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SAMELSON PRODUCTS IN SPACES OF SELF-HOMOTOPY EQUIVALENCES

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1. Introduction. The homotopy groups of any group-like space are equipped with a Samelson product satisfying, up to sign, the identities of a graded Lie bracket. We shall compute the Samelson product in two kinds of spaces of self-homotopy equivalences arising when adding a homotopy or a homology group to a space.

First, let $A \longrightarrow X$ be a cofibration with a Moore space $M(G,n)$ as cofibre. For the monoid $\mathrm{aut}^A(X)$ of maps under A homotopic (rel. A) to the identity, the Samelson product is a pairing

$$\pi_{n+i}(G;X) \otimes \pi_{n+j}(G;X) \longrightarrow \pi_{n+i+j}(G;X)$$

of homotopy groups with coefficients [**1**] in G. Theorem 2.1 computes this pairing in terms of a homomorphism associated to $\alpha \in \pi_i(\mathrm{aut}^A(X))$. This homomorphism can be described as the boundary map $\pi_*(G;X) \longrightarrow \pi_{*+i}(G;X)$ of a certain fibration

$$\Omega^{i+1}X \longrightarrow E(\alpha) \longrightarrow X$$

naturally associated to α.

Dually, let $Y \longrightarrow B$ be a fibration with an Eilenberg-MacLane space $K(G,n)$ as fibre. For the space $\mathrm{aut}_B(Y)$ of maps over B homotopic (over B) to the identity, the Samelson product is a pairing

$$H^{n-i}(Y;G) \otimes H^{n-j}(Y;G) \longrightarrow H^{n-i-j}(Y;G)$$

of cohomology groups with local coefficients. Theorem 4.1 computes this pairing in terms of the differential $H^*(Y;G) \longrightarrow H^{*-i}(Y;G)$ in the Wang sequence for the fibration

$$Y \longrightarrow E(\alpha) \longrightarrow S^{i+1}$$

classified by the element $\alpha \in \pi_i(\mathrm{aut}_B(Y))$.

Both these formulas are reminiscent of the classical one [**3**] relating the Samelson and Pontryagin products.

I use Switzer's notation [**5**] for mapping spaces: If $u : U \longrightarrow V$ is a map, $i : T \longrightarrow U$ a cofibration, and $p : V \longrightarrow W$ a fibration, $F_u(U,T;V,W)$ is the space of all maps $v : U \longrightarrow V$ with $vi = ui$ and $pv = pu$.

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2. Self-maps of Moore cofibrations. Let A be a connected space, G an abelian group, $n \geqq 3$ an integer, and $k' : M \longrightarrow A$ a map of the Moore space $M = M(G, n-1)$ into A. The mapping cone, X, of k' is the push-out

$$\begin{array}{ccc} M & \xrightarrow{k'} & A \\ \downarrow & & \downarrow i \\ TM & \xrightarrow{h} & X \end{array}$$

of k' and the inclusion of M into the top of the cone $TM = {}^{I\times M}/_{0\times M}$.

Composition of maps makes $F_1(X, A; X)$, i.e., the space of maps u such that

$$\begin{array}{ccc} & A & \\ {}^{i}\swarrow & & \searrow^{i} \\ X & \xrightarrow{u} & X \end{array}$$

commutes, into a topological monoid. The component F_1 containing the identity of X is even (homotopy equivalent to) a group-like space ([7], Theorem 2.4, p. 462) and thus equipped with a Samelson product

$$\langle\ ,\ \rangle : \pi_i(F_1) \otimes \pi_j(F_1) \longrightarrow \pi_{i+j}(F_1).$$

The purpose of this section is to describe this Samelson product.

Let $(Z, *)$ be any based connected space and $[Z, *; F_1]$ the group of homotopy classes (rel. $*$) of based maps of Z into F_1.

LEMMA 2.1. *There exists a natural bijection*

$$[Z, *; F_1] \leftrightarrow \pi_n(G; F_*(Z, *; X))$$

which is an isomorphism of abelian groups if Z is a coH-space.

Proof. There are homotopy equivalences

$$F_1(X, A; X) \xrightarrow{\bar{h}} F_h(TM, M; X) \xleftarrow{\cdot h} F_*(TM, M; X).$$

The first of these maps, right-composition with h, is even a homeomorphism by the universal property of push-out. The second map is right-multiplication by h with respect to the action

$$F_*(TM, M; X) \times F_*(TM, *; X) \longrightarrow F_*(TM, *; X)$$

induced by the coaction $TM \longrightarrow \Sigma M \vee TM$.

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Hence

$$[Z,*;F_1] = [Z,*;F_*(TM,M;X)]$$

and by adjointness the right hand set can be identified to

$$\pi_0 F_*(TM,M;F_*(Z,*;X)) = \pi_n(G;F_*(Z,*;X)).$$

See [**1**] for the definition of homotopy groups with coefficients.

