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TENSOR PRODUCTS AND THE SPLITTING OF ABELIAN GROUPS

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0. Introduction. In [**2**], Irwin, Khabbaz, and Rayna discuss the splitting problem for abelian groups through the use of the tensor product. Throughout the paper they make a basic assumption, namely, that the torsion subgroup contains but one primary component. Under this restriction they introduce the concept of "splitting length", which is a positive integer indicator of how far a group is from splitting. The results obtained along these lines may be extended to groups whose torsion subgroups contain any finite number of primary components by applying the work of Oppelt [**4**].

Irwin, Khabbaz, and Rayna [**2**] define the notion of a p-sequence and show that for groups A where $T(A)$ is p-primary and $A/T(A)$ has rank one, the existence of a torsion-free element with a p-sequence is sufficient for the group to split.

In this paper we investigate the role of the tensor product in the splitting problem where the torsion subgroup has an infinite number of primary components. We also discuss an extension of p-sequences to abelian groups with arbitrary torsion-free rank and arbitrary torsion subgroups.

In § 1 we introduce necessary notation and terminology.

In § 2 the main theorem, Theorem 2.11, gives conditions for the splitting of $A_1 \otimes \ldots \otimes A_n$ to be equivalent to the splitting of all the A_i, $1 \leqq i \leqq n$, with Corollary 2.14 determining a class of groups with every member of the class having splitting length one or infinity.

In § 3 we extend the concept of p-sequences and indicate in Theorem 3.4 the relationship between the existence of sufficiently many p-sequences and the question of splitting.

1. Preliminaries. Throughout, group will mean abelian group.

For a group G, we denote its torsion subgroup by $T(G)$. $\bar{G}$ will designate $G/T(G)$ and for $g \in G$, $\bar{g} = g + T(G)$. The phrase "G splits" will mean that $T(G)$ is a direct summand of G.

Given a group G, the rank of G will always mean the torsion-free rank of G; and for a prime p and an element $g \in G$, $h_p(g)$ will stand for the p-height of g in G. For a subset S of G, (S) will denote the subgroup of G generated by S.

If for each λ in an index set Λ we have a group A_λ, $\sum_\Lambda A_\lambda$ will denote the (weak, discrete) direct sum and $\prod_\Lambda A_\lambda$ the (strong, complete) direct product of the A_λ.

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The notation $\mathbf{Z}$, $\mathbf{Z}^+$, $\mathbf{Q}$ will designate the integers, positive integers, and rational numbers, respectively, while $\mathbf{Q}_p = \{a/p^i: a, i \in \mathbf{Z}\}$, $\mathbf{Q}_{-p} = \{a/b: a, b \in \mathbf{Z}, (b, p) = 1\}$, and $\mathbf{P} = \{p: p \in \mathbf{Z}^+, p \text{ a prime}\}$. Also, $\mathbf{Z}(p^k)$ will be used for the cyclic group of order p^k.

For the remainder of the paper we fix an infinite subset Π of $\mathbf{P}$ and Λ any non-empty subset of $\mathbf{P}$.

2. Tensor products and splitting.

Definition 2.0. (i) We call a torsion group T Λ-primary if the p-primary component of T is zero for $p \notin \Lambda$ and not zero for $p \in \Lambda$.

(ii) A group A is said to be Λ-divisible if A is p-divisible for every $p \in \Lambda$ and Λ-nondivisible if A is not p-divisible for any $p \in \Lambda$.

(iii) We call a subgroup S of A Λ-pure in A if for every $p \in \Lambda$, $n \in \mathbf{Z}^+$, $p^n S = S \cap p^n A$.

In [**2**], Irwin, Khabbaz, and Rayna prove the theorem:

THEOREM 2.1. *Suppose that M is a group where $M/T(M)$ is not p-divisible and A is any group with $T(A)$ p-primary. Then $M \otimes A$ splits implies that A splits.*

In considering this theorem for a group A with $T(A)$ Π-primary one might wish to replace "$M/T(M)$ not p-divisible" by "$M/T(M)$ not Π-divisible" or "$M/T(M)$ Π-nondivisible." The following example shows that these assumptions will not yield a generalization.

