Glasgow Math. J. **45** (2003) 389–400. © 2003 Glasgow Mathematical Journal Trust.
DOI: 10.1017/S0017089503001307. Printed in the United Kingdom

NON-AMENABLE TYPE K EQUATIONS OVER GROUPS

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(Received 19 April, 2002; accepted 3 October, 2002)

Abstract. We define a class of equations that are not amenable but are type K and are therefore solvable over torsion-free groups. Moreover, we show that these new equations are solvable over all groups.

2000 *Mathematics Subject Classification.* 20E05.

1. Introduction. In his paper [**12**], Klyachko proved a beautiful result about tesselations of the 2-sphere. There were two things that made this result so enticing. First, the context is easily accessible. Very roughly, this result states that a system of cars driving continuously along a connected graph of one-lane roads on a sphere can not avoid collisions. As Klyachko suggests, this fact would be appropriate “as a problem for a school mathematical tournament” [**12**]. Secondly, the result leads to the proof of the Weak Kervaire Conjecture for torsion-free groups. That is to say, if G is a non-trivial torsion-free group and $e \in G * \langle t \rangle$, then $\langle G, t \mid e \rangle$ is not trivial. (It is not hard to see that this question is only interesting in the case where t has exponent sum 1 in e.)

Since Howie used relative diagrams in [**11**] to prove that all length three equations are solvable over all groups, it has become standard to apply results about tesselations of the 2-sphere to the study of equations over groups. This was Klyachko’s approach: to translate his geometric result into a group-theoretic theorem. He assumed there was a relative diagram that gave a counter-example to the Weak Kervaire conjecture for the equation e over the group G. He then organized a motion for a set of cars on this relative diagram. A collision point, whose existence is assured by his geometric result, established a relation of the form a^k in the group G, whereby G must have torsion.

Since Klyachko’s paper in 1995, several authors have generalized his results by generalizing his methods. Fenn and Rourke ([**7**] and [**8**]) organized motions for more complicated equations. They named the class of equations to which they applied their methods *amenable equations*. The precise definition of amenable is quite complicated. However, amenable equations are those that have an eventual derivative of the form $at^{-1}bt^m$ or $[\prod_{i=1}^{k}(a_it^{-1}b_it)]t^n$ for some positive integers m, k and n.

Independently, Richard Goldstein and the author reached similar results in [**4**]. The objects of study here were relative pictures, the duals of relative diagrams, in which regions yield group relations. Consistent regions were defined to be those which gave relations of the form a^k. An equation is said to be *type* K if any relative picture for that equation has at least two consistent regions. It follows immediately from Howie’s original work that type K equations are solvable over torsion-free groups. To show that consistent regions exist on relative pictures for certain equations, discrete criteria for acceptable simple closed paths were defined. (These paths coincided with transverse paths used in Klyachko's original proof.) A minimal acceptable path was the boundary of a consistent region. While the flavor seemed a little different, the set of equations for which the techniques in [**4**] were applicable was precisely the amenable equations defined by Fenn and Rourke. This begs the following question: *Are there type K equations which are not amenable*?

In this note, we answer this in the affirmative. For example, we show that the equation $e_0 = a_1t^3a_2t^{-1}a_3ta_4t^{-2}a_5ta_6t^{-1}a_7t^3a_8t^{-2}$ is type K. This equation is not amenable as its derivatives are of the form $e_0' = AT^2BT^{-1}CT^2DT^{-1}$ and $e_0'' = AT^2$. In fact, we show that e_0 satisfies a stronger condition which implies that every relative picture for e_0 has a two consistent regions which yield relations of the form aa^{-1}. Such relative pictures are not reduced in the sense of Sieradski [**16**]. So, for any group G, $\langle G, t \mid e_0 \rangle$ is relatively aspherical as defined in [**1**]. It follows that e_0 is solvable over all groups.

We end this discussion saying that the geometric results found in this article follow from those found in [**3**] and feel like the discrete techniques used in [**4**]. It would be interesting to know if it is possible to recapture these results using Klyachko's car-crash methods.

2. Definitions. An *equation* in the variable t with coefficients $S = \{a_i | 1 \leq i \leq k\}$ is a string of the form $e = a_1t^{r_1}a_2t^{r_2} \dots a_kt^{r_k}$. We will always assume that no coefficient occurs more than once in a given equation, although different coefficents might eventually represent the same element of some group. The *exponent sum* $\sigma(e)$ and the *length* $|e|$ are defined by $\sigma(e) = r_1 + r_2 + \dots + r_k$ and $|e| = |r_1| + |r_2| + \dots + |r_k|$ respectively. We will assume that the equations discussed throughout this article satisfy $\sigma(e) \geq 0$.

The *shape* of e is the element $\phi(e) = t^{r_1}t^{r_2} \dots t^{r_k}$ of the free monoid $M[t, t^{-1}]$. (Here, there is no cancellation.) In what follows, we will need to discuss specific classes of shapes of words and subwords. We list them here for convenience.

