DISCRETE SEMI-ORDERED LINEAR SPACES

ISRAEL HALPERIN AND HIDEGORO NAKANO

1. Introduction. Let R be a *semi-ordered linear space*, that is, a *vector lattice* in Birkhoff's terminology [2]. An element $a \in R$ is said to be *discrete*, if for every element $x \in R$ such that $|x| \leqslant |a|$ there exists a real number α for which $x = \alpha a$. For every pair of discrete elements $a, b \in R$ we have $|a| \cap |b| = 0$ or there exists a real number α for which $b = \alpha a$ or $a = \alpha b$. Because, putting

$$c = |a| \cap |b|,$$

we have $c = \alpha a$, $c = \beta b$ for some real numbers α, β.

A system of elements $a_\lambda \in R (\lambda \in \Lambda)$ is said to be *complete*, if $|x| \cap |a_\lambda| = 0$ for all $\lambda \in \Lambda$ implies $x = 0$. R is said to be *universally continuous* if for every system of positive elements $a_\lambda \in R (\lambda \in \Lambda)$ there exists $\bigcap_{\lambda \in \Lambda} a_\lambda$ (*conditionally complete* in Birkhoff's terminology [2]).

Definition. A semi-ordered linear space R is said to be *discrete*, if R is universally continuous and has a complete system of discrete elements.

Let R be universally continuous. We shall use the notation $a_\lambda \downarrow_{\lambda \in \Lambda} a$ to mean: $a = \bigcap_{\lambda \in \Lambda} a_\lambda$ and for all $\lambda_1, \lambda_2 \in \Lambda$ there exists $\lambda \in \Lambda$ with $a_\lambda \leqslant a_{\lambda_1} \cap a_{\lambda_2}$. A linear functional L on R is said to be *universally continuous*, if

$$R \ni a_\lambda \downarrow_{\lambda \in \Lambda} 0 \quad \text{implies} \inf_{\lambda \in \Lambda} |L(a_\lambda)| = 0.$$

The totality of universally continuous linear functionals on R is said to be the *conjugate space* of R and denoted [5] by $\overline{R}$. R is said to be *semi-regular*, if R is universally continuous and $\bar{x}(a) = 0$ for all $\bar{x} \in \overline{R}$ implies $a = 0$.

Let R be semi-regular. A sequence of elements $a_\nu \in R$ ($\nu = 1, 2, \ldots$) is said to be w-*convergent* to $a \in R$, if

$$\lim_{\nu \to \infty} \bar{x}(a_\nu) = \bar{x}(a) \qquad \text{for every } \bar{x} \in \overline{R}$$

and then we write $\text{w-}\lim_{\nu \to \infty} a_\nu = a$.

A sequence $a_\nu \in R$ ($\nu = 1, 2, \ldots$) is said to be |w|-*convergent* to $a \in R$, if

$$\lim_{\nu \to \infty} \bar{x}(|a_\nu - a|) = 0 \qquad \text{for every } \bar{x} \in \overline{R},$$

and then we write $|\text{w}|\text{-}\lim_{\nu \to \infty} a_\nu = a$.

In a semi-ordered linear space R we have *order convergence*, i.e., we write $\lim_{\nu \to \infty} a_\nu = a$, if there exists a sequence of elements $R \ni l_\nu \downarrow_{\nu=1}^{\infty} 0$ such that

Received March 31, 1950.

https://doi.org/10.4153/CJM-1951-034-4 Published online by Cambridge University Press

$$|a_\nu - a| \leqslant l_\nu \qquad (\nu = 1, 2, \ldots).$$

Kantorovitch [3] introduced *star convergence*, i.e., we write $\text{s-}\lim_{\nu\to\infty} a_\nu = a$ if every partial sequence from $a_\nu \in R$ $(\nu = 1, 2, \ldots)$ contains a partial sequence which is order convergent to a.