Example 2.2. Let $M = \sum_{q \in \Pi} \mathbf{Q}_{-q}$ and $A = \prod_{p \in \Pi} \mathbf{Z}(p)$. Now

$$\mathbf{Q}_{-q} \otimes A \cong (\mathbf{Q}_{-q} \otimes \mathbf{Z}(q)) \oplus \left(\mathbf{Q}_{-q} \otimes \prod_{p \in \Pi - \{q\}} \mathbf{Z}(p)\right) \cong \mathbf{Z}(q) \oplus \left(\mathbf{Q}_{-q} \otimes \prod_{p \in \Pi - \{q\}} \mathbf{Z}(p)\right)$$

and $\mathbf{Q}_{-q} \otimes \prod_{p \in \Pi - \{q\}} \mathbf{Z}(p)$ is torsion free since $\mathbf{Q}_{-q}$ is Π-$\{q\}$-divisible and $T(\prod_{p \in \Pi - \{q\}} \mathbf{Z}(p))$ is Π-$\{q\}$-primary. Thus $\mathbf{Q}_{-q} \otimes A$ splits. Moreover, $M \otimes A = \sum_{q \in \Pi} \mathbf{Q}_{-q} \otimes A \simeq \sum_{q \in \Pi} (\mathbf{Q}_{-q} \otimes A)$ and hence splits. Note that M is Π-nondivisible and yet it is well-known that A does not split.

The proof of Theorem 2.1 uses the fact that every group contains a p-basic subgroup. Unfortunately, the concept of p-basic subgroups does not extend to Λ-basic subgroups if Λ has more than one element. As pointed out in [**1**], p-basic subgroups exist for each prime p. However, they need not be isomorphic for distinct primes. For a simple example consider the group $\mathbf{Q}_p$. $\{0\}$ is the unique p-basic subgroup of $\mathbf{Q}_p$ while (1) is a q-basic subgroup for any $q \neq p$. Therefore $\mathbf{Q}_p$ cannot have a $\{p, q\}$-basic subgroup for $q \neq p$.

The following theorem and corollary seem to yield the proper generalizations of Theorem 2.1:

THEOREM 2.3. *Let A be any group with $T(A)$ Π-primary. Suppose that M is a torsion free group containing an element m_0 such that for every $p \in \Pi$, $h_p(m_0) = 0$ and $M/(m_0)$ is Π-divisible. Then $M \otimes A$ splits if and only if A splits.*

Proof. Suppose that $M \otimes A$ splits. Let $S = (m_0)$. We claim that S is Π-pure in M so that M/S has no Π-torsion.

Suppose that for $m \in M$, $p \in \Pi$, $\alpha, \beta \in \mathbf{Z}^+$, $k \in \mathbf{Z}$ with $(k, p) = 1$ we have $p^\alpha m = p^\beta k m_0$.

Case 1. $\alpha > \beta$. Now $p^\beta(p^{\alpha-\beta}m - km_0) = 0$ implies that $p^{\alpha-\beta}m = km_0$ since M is torsion free. But then $h_p(m_0) = h_p(km_0) \geqq \alpha - \beta > 0$, which is a contradiction.

Case 2. $\beta \geqq \alpha$. $p^\alpha(p^{\beta-\alpha}km_0 - m) = 0$ implies that $m = p^{\beta-\alpha}km_0 \in S$.

Now the short exact sequence $0 \to S \to M \to M/S \to 0$ yields the following exact sequence

$$\operatorname{Tor}(M/S, A) \to S \otimes A \xrightarrow{e} M \otimes A \to (M/S) \otimes A \to 0.$$

Since M/S has no Π-torsion and $T(A)$ is Π-primary, $\operatorname{Tor}(M/S, A) = 0$.

As in [**2**], we have that $e[T(S \otimes A)] = T(M \otimes A)$. Since $M \otimes A$ splits and $e[T(S \otimes A)] = T(M \otimes A)$, $e(S \otimes A) \simeq S \otimes A \simeq A$ also splits.

The other implication being obvious, this completes the proof.

COROLLARY 2.4. *Let A be any group with $T(A)$ Π-primary. Suppose that M is a group containing an element m_0 with $h_p(\bar{m}_0) = 0$ for all $p \in \Pi$ and $\bar{M}/(\bar{m}_0)$ Π-divisible. Then $M \otimes A$ splits implies that A splits.*

Proof. Consider $0 \to T(M) \to M \to \bar{M} \to 0$. This yields $0 \to T(M) \otimes A \to M \otimes A \to \bar{M} \otimes A \to 0$, since $\bar{M}$ is torsion free.

