4.13) for $k = 1$.

Again, equating the coefficients of $q^{25(5n+1)}$ from both sides of (4.27),

$$5r_3(25n+5) = 6r_3(5^2(25n+5)) - r_3(25^2(25n+5)),$$

which, with the aid of (4.30), reduces to

$$31r_3(25n+5) = r_3(25^2(25n+5)),$$

which is nothing but (4.13) with $k = 2$. We complete the proof of (4.13) by mathematical induction.

We now prove (4.14). Equating the coefficients of $q^{5n+2}$ from both sides of (4.27),

$$6r_3(25n+10) = r_3(25(25n+10)), \tag{4.31}$$

which is (4.14) for $k = 1$.

Again, equating the coefficients of $q^{25(5n+2)}$ from both sides of (4.27),

$$5r_3(25n+10) = 6r_3(5^2(25n+10)) - r_3(25^2(25n+10)),$$

which, by (4.31), reduces to

$$31r_3(25n+10) = r_3(25^2(25n+10)),$$

which is (4.14) with $k = 2$. Now the proof of (4.13) can be completed by mathematical induction.

In a similar fashion, equating the respective coefficients of $q^{5n+3}$ and $q^{5n+4}$ from both sides of (4.27), and proceeding as in the proofs of (4.13) and (4.14), we can prove (4.15) and (4.16). Thus, we complete the proof of the lemma. □

Proof of Theorem 4.2. Replacing $n$ by $8n+4$ in (4.11),

$$5r_3(8(5n+2)+5) = r_3(8(125n+65)+5). \tag{4.32}$$

Employing (3.29) in (4.32), we readily deduce (4.5).

Next, replacing $n$ by $8n+5$ in (4.12), and then using (3.29), we deduce (4.6).

Again, replacing $n$ by $8n$ in (4.13),

$$\left(\frac{5^{k+1}-1}{4}\right)r_3(8(25n)+5) = r_3\left(8\left(5^{2k+2} + \frac{5^{2k+1}-5}{8}\right)+5\right),$$

which implies (4.7) with the aid of (3.29).

Similarly, replacing $n$ by $8n+3$, $8n+6$, and $8n+1$ in (4.14), (4.15) and (4.16) respectively, and then employing (3.29), we deduce (4.9), (4.10) and (4.8) respectively, to finish the proof. □

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## References


[1] B. C. Berndt, *Ramanujan's Notebooks, Part III* (Springer, New York, 1991).
[2] S. Cooper and M. D. Hirschhorn, 'Results of Hurwitz type for three squares', *Discrete Math.* **274** (2004), 9–24.
[3] F. Garvan, D. Kim and D. Stanton, 'Cranks and $t$-cores', *Invent. Math.* **101** (1990), 1–17.
[4] M. D. Hirschhorn and J. A. Sellers, 'Two congruences involving 4-cores', *Electron. J. Combin.* **3**(2) (1996), R10.
[5] M. D. Hirschhorn and J. A. Sellers, 'Some amazing facts about 4-cores', *J. Number Theory* **60** (1996), 51–69.
[6] M. D. Hirschhorn and J. A. Sellers, 'On representations of a number as a sum of three squares', *Discrete Math.* **199** (1999), 85–101.
[7] K. Ono and L. Sze, '4-core partitions and class numbers', *Acta Arith.* **80** (1997), 249–272.



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