For general Z, the bijection of Lemma 2.1, in the following always denoted by a double arrow $\leftrightarrow$, does not preserve the group structure. It is, however, natural in the sense that

$$\begin{array}{ccc} [Z_1,*;F_1] & \longleftrightarrow & \pi_n(G;F_*(Z_1,*;X)) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle \pi_n(\bar{f})} \\ [Z_2,*;F_1] & \longleftrightarrow & \pi_n(G;F_*(Z_2,*;X)) \end{array}$$

commutes for any based map $f : Z_1 \longrightarrow Z_2$. Thus $[Z,*;F_1]$ supports two natural group structures, one of which is abelian. If Z is a coH-space, e.g., a sphere, the two group structures coincide.

COROLLARY 2.2. *For* $i > 0, \pi_i(F_1) = \pi_{n+i}(G;X)$.

In view of this corollary, I allow myself to confuse a map $\alpha : (S^i,*) \longrightarrow (F_1,1)$ with its homotopy class in either $\pi_i(F_1)$ or $\pi_{n+i}(G;X)$. The Samelson product, for instance, can then be considered as a bilinear map

$$\langle \quad , \quad \rangle : \pi_{n+i}(G;X) \otimes \pi_{n+j}(G;X) \longrightarrow \pi_{n+i+j}(G;X).$$

I shall now describe this map.

Let $j_1 : S^i \vee S^j \longrightarrow S^i \times S^j$ be the inclusion of the wedge into the product of two spheres. Since j_1 is a cofibration

$$\bar{j}_1 : F_*(S^i \times S^j,*;X) \longrightarrow F_*(S^i \vee S^j,*;X) = F_*(S^i,*;X) \times F_*(S^j,*;X)$$

is a fibration with fibre

$$F_*(S^i \times S^j, S^i \vee S^j;X) = F_*(S^i \wedge S^j,*;X).$$

The long exact sequence for this fibration breaks up into short exact sequences

$$\begin{array}{c} 0 \longrightarrow \pi_{n+i+j}(G,X) \xrightarrow{\kappa} \pi_n(G;F_*(S^i \times S^j,*;X)) \xrightarrow{\pi_n(\bar{j}_1)} \pi_{n+i}(G;X) \times \pi_{n+j}(G;X) \longrightarrow 0 \\ \updownarrow \\ [S^i \times S^j,*;F_1] \end{array}$$

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Provided the abelian group structure is used, $\pi_n(\bar{p}_1)+\pi_n(\bar{p}_2)$, with $p_1 : S^1 \times S^j \longrightarrow S^i, p_2 : S^i \times S^i \longrightarrow S^j$ the projections, is a splitting, and thus

$$\lambda = 1 - \pi_n(\bar{p}_1)\pi_n(\bar{j}_1) - \pi_n(\bar{p}_2)\pi_n(\bar{j}_1) : [S^i \times S^j, *; F_1] \longrightarrow \pi_{n+i+j}(G;X)$$

is a homomorphism extending the identity on the subgroup $\pi_{n+i+j}(G;X)$. (With the other group structure on the middle term, the above short exact sequence is in general not split; indeed, the Samelson product is the obstruction to the existence of a splitting [**6**], ([**7**], X. 5)).

For $\alpha \in \pi_i(F_1) = \pi_{n+i}(G;X)$ and $\beta \in \pi_j(F_1) = \pi_{n+j}(G;X)$, I now define

$$\theta(\alpha)(\beta) = \lambda(\alpha \times \beta)$$

where the cross product

$$[S^i, *; F_1] \times [S^j, *; F_1] \xrightarrow{\times} [S^i \times S^j, *; F_1 \times F_1] \longrightarrow [S^i \times S^j, *; F_1]$$

is the one induced by the product on F_1. It is proved below that $\theta(\alpha) : \pi_{n+i}G;X) \longrightarrow \pi_{n+i+j}(G;X)$ is a homomorphism. The next section contains the proof of

THEOREM 2.3. *If* $\alpha \in \pi_{n+i}(G;X), \beta \in \pi_{n+j}(G;X)$, *then*

$$\langle \alpha, \beta \rangle = \theta(\alpha)\beta - (-1)^{ij}\theta(\beta)\alpha.$$

It will be convenient to have a description of the cross product relative to the bijection $\leftrightarrow$ of Lemma 2.1. On the level of spaces, the cross product

$$\times : F_*(S^i, *; F_1) \times F_*(S^j, *; F_1) \longrightarrow F_*(S^i \times S^j, *; F_1)$$

takes (α, β) to the map $(\alpha \times \beta)(s,t) = \alpha(s)\beta(t), (s,t) \in S^i \times S^j$. Let

$$\pi_n(\alpha \times \quad) : \pi_n(G; F_*(S^j, *; X)) \longrightarrow \pi_n(G; F_*(S^i \times S^j, *; X))$$

be the homomorphism induced by taking the cross product with α.