If $\phi(e)$ is the empty string in $M = M[t, t^{-1}]$, we say that e has the *empty shape*.

If $\phi(e) = t^m$ for some $m > 0$ we say e has a *positive shape of degree m*; if $\phi(e) = t^{-m}$ for some $m > 0$ we say that e has a *negative shape of degree m*.

If e has a shape which is positive, negative or empty we will say that e is *stable*.

If $\phi(e) = (tt^{-1})^m$ for some $m > 0$, then e has a *positively alternating shape of degree m*; if $\phi(e) = (t^{-1}t)^m$ we say that e has a *negatively alternating shape of degree m*.

If $\phi(e) = W^m$ for some non-empty shape W in M and some $m \geq 2$, then we say that the shape of e is a *proper power*.

We consider e as an equation over the group G by assigning group elements to the coefficients. That is to say, an *equation over G* is a pair (e, α) where $e = a_1t^{r_1}a_2t^{r_2} \dots a_kt^{r_k}$ is an equation and $\alpha: S \to G$ is a function satisfying: for all i (modulo k), if $r_{i-1}r_i < 0$, then $\alpha(a_i) \neq 1$. Such a function is called a *proper assignment*. For ease of notation, define $g_i = a_i$ and $\hat{e} = g_1t^{r_1}g_2t^{r_2} \dots g_kt^{r_k}$. We say (e, α) is solvable over G if the cannonical homomorphism $G \to [G * \langle t \rangle]/N(\hat{e})$ is injective. We refer to this factor group by the relative presentation $\langle G, t \mid \hat{e} \rangle$. More generally, we say e is solvable over G if (e, α) is solvable over G for every proper assignment α.

A basic question in the study of equations over groups is: Which equations e are solvable over which groups G? Most of this work is driven by two conjectures. Levin's conjecture predicts that if G is a torsion-free group, then every equation is solvable over G. The Kervaire-Laudenbach-Howie conjecture states that if $\sigma(e) \neq 0$, then e is solvable over any group.

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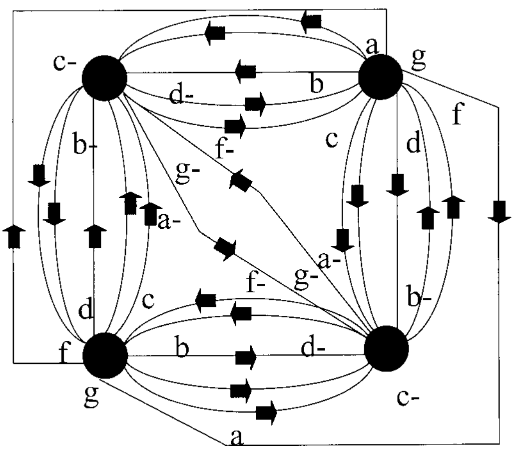

Figure 1

3. General results. In [**11**], Howie used relative diagrams to show that equations satisfying $|e| = 3$ are solvable over all groups. Since then, many authors have used relative diagrams, and the dual notion of relative pictures, to investigate equations over groups. Bogley and Pride give a thorough treatment of this approach in [**1**]. We will assume the reader has some familiarity with these techniques and limit ourselves to a brief explanation.

Given an equation e, an *e-graph* is a connected, directed graph embedded in S^2 some of whose corners are labeled with the coefficients of e and their inverses. We require that as one reads counter-clockwise around any vertex v from a preferred corner, reading t for each edge emanating from v, reading t^{-1} for each edge directed toward v and reading the appropriate coefficient for each corner, one either reads the equation e (in which case v is a positive vertex) or (some conjugate of) e^{-1} (when v is a negative vertex). Figure 1 shows an e_1-graph where $e_1 = at^3bt^{-2}ct^3dt^{-2}ftgt^{-1}$.

If Γ is an e-graph and Δ is a region of Γ, then we get a word $\omega(\Delta)$ by reading clockwise around the corners of Δ. Similarly, if we read counter-clockwise around the corners of Δ, we get another word $\omega^*(\Delta)$. These are well-defined up to cyclic conjugation. If the coefficients of e represent elements of the group G, then we will refer to the corresponding elements of G as $\omega_G(\Delta)$ and $\omega^*_G(\Delta)$.

We now translate Howie's Thoerem into the context of relative pictures.

THEOREM 1 (Howie). *Let e be an equation over the group G. Then e is not solvable over G if and only if there exists an e-graph Γ with a specified region Δ_0 satisfying:*

(1) $\omega^*_G(\Delta_0) \neq 1$*;*
(2) *if* $\Delta \neq \Delta_0$*, then* $\omega_G(\Delta) = 1$*; and*
(3) *for any region Δ, every cyclic conjugate of $\omega(\Delta)$ is freely reduced.*

An e-graph that does not satisfy the third item in the above theorem, is said to be *reducible*. In the language of Bogley and Pride, such cancellation corresponds to a *dipole*. In [**1**], it is proven that if every e-graph has a dipole, then e is solvable over any group G.