We have furthermore *individual convergence* [7] i.e., we write $\text{ind-}\lim_{\nu\to\infty} a_\nu = a$, if

$$\lim_{\nu\to\infty} (a_\nu \frown x) \smile y = (a \frown x) \smile y \qquad \text{for all } x, y \in R;$$

and *star individual convergence*, i.e., we write $\text{s-ind-}\lim_{\nu\to\infty} a_\nu = a$ if every partial sequence from $a_\nu \in R (\nu = 1, 2, \ldots)$ contains a partial sequence which is individually convergent to a.

The purpose of this paper is to prove the

THEOREM. *Each of the following is necessary and sufficient in order that R should be discrete.*

(A) R is semi-regular and w-convergence coincides with |w|-convergence.
(B) R is semi-regular and star individual convergence coincides with individual convergence.
(C) R is semi-regular and |w|-convergence implies individual convergence.

The letters (A), (B), (C) will be used for reference throughout the paper, and R will denote a semi-ordered linear space.

2. LEMMA 1.[1] *If R is discrete, then R is semi-regular and* w*-convergence coincides with* |w|*-convergence, that is,*

$$\text{w-}\lim_{\nu\to\infty} x_\nu = 0 \quad \textit{implies} \quad \text{w-}\lim_{\nu\to\infty} |x_\nu| = 0.$$

Proof. If R is discrete, then R is universally continuous by definition. Furthermore R is semi-regular, because for every discrete element $a \neq 0$ we obtain a linear functional $\bar{a}$ in $\overline{R}$ as

$$[a]x = \bar{a}(x)a \qquad (x \in R)$$

for the projector (cf. [4]) $[a]$ of a.

Let $0 \leqslant a_\lambda \in R$ $(\lambda \in \Lambda)$ be a complete system of discrete elements. Then we have obviously

$$\bigcap (1 - [a_{\lambda_1} + \ldots + a_{\lambda_\kappa}]) = 0$$

for all finite numbers of elements $\lambda_1, \ldots, \lambda_\kappa \in \Lambda$. Therefore we have by definition

$$\bigcap \bar{a}(1 - [a_{\lambda_1} + \ldots + a_{\lambda_\kappa}]) = 0$$

for every positive $\bar{a} \in \overline{R}$.

[1]From Lemma 1 we conclude immediately that in l_1 space weak convergence coincides with norm convergence, as was proved first by J. Schur [9].

We assume that $x_\nu \in R$ ($\nu = 1, 2, \ldots$) is w-convergent to zero but not |w|-convergent to zero and derive a contradiction. We can suppose that for some positive $\bar{a} \in \bar{R}$ the inequality $\bar{a}(|x_\nu|) > 2$ holds for an infinite number of ν, hence

$$\bar{a}(x_\nu^+) > 1 \text{ or } \bar{a}(x_\nu^-) > 1$$

for an infinite number of ν. Replacing x_ν by $-x_\nu$ if necessary, we can suppose $\bar{a}(x_\nu^+) > 1$ for an infinite number of ν and hence (using only these x_ν) for all x_ν.

Now for each $\mu = 1, 2, \ldots$ define x_μ and a projector

$$P_\mu = [a_{\mu_1} + \ldots + a_{\mu_\kappa}]$$

(with a finite number of indices $\mu_1, \ldots, \mu_\kappa \in \Lambda$, $\kappa = \kappa(\mu)$) by induction on μ so that:

$$\text{(i)} \qquad \bar{a}((\mathbf{U}_{\rho<\mu}P_\rho)|x_\mu|) < \tfrac{1}{5},$$

$$\text{(ii)} \qquad P_\mu \mathbf{U}_{\rho<\mu} P_\rho = 0,$$

$$\text{(iii)} \qquad \bar{a}((1 - \mathbf{U}_{\rho\leqq\mu}P_\rho)|x_\mu|) < \tfrac{1}{5}.$$

Set $Q_\mu = [P_\mu x_\mu^+]$ and $Q = \mathbf{U}_{\mu=1}^{\infty} Q_\mu$. Then $\bar{a}Q$ is in $\bar{R}$, yet $\bar{a}Q(x_\mu) > \frac{1}{5}$ for all μ, contradicting the assumption $\lim_{\mu\to\infty} \bar{a}Q(x_\mu) = 0$.