$T(M) \otimes A$ is canonically contained in $T(M \otimes A)$. Thus $M \otimes A$ splits implies that $M \otimes A/(T(M) \otimes A) \simeq \bar{M} \otimes A$ splits. Now, by Theorem 2.3, A splits.

To show the usefulness of Theorem 2.3 we give

Example 2.5. Even though $\sum_{q \in \Pi} \mathbf{Q}_{-q} \otimes \prod_{p \in \Pi} \mathbf{Z}(p)$ splits, by Example 2.2, $\prod_{q \in \Pi} \mathbf{Q}_{-q} \otimes \prod_{p \in \Pi} \mathbf{Z}(p)$ does not split since $1 \in \prod_{q \in \Pi} \mathbf{Q}_{-q}$ defined by $1|_q = 1$ for all $q \in \Pi$ has zero height at every prime in Π, $\prod_{q \in \Pi} \mathbf{Q}_{-q}/(1)$ is divisible and $\prod_{p \in \Pi} \mathbf{Z}(p)$ does not split.

Due consideration of the question of splitting and Theorem 2.3 might lead one to the conclusion that if $T(A)$ is Π-primary and M is torsion free with an element m_0 having the property that for some infinite subset σ of Π, $h_p(m_0) = 0$ $p \in \sigma$ and $M/(m_0)$ is σ-divisible, then $M \otimes A$ splits implies that A splits. The following examples show that this is not the case and all primes in Π are relevant to this question.

Example 2.6. Let σ be an infinite subset of Π with $\Pi - \sigma$ infinite.

Let $M = \sum_{\Pi - \sigma} \mathbf{Q}_{-q} \oplus \prod_\sigma \mathbf{Q}_{-p}$ and $A = \prod_{\Pi-\sigma} \mathbf{Z}(q) \oplus \sum_\sigma \mathbf{Z}(p)$. The element $(0, 1) \in M$, 1 as before, has height 0 at $p \in \sigma$ and $M/((0, 1))$ is σ-divisible.

One can easily check from preceding results that $M \otimes A$ splits whereas A does not.

Example 2.7. Let σ be a subset of Π with $\Pi - \sigma = \{p_1, p_2, \ldots, p_n\}$, $n \geqq 1$. For $1 \leqq i \leqq n$ let B_i be any nonsplitting group with $T(B_i)$ p_i-primary.
Let $M = \prod_\sigma \mathbf{Q}_{-q}$ and $A = \sum^n{}_{i=1} B_i \oplus \sum_\sigma \mathbf{Z}(p)$. Then $M \otimes A$ splits since $T(\sum^n{}_{i=1} B_i)$ is $\Pi - \sigma$-primary and M is $\Pi - \sigma$-divisible, but A does not split.

In Example 2.7, failure of the implication "$M \otimes A$ splits $\Rightarrow A$ splits" is due to the fact that M is $\Pi - \sigma$-divisible. $\Pi - \sigma$-nondivisibility leads to a better theorem.

THEOREM 2.8. *Let A be any group with $T(A)$ Π-primary. Suppose that σ is a subset of Π with $\Pi - \sigma$ finite. Suppose that M is a group with $m_0 \in M$ such that $h_p(\bar{m}_0) = 0$ for all $p \in \sigma$, $\bar{M}/(\bar{m}_0)$ is σ-divisible, and $\bar{M}$ is $\Pi - \sigma$-nondivisible. Then $M \otimes A$ splits implies that A splits.*

Proof. (i) As in Corollary 2.4, the proof of the case for M a mixed group follows from the case for M torsion free, so we assume that M is torsion free.
(ii) Suppose that $M \otimes A$ splits. Write $T(A) = \sum_\Pi A_p = \sum_{\Pi-\sigma} A_p \oplus \sum_\sigma A_p$. Let $\sum A_p$ represent either $\sum_{\Pi-\sigma} A_p$ or $\sum_\sigma A_p$. Consider $0 \to \sum A_p \to A \to A/\sum A_p \to 0$. We have $0 \to M \otimes \sum A_p \to M \otimes A \to M \otimes (A/\sum A_p) \to 0$, since M is torsion free.
Again, $M \otimes \sum A_p$ is canonically contained in $T(M \otimes A)$ so that $M \otimes A$ splits gives that $M \otimes A/(M \otimes \sum A_p) \simeq M \otimes (A/\sum A_p)$ splits.
(iii) Now by Theorem 2.3 $A/\sum_{\Pi-\sigma} A_p$ splits and by [**2**] and [**4**] $A/\sum_\sigma A_p$ splits.
The proof is now completed by

LEMMA 2.9. *If $T(A) = B \oplus C$ and A/B and A/C both split, then A splits.*

Proof. Note that $B \cap C = 0$. Since $(B + C)/C$ is a summand of A/C, B is a summand of A, by [**3**, lemma 6, p. 18]. Similarly, C is a summand of A. Thus, A splits.