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We note that if (e, α) is an equation over the group G and Γ is an e-graph satisfying the above theorem, the only region which might have degree one is Δ_0. This is because α is proper.

If Γ is an e-graph, a region Δ is said to be a *consistent region* if its boundary is consistently directed and there is some coefficient a of e so that each corner of Δ is labeled either a or a^{-1}. As an example, the e-graph in Figure 1 has two consistent regions labeled g^2 and g^{-2}. We see that if e is an equation over the group G and Δ is a consistent region then $\omega_G(\Delta) = a^k$ for some $a \in G$ and some integer k. We assume a proper labeling, so $a \neq 1$ in G. Therefore, either G has torsion; or, $k = 0$ whence Γ is reducible. We say that e is *type* K if every e-graph has at least two consistent regions. This discussion proves the following, found in [**4**]:

FACT 1. *If e is a type* K *equation, then e is solvable over every torsion-free group.*

We now define a new condition, stronger than type K. Let Γ be an e-graph and let Δ be a region of Γ. We say that Δ is *dipolar* if it is a consistent region with exactly two corners, one of which is adjacent to a positive vertex and one of which is adjacent to a negative vertex. We see that if Δ is dipolar, then $\omega(\Delta) = aa^{-1}$ for some coefficient a of e.

We say that the equation e is *type* K* if every e-graph has at least two distinct regions each of which is either degree one or dipolar. Since both dipolar regions and degree one regions are consistent, we see that if e is type K*, then it is type K.

Moreover, we see that if e is type K*, then there are no reduced proper e-graphs. The following fact follows immediately.

FACT 2. *If e is type* K*, *then e is solvable over every group.*

4. Magnus derivatives. In this section we discuss Magnus derivatives. An algebraic discussion of derivatives can be found in [**13**]. We will prefer to follow the more geometric approach presented in [**4**].

An equation e is said to be in *standard form* if its shape $\phi(e) = t^{r_1}t^{r_2}\dots t^{r_k}$ satisfies if $k \geq 2$, then $r_1 > 0$, $r_k < 0$ and $r_i r_{i+1} < 0$. Every equation is conjugate to one in standard form. That is to say, for an equation e whose shape is not stable, there are positive integers $m_1, m_2, \dots, m_l$ and $n_1, n_2, \dots n_l$ so that e is conjugate to an equation whose shape is $t^{m_1}t^{-n_1}t^{m_2}t^{-n_2}\dots t^{m_l}t^{-n_l}$. We say that the derivative of e, e' is an equation whose shape is $T^{m_1-1}T^{-(n_1-1)}T^{m_2-1}T^{-(n_2-1)}\dots T^{m_l-1}t^{-(n_l-1)}$. In the case where e is stable, we establish the convention that e' has the same shape as e.

We have defined derivatives in terms of shapes of equations (as opposed to just equations) and up to cyclic conjugacy in order to simplify the discussion. If some of the exponents in e are 1 or -1, then the corresponding occurences of T in e' do not appear. After computing the shape of e', we fill in coefficients as necessary. We have also changed variables from t to T when passing to the derivatives. This is a notational convenience.

Let us look at an example. Let $e_0 = a_1t^3a_2t^{-1}a_3ta_4t^{-2}a_5ta_6t^{-1}a_7t^3a_8t^{-2}$. It is in standard form and has shape $\phi(e_0) = t^3t^{-1}tt^{-2}tt^{-1}t^3t^{-2}$. Therefore, the derivative of e_0 has shape $T^2T^{-1}T^2T^{-1}$. We may then take the derivative of e_0 to be $e'_0 = ATBTCT^{-1}DTETFT^{-1}$.

It turns out that we will not be interested in coefficients of an equation e between like powers of the variables. The corresponding corners on e-graphs are either sink corners or source corners. These corners will not appear in consistent regions which

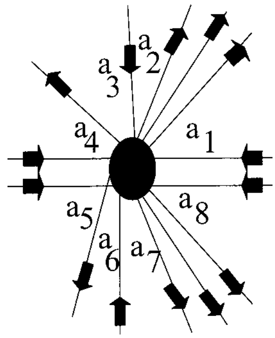

Figure 2

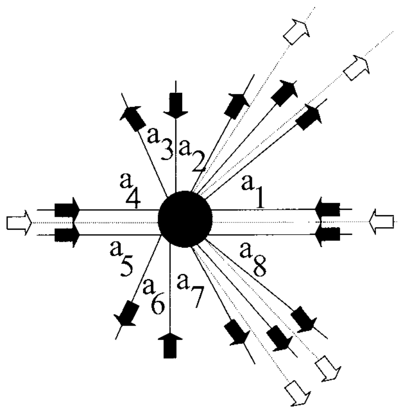

Figure 3

have consistently directed boundaries. So, we will leave these coefficients out of the derivative. There will be no loss of generality in doing so. In our example then, we would take $e'_0 = AT^2BT^{-1}CT^2DT^{-1}$.