LEMMA 2. *Let R be semi-regular. For a positive $p \in R$, if*

$$\text{w-}\lim_{\nu\to\infty} x_\nu = 0, \ |x_\nu| \leqslant p \qquad (\nu = 1, 2, \ldots)$$

implies $\text{w-}\lim_{\nu\to\infty} |x_\nu| = 0$, *then the normal manifold $[p]R$ is discrete.*

Proof. If $[p]R$ is not discrete, then there exists an element p_0 which we choose to denote also by $p(0, 1)$, such that $0 \neq [p_0] \leqslant [p]$, $[p_0]R$ has no discrete element except 0, and furthermore $[p_0]R$ is regular, i.e., there exists a positive $\bar{a} \in \bar{R}$ such that if ($0 \leqslant x \in R$)

$$\bar{a}(x) = 0 \text{ implies } [p_0]\,x = 0.$$

For such a positive $\bar{a} \in \bar{R}$, we see easily that there exist two elements $p(0, 2^{-1})$, $p(2^{-1}, 1)$ such that

$$[p_0] = [p(0, 1)] = [p(0, 2^{-1})] + [p(2^{-1}, 1)],$$
$$\bar{a}([p(0, 2^{-1})]p) = \bar{a}([p(2^{-1}, 1)]p).$$

Thus we obtain by induction elements

$$p(\mu 2^{-\nu}, (\mu + 1)2^{-\nu}) \qquad (\mu = 0, 1, 2, \ldots, 2^\nu - 1; \nu = 1, 2, \ldots)$$

such that

https://doi.org/10.4153/CJM-1951-034-4 Published online by Cambridge University Press

$$[p(\mu 2^{-\nu}, (\mu+1)2^{-\nu})]$$
$$= [p(2\mu 2^{-\nu-1}, (2\mu+1)2^{-\nu-1})] + [p((2\mu+1)2^{-\nu-1}, 2(\mu+1)2^{-\nu-1})],$$
$$\bar{a}([p(2\mu 2^{-\nu-1}, (2\mu+1)2^{-\nu-1})]p)$$
$$= \bar{a}([p((2\mu+1)2^{-\nu-1}, 2(\mu+1)2^{-\nu-1})]p).$$

Putting $x_\nu = \sum_{\mu=0}^{2^\nu-1} (-1)^\mu [p(\mu 2^{-\nu}, (\mu+1)2^{-\nu})]p$, we have

$$|x_\nu| = [p_0]p \qquad (\nu = 1, 2, \ldots)$$

and hence naturally

$$\lim_{\nu\to\infty} \bar{a}(|x_\nu|) = \bar{a}([p_0]p) \neq 0. \tag{2.1}$$

On the other hand we can prove

$$\lim_{\nu\to\infty} \bar{b}(x_\nu) = 0 \qquad \text{for every } \bar{b} \in \bar{R}.$$

This can be done as follows: For a positive $\bar{b} \in \bar{R}$, define a function of a real variable $\bar{b}(t)$, $0 \leqslant t \leqslant 1$, by

$$\bar{b}(t) = \bar{b}\Big(\bigcup_{\mu 2^{-\nu} \leqq t} [p((\mu-1)2^{-\nu}, \mu 2^{-\nu})]p\Big).$$