LEMMA 2.4. *The diagram*

$$\begin{array}{ccc} [S^j, *; F_1] & \xrightarrow{\alpha\times} & [S^i \times S^j, *; F_1] \\ \downarrow & & \downarrow \\ \pi_n(G; F_*(S^j, *; X)) & \xrightarrow{\pi_n(\alpha\times \quad)+\pi_n(\bar{p}_1)\alpha} & \pi_n(G; F_*(S^i \times S^j, *; X)) \end{array}$$

commutes.

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Proof. Let

$$m : F_*(S^i \times S^j, *; F_*(TM, M; X)) \longrightarrow F_*(S^i \times S^j, *; F_h(TM, M; X))$$

be the map that takes $\zeta : S^i \times S^j \longrightarrow F_*(TM, M; X)$ to the map

$$m(\zeta)(s, t) = \zeta(s, t) \cdot \alpha(s)h, \quad (s, t) \in S^i \times S^j.$$

Then the diagrams

$$\begin{array}{ccc}
 & & [S^i \times S^j, *, F_*] = \pi_n(G; F_*(S^i \times S^j, *; X)) \\
 & \swarrow^{\cdot h} & \\
[S^i \times S^j, *; F_h] & & \Big\uparrow + \pi_n(\bar{p}_1)\alpha \\
 & \nwarrow_{m} & \\
 & & [S^i \times S^j, *; F_*] = \pi_n(G; F_*(S^i \times S^j, *; X))
\end{array}$$

and

$$\begin{array}{ccccccc}
F_*(S^j *; F_1) & \xrightarrow{\bar{h}} & F_*(S^j *; F_h) & \xrightarrow{\cdot h} & F_*(S^j *; F_*) & = & F_*(TM, M; F_*(S^j, *; X)) \\
\alpha\times\Big\downarrow & & \alpha\times\Big\downarrow & & \alpha\times\Big\downarrow & & \underline{\alpha\times}\Big\downarrow \\
F_*(S^i \times S^j, *; F_1) & \xrightarrow{\bar{h}} & F_*(S^i \times S^i, *; F_h) & \xleftarrow{m} & F_*(S^i \times S^i, *; F_*) & = & F_*(TM, M; F_*(S^i \times S^j, *; X))
\end{array}$$

commute. (The two middle vertical arrows take β to the map $\alpha(s)\beta(t)$; $\cdot h$ is defined in the proof of Lemma 2.1.)

To prove the lemma, apply the functor π_0 to the second diagram using the first diagram to interpret the maps occurring in the lower horizontal line.

If $\alpha : (S^i, *) \longrightarrow (F_1, 1)$ is a map, the adjoint of α is the map

$$\mathrm{ad}(\alpha) : X \longrightarrow F_*(S^i; X)$$

given by $\mathrm{ad}(\alpha)(x)(s) = \alpha(s)(x)$. (Note that $\mathrm{ad}(\alpha)$ takes X into the component containing the constant map.) Form the pull-back

$$\begin{array}{ccc}
E(\alpha) & \longrightarrow & F_*(E^{i+1}; X) \\
\Big\downarrow & & \Big\downarrow \\
X & \xrightarrow{\mathrm{ad}(\alpha)} & F_*(S^i; X)
\end{array}$$

along $\mathrm{ad}(\alpha)$ of the restriction fibration determined by $S^i \hookrightarrow E^{i+1}$, the $(i+1)$-dimensional disc. As the fibre of $E(\alpha) \longrightarrow X$ is $\Omega^{i+1}X$, its long exact homotopy sequence

$$\cdots \longrightarrow \pi_q(G; E(\alpha)) \longrightarrow \pi_q(G; X) \longrightarrow \pi_{q+i}(G; X) \longrightarrow \pi_{q-1}(G; E(\alpha)) \longrightarrow \cdots$$

contains a boundary map which raises degrees by i.

COROLLARY 2.5. *If $q \geqq n$, the above boundary map*

$$\pi_q(G;X) \longrightarrow \pi_{q+i}(G;X)$$

is equal to $\theta(\alpha)$.

Proof. The boundary map in question is $\partial\pi_{n+j}(\mathrm{ad}(\alpha))$ where

$$\pi_n(\mathrm{ad}(\alpha)) : \pi_{n+j}(G;X) \longrightarrow \pi_{n+j}(G;F_*(S^i;X)) = \pi_n(G;F_*(S^i \times (S^j,*);X))$$

is induced by $\mathrm{ad}(\alpha)$ and ∂ is the boundary map of the fibration

$$F_*((E^{i+1},S^i) \times (S^j,*);X) \longrightarrow F_*(E^{i+1} \times (S^j,*);X) \longrightarrow F_*(S^i \times (S^j,*);X).$$