Let us look at what the remaining coefficients of the derivative represent. This is most easily seen geometrically. As an example, let us revisit the equation $e_0 = a_1t^3a_2t^{-1}a_3ta_4t^{-2}a_5ta_6t^{-1}a_7t^3a_8t^{-2}$. Assume that we have some e_0-graph Γ. Then every vertex of Γ looks like the star graph in Figure 2 or its inverse.

We add dotted edges to these star graphs as follows: in each source corner, we add a dotted edge directed away from the center; in each sink corner, we add a dotted edge directed toward the center.

If we now remove the original edges and add coefficient labels to corners, we get star graphs corresponding to the derivative of e_0. So, we see that the new coefficients of e'_0 correspond to maximal alternating shaped subwords of (cyclic conjugates of) e_0. Specifically: $A \equiv t^{-1}a_1t$; $B \equiv ta_2t^{-1}a_3ta_4t^{-1}$; $C \equiv t^{-1}a_5ta_6t^{-1}a_7t$; and $D \equiv ta_8t^{-1}$.

This generalizes to the following.

Remark. Let e' be a derivative of e. If e' contains a subword of the form TAT^{-1}, then the coefficient A corresponds to a subword of e with a positively alternating shape.

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If e' contains a subword of the form $T^{-1}BT$, then the coefficient B corresponds to a subword of e with a negatively alternating shape.

Next, we note how length and exponent sum behave under derivatives. We shall state this as a lemma whose proof is left to the reader.

LEMMA 1. *If e' is a derivative of e, then $\sigma(e') = \sigma(e)$. Also, $|e'| \leq |e|$ with equality holding if and only if e is stable.*

Every non-stable equation has an eventual derivative that is stable. The *height*, $h(e)$, of e is defined recursively: if e is stable, then $h(e) = 0$; and, if e is not stable, then $h(e) = h(e') + 1$. If $h(e) = 1$, then e is said to be *pre-stable*.

We spend the remainder of this section describing the possible shapes of a pre-stable equation e. Note that e is pre-stable exactly when e has occurences of both t and t^{-1} and no cyclic conjugate of e has consecutive occurences of t^{-1}.

Let e be a pre-stable equation e with $\sigma(e) = k > 0$. Then there exists $\gamma = (r_1, r_2, \ldots, r_k)$, a k-tuple of non-negative integers not all of which are zero, so that the shape of e is conjugate to $(t^{-1}t)^{r_1} t (t^{-1}t)^{r_2} t \ldots (t^{-1}t)^{r_k} t$. We call the pre-stable equation with this shape e_γ. Moreover, if e is an equation with height h and e_γ is a $(h-1)$-th derivative of e we say that γ is the *pre-stable form* of e. This is well defined up to cyclic conjugation of the coordinates of γ.

If $\sigma(e) = 0$ and e is pre-stable, then e has shape conjugate to $\phi(e) = (t^{-1}t)^r$ for some $r > 0$. We will denote such an equation $e_{[r]}$. If $e_{[r]}$ is the eventual derivative of e then we will say that $[r]$ is the pre-stable form of e.

With this new terminology, we classify amenable equations as precisely those whose prestable forms are either: $\gamma = (r_1, 0, 0, \ldots, 0)$ for some $r_1 > 0$ or [**1**].

5. Inscribing e'-graphs on e-graphs. In this section, we will discuss how we can use an e-graph to create an e'-graph and how we can use the resulting e'-graph to learn something about the original e-graph. This will help us in two ways. First, we will be able to prove that if e' is type K*, then e is type K*. Secondly, this is the technique that we will exploit to prove the main result of this paper: that there are type K* equations which are not amenable. We recommend that the interested reader compare this discussion with those of [**3**], [**4**] and [**5**].

Let e be an equation with derivative e' and let us assume that $|e'| \geq 2$. Let Γ be an e-graph. As described in section 2 of [**4**], we may create an e' graph using the vertices of Γ as follows. In each region whose boundary is not consistently directed, there are as many source corners as sink corners. We add a dotted edge directed from each source corner to a sink corner. We do this in such a way that every source corner and each sink corner is used exactly once and so that the added edges do not cross.

If we then delete all of the original edges from Γ and label the corners with appropriate coefficients, we have what might be an e'-graph with the caveat that it might not be connected. For what follows, we will not delete the old edges and instead make use of the new graph Γ' which has both solid edges (corresponding to occurences of t in e) and dotted edges (corresponding to occurences of T in e'). Every vertex of Γ' is adjacent to both solid and dotted edges.