Then it is not difficult to see that $\bar{b}(t)$ is absolutely continuous:

$$\bar{b}(t) = \int_0^t g(s)ds \qquad (0 \leqslant t \leqslant 1)$$

for some summable function $g(s)$. Now

$$\lim_{\nu\to\infty}\Big(\sum_{\text{odd }\mu}\int_{\mu 2^{-\nu}}^{(\mu+1)2^{-\nu}} g(s)ds\Big) = \lim_{\nu\to\infty}\Big(\sum_{\text{even }\mu}\int_{\mu 2^{-\nu}}^{(\mu+1)2^{-\nu}} g(s)ds\Big) = \tfrac{1}{2}\int_0^1 g(s)ds.$$

This is easily proved for continuous $g(s)$ and easily extended to all summable $g(s)$ (cf. [1]). Now the above shows that

$$\lim_{\nu\to\infty} \bar{b}(x_\nu^+) = \lim_{\nu\to\infty} \bar{b}(x_\nu^-)$$

and hence $\lim_{\nu\to\infty} \bar{b}(x_\nu) = 0$ for every positive $\bar{b} \in R$. Therefore we have w-$\lim_{\nu\to\infty} x_\nu = 0$ but not $|$w$|$-$\lim_{\nu\to\infty} x_\nu = 0$ (by (2.1) contradicting the assumption.

In this proof, let y_γ $(\gamma = 1, 2, \ldots)$ be the sequence consisting of all elements

$$[p(\mu 2^{-\nu}, (\mu+1)2^{-\nu})]p \qquad (\mu = 0, 1, 2, \ldots, 2^\nu - 1;\ \nu = 1, 2, \ldots).$$

Then every partial sequence from $y_\gamma (\gamma = 1, 2, \ldots)$ contains a partial sequence $y_{\gamma_\nu} (\nu = 1, 2, \ldots)$ such that

$$\sum_{\nu=1}^{\infty} \bar{a}(y_{\gamma_\nu}) < +\infty.$$

Since $0 \leqslant y_{\gamma_\nu} \leqslant [p_0]p$ $(\nu = 1, 2, \ldots)$, putting $y_0 = \limsup_{\nu \to \infty} y_{\gamma_\nu}$ we conclude that $\bar{a}(y_0) = 0$, and hence $y_0 = 0$. Therefore we have $\text{s-}\lim_{\gamma \to \infty} y_\gamma = 0$, while $\limsup_{\gamma \to \infty} y_\gamma = [p_0]p \neq 0$. Thus we obtain further:

LEMMA 3. *If R is semi-regular and for a positive $p \in R$ if*

$$\text{s-}\lim_{\nu \to \infty} x_\nu = 0, \ |x_\nu| \leqslant p \qquad (\nu = 1, 2, \ldots)$$

implies $\lim_{\nu \to \infty} x_\nu = 0$, *then the normal manifold $[p]R$ is discrete.*

Conversely we have

LEMMA 4. *If R is discrete, then*

$$\text{s-ind-}\lim_{\nu \to \infty} x_\nu = 0 \textit{ implies } \text{ind-}\lim_{\nu \to \infty} x_\nu = 0.$$

Proof. Let $a_\lambda (\lambda \in \Lambda)$ be a complete system of discrete elements. If

$$\text{s-}\lim_{\nu \to \infty} x_\nu = 0, \ |x_\nu| \leqslant p \qquad (\nu = 1, 2, \ldots),$$

then we have obviously

$$\lim_{\nu \to \infty} [a_\lambda]|x_\nu| = 0 \qquad \text{for every } \lambda \in \Lambda.$$

Putting $x_0 = \limsup_{\nu \to \infty} |x_\nu|$, we have

$$[a_\lambda]x_0 = \limsup_{\nu \to \infty} [a_\lambda]\,|x_\nu| = 0 \qquad \text{for every } \lambda \in \Lambda.$$

Since $a_\lambda (\lambda \in \Lambda)$ is a complete system in R, we obtain then $x_0 = 0$. Therefore we have $\lim_{\nu \to \infty} x_\nu = 0$.