As shown by the commutative diagram

$$\begin{array}{ccccc}
 & & \pi_nF_*(S^j,*;X) \searrow^{\pi_n(\bar{p}_2)} & & \\
 & & \| & & \\
\pi_nF_*(E^{i+1}\times(S^j,*);X) & \longrightarrow & \pi_nF_*(S^i\times(S^j,*);X) & \xrightarrow{\partial} & \pi_{n-1}F_*((E^{i+1},S^i)\times(S^j,*);X) \\
 & & \nwarrow^{\kappa} & & \| \\
 & & & & \pi_nF_*((S^i,*)\times(S^j,*);X)
\end{array}$$

where the upward slanted arrow is induced by an inclusion, ∂ is zero on im $\pi_n(\bar{p}_2)$ and the identity on $\pi_{n+i+j}(G;X)$; so is λ. The homomorphism $\pi_{n+j}(\mathrm{ad}(\alpha))$ has an alternative description provided by the commutative diagram

$$\begin{array}{ccc}
\pi_n(G;F_*(S^j,*;X)) & \xrightarrow{\pi_n(\alpha\times\ \ )} & \pi_n(G;F_*(S^i\times S^j,*;X)) \\
\| & & \uparrow \\
\pi_{n+j}(G;X) & \xrightarrow{\pi_{n+j}(\mathrm{ad}(\alpha))} & \pi_{n+j}(G;F_*(S^i;X))
\end{array}$$

Consequently,

$$\theta(\alpha) = \lambda(\pi_n(\alpha\times\ \ ) + \pi_n(\bar{p}_1)\alpha) = \lambda\pi_n(\alpha\times\ \ ) = \partial\pi_{n+j}(\mathrm{ad}(\alpha))$$

where the first equality is Lemma 2.4.

Corollary 2.5 offers a description of $\theta(\alpha)$ which stresses the duality between this section and Section 4.

COROLLARY 2.6. *$\theta(\alpha) = \lambda\pi_n(\alpha\times\ \ )$ is a homomorphism.*

Since F_1 is group-like, the identity map $1 \in [F_1, 1; F_1]$ has an inverse $J \in [F_1, 1; F_1]$. As $J_*(\gamma) = \gamma^{-1}$ for general $\gamma \in [Z, *; F_1]$, we have in particular $J_*(\alpha) = -\alpha, J_*(\beta) = -\beta$. Thus

COROLLARY 2.7. $\theta(\alpha)(J_*\beta) = -\theta(\alpha)\beta$.

A little more effort is required to establish

LEMMA 2.8. $\theta(J_*\alpha)\beta = -\theta(\alpha)\beta$.

Proof. $J_*(\alpha) = -\alpha = \alpha q$ for a degree -1 self-map q of S^i. With the notation from the proof of Corollary 2.5, we have

$$\begin{aligned}\theta(J_*\alpha) = \theta(\alpha q) &= \partial\pi_n(\mathrm{ad}(\alpha q)) = \partial\pi_{n+j}(\bar{q})\pi_n(\mathrm{ad}(\alpha)) \\ &= \pi_{n+j-1}(\bar{q})\partial\pi_n(\mathrm{ad}(\alpha)) = -\partial\pi_n(\mathrm{ad}(\alpha)) = -\theta(\alpha).\end{aligned}$$

Finally, let $\eta(\beta, \alpha) : S^i \times S^j \longrightarrow F_1$ be the map $\eta(\beta, \alpha)(s, t) = (J\beta)(t)\alpha(s), s \in S^i, t \in S^j$.

LEMMA 2.9. $\lambda\eta(\beta, \alpha) = -(-1)^{ij}\theta(\beta)\alpha$.

Noting that $\eta(\beta, \alpha)\tau = J(\beta) \times \alpha$, where $\tau : S^j \times S^i \longrightarrow S^i \times S^j$ interchanges the coordinates, the proof becomes similar to that of Lemma 2.7. The sign comes in because the self-map of $S^j \wedge S^i = S^{i+j} = S^i \wedge S^j$ induced by τ has degree $(-1)^{ij}$.

3. The commutator. The commutator is the map

$$\Phi : F_1 \times F_1 \longrightarrow F_1, \quad (u, v) \longrightarrow u \circ v \circ J(u) \circ J(v),$$

and the Samelson product of $\alpha \in \pi_i(F_1)$ and $\beta \in \pi_j(F_1)$ is the unique homotopy class such that the diagram

$$\begin{array}{ccc} S^i \times S^j & \xrightarrow{\alpha\times\beta} & F_1 \times F_1 \\ \downarrow & & \downarrow \Phi \\ S^i \wedge S^j & \xrightarrow{\langle\alpha,\beta\rangle} & F_1 \end{array}$$

commutes up to homotopy. This section contains the computation of $\langle\alpha, \beta\rangle$.