The dotted edges of Γ' divide the ambient 2-sphere into subsets which we call *meta-regions*. So, there are no vertices interior to a meta-region. In fact, the only parts of Γ' in the interior of a meta-region are the interiors of solid edges. If a meta-region is topologically equivalent to a disk, we call it a *meta-disk*.

Note that, if D is a meta-disk whose boundary is consistently directed, then D properly contains regions of Γ each of whose boundaries is consistently directed. We will say that a meta-disk is *consistent* if it corresponds to a consistent region of Γ' when viewed as an e'-graph. Similarly, we will say that a meta-disk is *dipolar* if it corresponds to a dipolar region of Γ'.

The following fact was the key to the proof in [**4**] that if e' is a type K equation, then so is e.

FACT 3. *Every consistent meta-disk of Γ' properly contains a consistent region of Γ.*

In fact, in the specific case where the meta-disk is degree one (when it is bounded by a dotted loop), then it must contain a consistent region of degree one.

Along the same vein, we will prove the following.

FACT 4. *Let D be a dipolar meta-disk of Γ'. Then either D contains at least two regions of Γ which are degree one; or, every region of Γ which is contained in D is dipolar.*

Proof. Since D is dipolar, it is adjacent to two vertices of Γ', one positive and one negative. Let us call them v and w respectively. Moreover, there is some subword of e' of the form TAT^{-1}, so that the corner of D at v would be labeled A and the corner of D at w would be labeled A^{-1}. As described in the remark in the last section, A corresponds to some subword of e whose shape is alternating of degree k say. So, each corner of D contains $2k$ solid edges alternating with respect to their direction. There are only two such corners. It is easy to see that if it is not the case that every edge is adjacent to both v and w creating $2k-1$ dipolar regions of Γ, then there must be at least two loops.

FACT 5. *Let e be an equation and let e' be a derivative of e. If e' is type* K^**, then e is type* K^*.

Proof. Let Γ be an e-graph. We create a new graph Γ' as above. Since e' is type K^*, there are at least two meta-disks each of which is either dipolar or degree one. Each of these contains either a dipolar region of Γ or a degree one region of Γ. The result follows.

6. Type K* equations of exponent sum 2. In this section, we define a class of equations that are not amenable and prove that they are type K^*. As usual, we need only describe their shape. We begin by defining certain subshapes:

For any positive integer k, define α_k to be the positively alternating shape $\alpha_k = (tt^{-1})^k$.

For any positive integer l, define β_l to be the negatively alternating shape $\beta_l = (t^{-1}t)^l$.

Fix an even positive integer m. Let $Z_1 = (r_1, r_2, \ldots, r_m)$ and $Z_2 = (s_1, s_2, \ldots, s_m)$ be two m-tuples of positive integers.

We then define the equation $e_{(Z_1,Z_2)}$ to have shape:

$$\phi\big(e_{(Z_1,Z_2)}\big) = \alpha_{r_1}\beta_{r_2}\ldots\alpha_{r_{m-1}}\beta_{r_m}t\alpha_{s_1}\beta_{s_2}\ldots\alpha_{s_{m-1}}\beta_{s_m}t$$

We see that for any (Z_1, Z_2), $\sigma(e_{(Z_1,Z_2)}) = 2$.

Before we introduce the main theorem of this section, let us look at our example equation, $e_0 = a_1t^3a_2t^{-1}a_3ta_4t^{-2}a_5ta_6t^{-1}a_7^3a_8t^{-2}$. We see that the shape of e_0,

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$\phi(e_0) = t^3t^{-1}tt^{-2}tt^{-1}t^3t^{-2}$ is conjugate to $(tt^{-1})^2(t^{-1}t)^2t(tt^{-1})(t^{-1}t)t$ which is the shape of $e_{(Z_1,Z_2)}$ where $m = 2$ and $Z_1 = (2, 2)$ and $Z_2 = (1, 1)$.

The article [**3**] investigated pre-stable equations of the form:

$$\epsilon_{(n,n)} = A_1T^{-1}A_2TA_3T^{-1}A_4T\ldots A_{n-1}T^{-1}A_nTxTB_1T^{-1}B_2T\ldots B_{n-1}T^{-1}B_nTyT$$

We see that, taking $n = m/2$, $\epsilon_{(n,n)}$ is a derivative of $e_{(Z_1,Z_2)}$. Moreover, we see from the remark in section 4 that for each odd i, A_i corresponds to the positively alternating subword of $e_{(Z_1,Z_2)}$ whose shape is α_{r_i}; and, B_i corresponds to a subword shaped α_{s_i}. Similarly, for even i, A_i and B_i correspond to negatively alternating subwords shaped β_{r_i} and β_{s_i} respectively.