By virtue of Lemmas 1 and 2 we have: *the condition* (A) *is necessary and sufficient in order that* R *be discrete.* And furthermore, as an immediate consequence from Lemmas 3 and 4 we have: *the condition* (B) *is necessary and sufficient in order that* R *be discrete.*

Since $\text{s-}\lim_{\nu \to \infty} x_\nu = 0$ implies $|\text{w}|\text{-}\lim_{\nu \to \infty} x_\nu = 0$, as can be seen from the definitions, we obtain by Lemma 3:

LEMMA 5. *Let R be semi-regular. For a positive $p \in R$ if*

$$|\text{w}|\text{-}\lim_{\nu \to \infty} x_\nu = 0, \ |x_\nu| \leqslant p \qquad (\nu = 1, 2, \ldots)$$

implies $\lim_{\nu \to \infty} x_\nu = 0$, *then the normal manifold $[p]R$ is discrete.*

LEMMA 6. *If R is discrete, then*

$$|\text{w}|\text{-}\lim_{\nu \to \infty} x_\nu = 0 \textit{ implies } \text{ind-}\lim_{\nu \to \infty} x_\nu = 0.$$

Proof. It is sufficient to prove this for the case $x_\nu \geqslant 0$ $(\nu = 1, 2, \ldots)$. Now for fixed $p \geqslant 0$, let

$$x_p{}^* = \limsup_{\nu \to \infty} (x_\nu \frown p).$$

We need only prove that $x_p{}^* = 0$ for each $p \geqslant 0$. But for every discrete element $a \in R$ and any $\bar{a} \in \bar{R}$, it is easy to prove that $\bar{a}([a]x_p{}^*) = 0$. Hence $[a]x_p{}^* = 0$ for every discrete $a \in R$, implying that $x_p{}^* = 0$ as required.

By virtue of Lemmas 5 and 6 we obtain: *the condition* (C) *is necessary and sufficient in order that R be discrete.*

Remark 1. We can also prove the theorem algebraically without the use of classical integration theory (see [6]), if we apply some results obtained in an earlier paper [8].

Remark 2. The theorem is also valid with the following definition: R is discrete, if R is continuous and has a complete system of discrete elements, replacing the condition that R is semi-regular by the conditions that R is continuous and to every element $p \neq 0$ there exists $q \neq 0$ such that $[q] \leqslant [p]$ and $[q]R$ is regular.

References

[1] C. R. Adams and A. P. Morse, *Random sampling in the evaluation of a Lebesgue integral,* Bull. Amer. Math. Soc., vol. 45 (1939), 442-447.

[2] G. Birkhoff, *Lattice theory,* Amer. Math. Soc. Coll. Pub. 25 (1940).

[3] L. Kantorovitch, *Lineare halbgeordnete Räume,* Math. Sbornik, vol. 2 (44) (1937), 121-168.

[4] H. Nakano, *Teilweise geordnete Algebra,* Jap. J. Math., vol. 17 (1941), 425-511.

[5] ———, *Stetige lineare Funktionale auf dem teilweisegeordneten Modul,* J. Fac. Sci. Imp. Univ. Tokyo, vol. 4 (1942), 201-382.

[6] ———, *Discrete semi-ordered linear spaces* (in Japanese), Functional Analysis, vol. 1 (1947-9), 204-207.

[7] ———, *Ergodic theorems in semi-ordered linear spaces,* Ann. Math., vol. 49 (1948), 538-556.

[8] ———, *Modulared semi-ordered linear spaces,* Tokyo Math. Book Series, vol. 1 (Tokyo, 1950), §26, §27.

[9] J. Schur, *Über lineare Transformationen in der Theorie der unendlichen Reihen,* J. f. reine u. angew. Math., vol. 151 (1921), 79-111.

Queen's University and
Tokyo University

https://doi.org/10.4153/CJM-1951-034-4 Published online by Cambridge University Press