Let $S = S_1 \times S_2 \times S_3 \times S_4$ with $S_1 = S_3 = S^i$ and $S_2 = S_4 = S^j, i, j \geqq 1$. Following Whitehead [**6**], consider the stratification

$$\{*\} = P_0 \hookrightarrow P_1 \overset{i_{12}}{\hookrightarrow} P_2 \overset{i_{23}}{\hookrightarrow} P_3 \overset{i_{34}}{\hookrightarrow} P_4 = S$$

where $P_k, 0 \leqq k \leqq 4$, is the set of points in S with at least $4-k$ coordinates equal to the base point $*$. Thus

$$P_1 = \bigcup_{i=1}^{4} S_i = S_1 \vee S_2 \vee S_3 \vee S_4$$
$$P_2 = \bigcup_{\gamma \in \Gamma} S_\gamma$$

where $\Gamma = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$ and $S_\gamma = S_{\gamma(1)} \times S_{\gamma(2)}$. For $1 \leqq i \leqq 4$, consider the maps

$$p_i : P_2 \leftrightarrow S_i : j_i, \quad p_i j_i = 1,$$

with $p_i(s_1, s_2, s_3, s_4) = s_i$ and j_i the obvious inclusion. For $\gamma \in \Gamma$, consider the maps

$$p_\gamma : P_2 \leftrightarrow S_\gamma : j_\gamma, \quad p_\gamma j_\gamma = 1,$$

with $p_\gamma(s_1, s_2, s_3, s_4) = (s_{\gamma(1)}, s_{\gamma(2)})$ and j_γ the obvious inclusion.

The group $[P_2, *; F_1] \leftrightarrow \pi_n(G; F_*(P_2, *; X))$ is the middle term of a short exact sequence

$$0 \longrightarrow \pi_n(G; F_*(P_2, P_1; X)) \longrightarrow \pi_n(G; F_*(P_2, *; X)) \xrightarrow{\pi_n(\bar{i}_{12})} \pi_n(G; F_*(P_1, *; X)) \longrightarrow 0$$

where

$$\pi_n(G; F_*(P_2, P_1; X)) \cong \pi_{n+2i}(G; X) \oplus 4\pi_{n+i+j}(G; X) \oplus \pi_{n+2j}(G; X)$$
$$\pi_n(G; F_*(P_1, *; X)) \cong \pi_{n+i}(G; X) \oplus \pi_{n+j}(G; X) \oplus \pi_{n+i}(G; X) \oplus \pi_{n+j}(G; X).$$

Note that

$$\sum_{i=1}^{4} \pi_n(\bar{p}_i) : \pi_n(G; F_*(P_1, *; X)) \longrightarrow \pi_n(G; F_*(P_2, *; X))$$

is a splitting.

Now form the six endomorphisms $\pi_n(\bar{p}_\gamma)\pi_n(\bar{j}_\gamma)$ of $\pi_n(G; F_*(P_2, *; X))$.

LEMMA 3.1. $\sum_{\gamma \in \Gamma} \pi_n(\bar{p}_\gamma)\pi_n(\bar{j}_\gamma) = 1 + 2\sum_{i=1}^{4} \pi_n(\bar{p}_i)\pi_n(\bar{j}_i)$.

Proof. The maps $p_\gamma, j_\gamma, \gamma \in \Gamma$, restrict to maps

$$p_\gamma : P_1 \leftrightarrow P_1 \cap S_\gamma = S_{\gamma(1)} \vee S_{\gamma(2)} : j_\gamma$$

inducing

$$p_\gamma : P_2/P_1 \leftrightarrow S_{\gamma(1)} \wedge S_{\gamma(2)} : j_\gamma.$$

The collection of these consitute a pair of homeomorphisms

$$P_2/P_1 \leftrightarrow \bigvee_{\gamma\in\Gamma}(S_{\gamma(1)} \wedge S_{\gamma(2)})$$

inverse to each other. Thus the left hand side is the identity on the subgroup $\pi_n(G; F_*(P_2, P_1; X))$; so is the right hand side.

It remains to consider the two sides of the equality sign applied to the four subgroups im $\pi_n(\bar{p}_k)$, $1 \leqq k \leqq 4$. Since

$$\pi_n(\bar{p}_\gamma)\pi_n(\bar{j}_\gamma)\pi_n(\bar{p}_k) = \pi_n(\overline{p_k j_\gamma p_\gamma}) = \begin{cases} \pi_n(\bar{p}_k) & \text{if } k \in \gamma \\ 0 & \text{if } k \notin \gamma, \end{cases}$$

im $\pi_n(\bar{p}_k)$ is invariant under $\pi_n(\bar{p}_\gamma)\pi_n(\bar{j}_\gamma)$ and in fact

$$3 = \sum_{\gamma\in\Gamma} \pi_n(\bar{p}_\gamma)\pi_n(\bar{j}_\gamma) \Big| \text{ im } \pi_n(\bar{p}_k) : \text{im } \pi_n(\bar{p}_k) \longrightarrow \text{im } \pi_n(\bar{p}_k).$$