We wish to apply the geometric results found in [**3**] to the current situation. To do so, let us point out that the derivative of $\epsilon_{(n,n)}$ has shape T^2. So, if Γ is an $\epsilon_{(n,n)}$-graph and Γ' is obtained from Γ by adding dotted edges corresponing to the derivative, then the added edges form a finite collection of disjoint simple closed curves in S^2. At least two of the meta-regions must be meta-disks. The following lemma, adapted from [**3**], shows that each meta-disk must properly contain specific degree-two regions of Γ.

LEMMA 2. *Let n be a positive integer. Define*

$$\epsilon_{(n,n)} = A_1T^{-1}A_2T\ldots A_{m-1}T^{-1}A_mT^2B_1T^{-1}B_2T\ldots B_{m-1}T^{-1}B_mT^2.$$

Let Γ be an $\epsilon_{(n,n)}$-graph. Let Γ' be obtained by adding a set of dotted edges corresponding to occurrences of the variable in $\epsilon'_{(n,n)}$. Let D be a meta-disk of Γ'. Assume that D contains no loops of Γ. Then there exists some j with $0 \le j \le m+1$ and a family of regions Δ_i, $i = 1, 2, \ldots, m$ of Γ contained in D so that for $i \ne j$ each Δ_i has exactly two corners, one adjacent to a positive vertex, one adjacent to a negative vertex. Moreover, exactly one of the following is true:

(1) *for all $i < j$, $\omega(\Delta_i) = A_iA_i^{-1}$; or,*
(2) *for all $i < j$, $\omega(\Delta_i) = B_iB_i^{-1}$; or,*
(3) *for all $i < j$, $\omega(\Delta_i) = A_iB_i^{-1}$; or,*
(4) *for all $i < j$, $\omega(\Delta_i) = B_iA_i^{-1}$;*

and, exactly one of the following is true:

(5) *for all $i > j$, $\omega(\Delta_i) = A_iA_i^{-1}$; or,*
(6) *for all $i > j$, $\omega(\Delta_i) = B_iB_i^{-1}$; or,*
(7) *for all $i > j$, $\omega(\Delta_i) = A_iB_i^{-1}$; or,*
(8) *for all $i > j$, $\omega(\Delta_i) = B_iA_i^{-1}$.*

Proof. Consider the two cyclic subwords $X = TA_1T^{-1}A_2T\ldots A_{m-1}T^{-1}A_mT$ and $Y = TB_1T^{-1}B_2T\ldots B_{m-1}T^{-1}B_mT$ of $\epsilon_{(n,n)}$.

Let C be the dotted simple closed curve that bounds D. As we go around C, we alternately reach positive and negative vertices. If v is a positive vertex on C, then those edges adjacent to v contained in D correspond either to the subword X or Y. We will say that v is a positive X-vertex in the former case and a positive Y-vertex in the latter. Similarly, each negative vertex on C is either a negative X-vertex or a negative Y-vertex.

We measure the distance between vertices as the distance along C. So, if C runs through more than 2 vertices, given any vertex v, there are exactly two vertices which are distance 1 from v. The vertex which is immediately clockwise from v along C we call v^-; the other we call v^+.

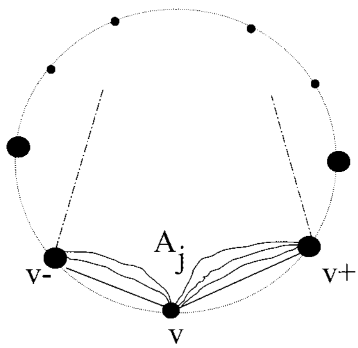

Figure 4

Two vertices v and w are *adjacent* if there is an edge of Γ contained in D which is adjacent to both v and w. The *spread* of a vertex v is the maximum distance from v to all vertices to which v is adjacent. There must be a vertex v whose spread is minimal. Since there are no loops in D, v must have spread 1.

Without loss of generality, we will assume that v is a positive X vertex, as the other three cases follow similarly. Now, each edge which is adjacent to v is also adjacent to either v^- or v^+. Since v is a positive vertex, both v^- and v^+ are negative vertices.

If there is a corner at v inside D which is interior to a region of Γ of degree greater than 2, there is exactly one such corner. In this case we pick j so that this corner is labeled A_j. Otherwise, either all edges adjacent to v are also adjacent to one of v^- or v^+. In these cases, we set $j = 0$ or $j = m + 1$ respectively.

Now, for each $i = 1, 2, \ldots, m$, we let Δ_i be the region of Γ adjacent to v containing the corner labeled A_i. The other corner of Δ_i is adjacent to v^- if $i > j$ and is adjacent to v^+ if $i < j$. So, if $i \neq j$, Δ_i has exactly two corners, one adjacent to a positive vertex and one adjacent to a negative vertex.