Since

$$\pi_n(\bar{p}_i)\pi_n(\bar{j}_i)\pi_n(\bar{p}_k) = \begin{cases} \pi_n(\bar{p}_k) & \text{if } i = k \\ 0 & \text{if } i \neq k, \end{cases}$$

im $\pi_n(\bar{p}_k)$ is invariant under $\pi_n(\bar{p}_i)\pi_n(\bar{j}_i)$ and

$$1 = \sum_{i=1}^{4} \pi_n(\bar{p}_i)\pi_n(\bar{j}_i) \Big| \text{ im } \pi_n(\bar{p}_k) : \text{im } \pi_n(\bar{p}_k) \longrightarrow \text{im } \pi_n(\bar{p}_k).$$

This proves the lemma.

The Samelson product of $\alpha \in \pi_i(F_1), \beta \in \pi_j(F_1)$ is

$$\langle \alpha, \beta \rangle = \lambda \pi_n(\bar{\Delta})\{\alpha, \beta\}$$

where

$$\{\alpha, \beta\} : S \longrightarrow F_1, \ (s_1, s_2, s_3, s_4) \longrightarrow \alpha(s_1) \circ \beta(s_2) \circ (J\alpha)(s_3) \circ (J\beta)(s_4)$$

and $\Delta : S^i \times S^j \longrightarrow S$ is the diagonal $\Delta(s, t) = (s, t, s, t)$. Choose a cellular approximation $\Delta_2 : S^i \times S^j \longrightarrow P_2$ such that $i_2\Delta_2 \simeq \Delta(\text{rel.} *), i_2 : P_2 \longrightarrow S$ the inclusion.

LEMMA 3.2.

$$\pi_n(\bar{\Delta})\{\alpha, \beta\} = \sum_{\gamma\in\Gamma} \pi_n(\bar{\Delta}_2)\pi_n(\bar{p}_\gamma)\pi_n(\bar{j}_\gamma)(\{\alpha, \beta\} \mid P_2).$$

Proof. This follows from the identity of Lemma 3.1 since

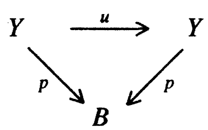
$$\pi_n(\bar{\Delta}_2)\left(\sum_{i=1}^{4}\pi_n(\bar{p}_i)\pi_n(\bar{j}_i)(\{\alpha,\beta\} \mid P_2)\right) = \pi_n(\bar{j}_1)(\alpha + J_*\alpha) + \pi_n(\bar{j}_2)(\beta + J_*\beta) = 0.$$

Lemma 3.2 implies that the Samelson product $\langle\alpha,\beta\rangle$ can be computed as

$$\begin{aligned}\langle\alpha,\beta\rangle &= \lambda\pi_n(\bar{\Delta})\{\alpha,\beta\}\\ &= \sum_{\gamma\in\Gamma}\lambda\pi_n(\bar{\Delta}_2)\pi_n(\bar{p}_\gamma)\pi_n(\bar{j}_\gamma)\pi_n(\bar{\imath}_2)\{\alpha,\beta\}\\ &= \theta(\alpha)\beta + \lambda\pi_n(\bar{p}_1)(\alpha-\alpha) + \theta(\alpha)(J_*\beta)\\ &+ \lambda\eta(\beta,\alpha) + \lambda\pi_n(\bar{p}_2)(\beta-\beta) + \theta(J_*\alpha)(J_*\beta)\\ &= \theta(\alpha)\beta - (-1)^{ij}\theta(\beta)\alpha.\end{aligned}$$

Corollary 2.7, Lemma 2.8, and Lemma 2.9 have been used for the last equality. This completes the proof of Theorem 2.1.

4. Self-maps of Eilenberg-MacLane fibrations. Let B be a connected space, G an abelian group, and $p : Y \longrightarrow B$ a (not necessarily orientable) fibration with an Eilenberg-MacLane space $K(G,n), n \geqq 1$, as fibre. The space $F_1(Y;Y,B)$, consisting of all maps u such that



commutes, is a topological monoid with composition of maps as multiplication. The subject of this section is the Samelson product

$$\langle\ ,\ \rangle : \pi_i(F_1) \otimes \pi_j(F_1) \longrightarrow \pi_{i+j}(F_1)$$

of the submonoid consisting of the identity component F_1 of $F_1(Y;Y,B)$.