Moreover, if v^+ is a negative X-vertex, for all $i < j$, $\omega(\Delta_i) = A_i A_i^{-1}$; if v^+ is a negative Y-vertex, for all $i < j$, $\omega(\Delta_i) = A_i B_i^{-1}$. Similarly, if v^i is a negative X-vertex, for all $i > j$, $\omega(\Delta_i) = A_i A_i^{-1}$; if v^- is a negative Y-vertex, for all $i < j$, $\omega(\Delta_i) = A_i B_i^{-1}$.

We are now ready to prove the main result of this section.

THEOREM 2. *Fix a positive even integer m and pick two m-tuples of positive integers: $Z_1 = (r_1, r_2, \ldots, r_{m-1}, r_m)$ and $Z_2 = (s_1, s_2, \ldots, s_{m-1}, s_m)$. Let $e_{(Z_1,Z_2)}$ have shape $\phi(e_{(Z_1,Z_2)}) = \alpha_{r_1}\beta_{r_2}\ldots\alpha_{r_{m-1}}\beta_{r_m}t\alpha_{s_1}\beta_{s_2}\ldots\alpha_{s_{m-1}}\beta_{s_m}t$. Assume there are k and l with $1 \leq k, l \leq m$ and $k \neq l$ so that $r_k \neq s_k$ and $r_l \neq s_l$. Then $e_{(Z_1,Z_2)}$ is type K^*.*

Proof. Let Γ be a $e_{(Z_1,Z_2)}$-graph. As before, add dotted edges corresponding to the derivative, $e'_{(Z_1,Z_2)}$ to get another graph Γ'. If, for the time being, we ignore the edges and labels from Γ we can consider Γ' to be an $e'_{(Z_1,Z_2)}$-graph. As per the previous lemma, we may add more dotted edges corresponding to the second derivative, $e''_{(Z_1,Z_2)}$ to obtain Γ''. Clearly, Γ'' has at least two meta-disks. It remains to show that each of these meta-disks contains either a degree one region or a dipolar region of Γ.

Let D be one of these meta-disks. Let j be the number and $\{\Delta_i\}$ be the family of degree two regions of Γ' whose existence is assured to us by the previous lemma. Now, at least one of k and l is different than j. Without loss of generality, we will assume $k \neq j$.

https://doi.org/10.1017/S0017089503001307 Published online by Cambridge University Press

Now, look at the meta-region Δ_k of Γ' and call the positive vertex to which it is adjacent v and the negative vertex w. We know $\omega(\Delta_k)$ is one of $A_kA_k^{-1}$, $B_kB_k^{-1}$, $A_kB_k^{-1}$, or $B_kA_k^{-1}$. If $\omega(\Delta_k) = A_kA_k^{-1}$ or $\omega(\Delta_k) = B_kB_k^{-1}$, then Δ_k is a dipolar region of Γ'. So by Fact 4, Δ_k contains either two loops or a dipolar region of Γ. If $\omega(\Delta_k) = A_kB_k^{-1}$ or $\omega(\Delta_k) = B_kA_k^{-1}$, then since the number of edges of Γ in Δ_k adjacent to v is different than the number adjacent to w, Δ_k must contain a loop of Γ.

The result follows.

It remains to point out that these equations are new type K equations. Let $e_{(Z_1,Z_2)}$ be as described in the statement of Theorem 2. Then the derivatives of $e_{(Z_1,Z_2)}$ have shapes $\phi(e'_{(Z_1,Z_2)}) = ((T^{-1}T)^mT)^2$ and $\phi(e''_{(Z_1,Z_2)}) = T^2$. So, $e_{(Z_1,Z_2)}$ has pre-stable shape $\gamma = (m/2, m/2)$. It follows that these equations are not amenable and are therefore the first examples of non-amenable type K equations.

7. Type K* equations of larger exponent sum. In this section, we prove that if $k \geq 6$ and if the k-tuple $\gamma = (r_1, r_2, \ldots, r_k)$ consists of distinct positive integers, then the pre-stable equation e_γ is type K*. Again, we note that pre-stable amenable equations are of the form e_γ where $\gamma = (r_1, 0, 0, \ldots, 0)$. It follows that the equations described in the following theorem are non-amenable.

THEOREM 3. *Fix $k \geq 6$ and let $r_1, r_2, \ldots, r_k$ be distinct positive integers. Let $e = \Pi_{i=1}^{k}[(\Pi_{s=1}^{r_i} a_{is}t^{-1}b_{is}t)c_it]$. Then e is type* K*.

Proof. Let Γ be an e-graph embedded in S^2.

As before we add dotted edges corresponding to the derivative to get the graph Γ' which has both solid and dotted edges. If for the moment, we ignore the solid edges of Γ', we have a graph embedded in S^2 each vertex of which has degree k. Since $k \geq 6$, there must be at least two meta-disks of Γ', that are degree at most two. Let D be a meta-disk of Γ' whose degree is at most 2. If D has degree 1, then D must contain a degree one region of Γ.