Let $a : (Z,C) \longrightarrow (F_1, 1)$ be any map. The adjoint of α is the map

$$\mathrm{ad}(\alpha) : Y \times Z \longrightarrow Y, (y,z) \longrightarrow \alpha(z)y.$$

Note that both $\mathrm{ad}(\alpha)$ and the projection p_1 onto Y can fill in the diagonal of the diagram

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$$\begin{array}{ccc} Y \times C & \xrightarrow{p_1} & Y \\ \downarrow & & \downarrow p \\ Y \times Z & \xrightarrow{pp_1} & B \end{array}$$

Thus we can associate to α the primary difference [**7**]

$$\delta^n(p_1, \mathrm{ad}(\alpha)) \in H^n(Y \times (Z, C); (pp_1)^*G)$$

where G now also denotes the system of local coefficients defined by the fibration p. In this way we obtain a map

$$[Z, C; F_1] \longrightarrow H^n(Y \times (Z, C); (pp_1)^*G)$$

which is a bijection and even an isomorphism of abelian groups if Z is a *coH*-space; cf. [**2**]. In particular,

$$\pi_i(F_1) \cong H^{n-i}(Y; G)$$

for $i > 0$ and the Samelson product becomes a bilinear map

$$\langle \ , \ \rangle : H^{n-i}(Y; G) \otimes H^{n-j}(Y; G) \longrightarrow H^{n-i-j}(Y; G)$$

of cohomology groups with local coefficients.

Consider now a homotopy class $\alpha \in \pi_i(F_1), i > 0$. Let

$$Y \longrightarrow E(\alpha) \longrightarrow S^{i+1}$$

be the fibration over S^{i+1} classified by α; i.e., with

$$\mathrm{ad}(\alpha) : Y \times S^i \longrightarrow Y$$

as its characteristic map. The total space is the push-out

$$\begin{array}{ccc} Y \times S^i & \xrightarrow{j_-} & Y \times E_-^{i+1} \\ j_+ \downarrow & & \downarrow \\ Y \times E_+^{i+1} & \longrightarrow & E(\alpha) \end{array}$$

of $j_-(y, s) = (\alpha(s)(y), s)$ and the inclusion j_+. $E_\pm^{i+1}$ are the two hemispheres of S^{i+1}. Hence the cohomology of $E(\alpha)$ can be computed from the Wang sequence

$$\cdots \longrightarrow H^q(E(\alpha); G) \longrightarrow H^q(Y; G) \xrightarrow{\theta(\alpha)} H^{q-i}(Y; G) \longrightarrow H^{q+1}(E(\alpha); G) \longrightarrow \cdots$$

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whose differential $\theta(\alpha)$ is the composite

$$H^q(Y) \xrightarrow{\mathrm{ad}(\alpha)^*} H^q(Y \times S^i) \xrightarrow{/s_i} H^{q-i}(Y)$$

where $/s_i$ is slant product [**4**] with a generator $s_i \in H_i(S^i; \mathbf{Z})$.

Writing $\beta\theta(\alpha)$ for $\theta(\alpha)$ applied to β, the main result of this section is

THEOREM 4.1. *If* $\alpha \in \pi_i(F_i) = H^{n-i}(Y; G), \beta \in \pi_j(F_1) = H^{n-j}(Y; G), i > 0, j > 0$, *then*

$$\langle \alpha, \beta \rangle = \alpha\theta(\beta) - (-1)^{ij}\beta\theta(\alpha).$$

The proof of this theorem occupies the rest of this paper. First some lemmas.

LEMMA 4.2. *If* $\alpha_1, \alpha_2 \in \pi_i(F_1), i > 0$, *then*

$$\theta(\alpha_1 + \alpha_2) = \theta(\alpha_1) + \theta(\alpha_2).$$

Proof. Let $\nu : S^i \longrightarrow S^i \vee S^i$ be a map such that $p_1\nu \simeq 1 \simeq p_2\nu$. The lemma then follows from the map of Serre spectral sequences induced by the commutative diagram

$$\begin{array}{ccc} E(\alpha_1 + \alpha_2) & \longrightarrow & E(\alpha_1) \cup_Y E(\alpha_2) \\ \downarrow & & \downarrow \\ \Sigma S^i & \xrightarrow{\Sigma\nu} & \Sigma(S^i \vee S^i) \end{array}$$

where Σ is the suspension functor and the total space to the right is $E(\alpha_1)$ and $E(\alpha_2)$ glued together along a common fibre Y.

COROLLARY 4.3. $\theta(J_*\alpha) = \theta(-\alpha) = -\theta(\alpha)$.

For α and β as in Theorem 4.1, recall the map

$$\beta \times \alpha : S^j \times S^i \longrightarrow F_1, \ (t, s) \longrightarrow \beta(t)\alpha(s)$$

introduced in Section 2.