If D has degree 2, then there are two vertices on the boundary of D: one positive, say v; and one negative, say w. The corner of D adjacent to v contains corners and germs of edges of Γ corresponding to some subword $t(\Pi_{s=1}^{r_i} a_{is}t^{-1}b_{is}t)$ for some $1 \leq i \leq k$; the corner of D adjacent to w contains corners and germs of edges of Γ corresponding to some subword $[t(\Pi_{s=1}^{r_j} a_{js}t^{-1}b_{js}t)]^{-1}$ for some $1 \leq j \leq k$.

If $i \neq j$, then $r_i \neq r_j$. So, the number of germs of edges in D at v is different than the number of germs at w. So, D must contain a degree one region of Γ. If $i = j$, then either D contains a degree one region of Γ or every region of Γ contained in D is dipolar.

It follows that e is type K*.

We remark that in the case where exactly one of the $r_i = 0$, the above proof can be adapted to show that e is solvable over all groups. However, we would not be able to draw the stronger conclusion that e is type K*.

8. Most equations are type K*. In this section, we give a heuristic argument that allows us to make the assertion that most equations are solvable over all groups.

In section 5, it was proved that if e' is type K*, then e is type K*. So, if all stable equations were type K*, then all equations would be type K*. Unfortunately, no stable equations are type K* although they are all solvable over all groups as proved in [**14**]. It remains to look at pre-stable equations.

In the last section, we proved that if $k \geq 6$ and the k-tuple $\gamma = (r_1, r_2, \ldots, r_k)$ consists of distinct positive integers, then the pre-stable equation e_γ is type K*. It follows that any equation that has a pre-stable form γ satisfying these conditions is also type K*.

Almost all non-negative integers k satisfy $k \geq 6$. Moreover, most k-tuples of non-negative integers have k distinct positive entries. Now, since most equations are non-stable and every non-stable has an eventual derivative which is pre-stable, we conclude that most equations are type K*. It follows that most equations are solvable over all groups.

We acknowledge that the above argument assumes a non-standard partial ordering on the set of equations. However, it can be argued that this partial ordering follows naturally from our line of inquiry: organizing equations first by exponent sum and then by height.

9. Equations that are not type K*. In this section we discuss some equations which are not type K*. We begin with a rather obvious example.

Let e be an equation whose shape is a proper power in M. Then there is an e-graph that has two vertices in which every region has degree two and no region is consistent. Therefore, such an e is not type K and so is not type K*.

In Theorem 1, we assumed that there were at least two different i's so that $r_i \neq s_i$. If for all i, we have $r_i = s_i$, then the equation $e_{(Z_1,Z_2)}$ has shape $\phi(e_{(Z_1,Z_2)}) = (\alpha_{r_1} \ldots \beta_{r_m})^2$. In this case, $e_{(Z_1,Z_2)}$ is neither type K nor type K*.

What if we have exactly one i so that $r_i \neq s_i$? For $m = 2$, $Z_1 = (1, 1)$ and $Z_2 = (1, 2)$ consider the equation $e_{(Z_1,Z_2)}$. It is conjugate to $e_1 = at^3bt^{-2}ct^3dt^{-2}ftgt^{-1}$. Figure 1 shows an e_1-graph which proves that e_1 is not type K*. We do note that this relative picture for e_1 has two consistent regions. It is not clear whether or not this must be the case for every e_1-graph. That is to say, it is not known whether e_1 is type K. This would be included in the following conjecture:

Conjecture. *If $\sigma(e) = 2$ and the shape of e is not a proper power, then e is type* K.

To put this conjecture into context, recall that we know if $\sigma(e) = 1$ and $|e| > 1$, then e is type K ([**12**], [**4**]). So, we see that this conjecture holds for exponent sum 1 equations.

How about equations of higher exponent sum? Let $e_2 = at^2bt^{-1}ct^3dt^{-1}$. Note that the shape of e_2 is not a proper power and $\sigma(e_2) = 3$. In fact, e_2 is a prestable equation conjugate to $e_{(1,1,0)}$. In [**4**] it was shown that there is an e_2-graph without any consistent regions. Therefore, we know that e_2 is not type K (and therefore, not type K*). So, we see that we should not expect this conjecture to hold for general equations of higher exponent sum.

In his Ph.D. thesis [**2**], Aaron Clark has defined a different class of equations that are type K and non-amenable. These include equations that have pre-stable shape $\gamma = (r_1, r_2)$ where $r_1 \neq r_2$. (It is not clear which of these equations are also type K*.) Therefore, the only outstanding cases for this conjecture are those equations whose shapes are not proper powers but that have eventual derivatives whose shapes are of the form $[(t^{-1}t)^r]^2$. Theorem 2 of section 6 describes a class of equations that have this property. The simplest equation for which this conjecture has not been verified is the equation $e_1 = at^3bt^{-2}ct^3dt^{-2}ftgt^{-1}$ described above.

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