LEMMA 4.4.

$$\beta\theta(\alpha) = \delta^n(p_1, \mathrm{ad}(\beta \times \alpha))/s_j \times s_i.$$

Proof. Let $s^j \in H^j(S^j; \mathbf{Z})$ be the dual generator of $s_j \in H_j(S^j; \mathbf{Z})$. Considering β as an element of $H^{n-j}(Y; G)$, we have

$$\beta \times s^j = \delta^n(p_1, \mathrm{ad}(\beta)) \in H^n(Y \times S^j).$$

The diagram

$$\begin{array}{ccccc}
H^{n-j}(Y) & \xrightarrow{\mathrm{ad}(\alpha)^*} & H^{n-j}(Y \times S^i) & \xrightarrow{/s_i} & \\
\downarrow{\times s^j} & & \downarrow{\times s^j} \quad (-1)^{ij} & & H^{n-i-j}(Y) \\
H^n(Y \times S^j) & \xrightarrow{(\mathrm{ad}(\alpha)\times 1)^*} & H^n(Y \times S^i \times S^j) & \xrightarrow{/s_i \times s_j} &
\end{array}$$

which commutes up to the sign indicated [**4**], shows that

$$\beta\theta(\alpha) = (-1)^{ij}\delta^n(\alpha(s)y, \beta(t)\alpha(s)y)/s_i \times s_j,$$

where the primary difference is between the two maps

$$(y, s, t) \longrightarrow \alpha(s)y \quad \text{and} \quad (y, s, t) \longrightarrow \beta(t)\alpha(s)(y)$$

of $Y \times S^i \times S^j$ into Y. Moreover,

$$\delta^n(y, \beta(t)\alpha(s)y)/s_i \times s_j = \delta^n(\alpha(s)y, \beta(t)\alpha(s)y)/s_i \times s_j$$

since $\delta^n(y, \alpha(s)y)/s_i \times s_j = 0$. Thus

$$\begin{aligned}
\beta\theta(\alpha) &= (-1)^{ij}\delta^n(y, \beta(t)\alpha(s)y)/s_i \times s_j \\
&= \delta^n(y, \beta(t)\alpha(s)y)/\tau_*(s_j \times s_i) \\
&= ((1 \times \tau)^*\delta^n(y, \beta(t)\alpha(s)y))/s_j \times s_i \\
&= \delta^n(p_1, \mathrm{ad}(\beta \times \alpha))/s_j \times s_i
\end{aligned}$$

by naturality of the slant product. Here, $\tau : S^j \times S^i \longrightarrow S^i \times S^j$ is the map that interchanges the two factors.

As in Section 2, consider the diagonal map $\Delta : S^i \times S^j \longrightarrow S$ of $S^i \times S^j$ into $S = S^i \times S^j \times S^i \times S^i$. Construct the map

$$\{\alpha, \beta\} : Y \times S \longrightarrow Y, (y, s_1, t_1, s_2, t_2) \longrightarrow \alpha(s_1) \circ \beta(t_1) \circ (J\alpha)(s_2) \circ (J\beta)(t_2)$$

and let also

$$\{\alpha, \beta\} = \delta^n(p_1, \{\alpha, \beta\}) \in H^n(Y \times S; G)$$

denote the primary difference of p_1 and $\{\alpha, \beta\}$. Then

$$\langle \alpha, \beta \rangle = ((1 \times \Delta)^*\{\alpha, \beta\})/s_i \times s_j.$$

I use this formula for the

Proof of Theorem 4.1. For any $s \in H^*(S;\mathbf{Z})$,

$$\Delta^*(s)/s_i \times s_j = \Delta^*\left(\sum_\gamma p_\gamma^* j_\gamma^*(s)\right) \Big/ s_i \times s_j$$

where

$$p_\gamma \leftrightarrow S_{\gamma(1)} \times S_{\gamma(2)} : j_\gamma$$

are the maps introduced in Section 3, and γ belongs to the set $\{(1,2),(1,4),(2,3),(3,4)\} \subset \Gamma$. Since

$$H^*(Y \times S;G) \cong H^*(Y;G) \otimes H^*(S;\mathbf{Z})$$

it follows that

$$((1\times\Delta)^*\{\alpha,\beta\})/s_i \times s_j = \left(\sum_\gamma (1\times\Delta)^*(1\times j_\gamma p_\gamma)^*\{\alpha,\beta\}\right) \Big/ s_i \times s_j$$

is the sum of the four terms

$$\delta^n(p_1, \mathrm{ad}(\alpha\times\beta))/s_i \times s_j = \alpha\theta(\beta)$$
$$\delta^n(p_1, \mathrm{ad}(\alpha\times J\beta))/s_i \times s_j = \alpha\theta(J_*\beta) = -\alpha\theta(\beta)$$
$$\delta^n(y, \beta(t)(J\alpha)(s)y)/s_i \times s_j = -(-1)^{ij}\beta\theta(\alpha)$$
$$\delta^n(p_1, \mathrm{ad}(J\alpha\times J\beta))/s_i \times s_j = (J_*\alpha)\theta(J_*\beta) = \alpha\theta(\beta)$$

according to Corollary 4.3 and Lemma 4.4. These four terms add up to

$$\langle\alpha,\beta\rangle = \alpha\theta(\beta) - (-1)^{ij}\theta(\alpha).$$

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