For M with rank one we obtain

COROLLARY 2.10. *Let A be any group with $T(A)$ Π-primary and σ a subset of Π with $\Pi - \sigma$ finite. Let M be a group of rank one. Then if there is an $m_0 \in M$ with $h_p(\bar{m}_0) = 0$ for all $p \in \sigma$ and $\bar{M}$ $\Pi - \sigma$-nondivisible, then $M \otimes A$ splits implies that A splits.*

Proof. It suffices to show that $\bar{M}/(\bar{m}_0)$ is σ-divisible, but this is clear since M has torsion-free rank one so $\bar{M}/(\bar{m}_0)$ is torsion with no σ-torsion.

We may now give the main theorem concerning $n \geqq 2$ factors in the tensor product.

THEOREM 2.11. *Suppose that $A_1, A_2, \ldots, A_n$ are groups with $T(A_i)$ Π_i-primary where Π_i, $1 \leqq i \leqq n$, are infinite subsets of P. For $1 \leqq i \leqq n$, let*

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$\sigma_i \subseteq \Pi_i$ with $\Pi_i - \sigma_i$ finite. Suppose that for each $i \in \{1, 2, \ldots, n\}$ we have $a_i \in A_i$ with $h_p(\bar{a}_i) = 0$ for all $p \in \bigcup_{j \neq i} \sigma_j$, $\bar{A}_i$ is $\bigcup_{j \neq i} (\Pi_j - \sigma_j)$-nondivisible, and $\bar{A}_i/(\bar{a}_i)$ is $\bigcup_{j \neq i} \sigma_j$-divisible. Then $A_1 \otimes \ldots \otimes A_n$ splits if and only if every A_i splits, $1 \leqq i \leqq n$.

Proof. The proof follows by induction using Theorem 2.8, the facts that

$$\bigcap_{i \neq n} \left(\bigcup_{j \neq i} \sigma_j \right) = \sigma_n$$

and

$$\bigcap_{i \neq n} \left(\bigcup_{j \neq i} (\Pi_j - \sigma_j) \right) = \Pi_n - \sigma_n,$$

and the following lemma:

LEMMA 2.12. *Let A and B be groups with $a \in A$, $b \in B$. If $h_p(a) = h_p(b) = 0$ for $p \in P$, then $h_p(a \otimes b) = 0$ in $A \otimes B$.*

Proof. From [**1**, p. 9, (g)] we know that $(A/pA) \otimes (B/pB) \simeq A \otimes B/p(A \otimes B)$. Since A/pA and B/pB are vector spaces over $\mathbf{Z}(p)$, $\bar{a}$ and $\bar{b}$ may be taken to be basis elements in A/pA and B/pB, respectively. Then $\overline{a \otimes b}$ is a basis element in $A \otimes B/p(A \otimes B)$ and thus $a \otimes b$ has zero p-height in $A \otimes B$.

If all $A_i = A$ in Theorem 2.11 we obtain

COROLLARY 2.13. *Suppose that A is any group with $T(A)$ Π-primary and $\sigma \subseteq \Pi$ with $\Pi - \sigma$ finite. If there is an $a \in A$ with $h_p(\bar{a}) = 0$ for all $p \in \sigma$, $\bar{A}/(\bar{a})$ is σ-divisible and $\bar{A}$ is $\Pi - \sigma$-nondivisible, then A does not split implies that $A \otimes \ldots \otimes A$ does not split for any finite number of factors.*

Note that all the theorems hold if we replace "$T(A)$ Π-primary" by "$T(A)$ σ-primary" for any $\sigma \subseteq \Pi$.

3. Λ-sequences and arbitrary rank. In this section of the paper we would like to extend the notion of p-sequences defined in [**2**] and examine their relation to splitting. We begin by writing [**2**, Lemma 3.1] to include an arbitrary number of primes.

LEMMA 3.1. *Let A be a group with $T(A)$ Λ-primary and suppose that B is a torsion free Λ-pure subgroup of A such that A/B is torsion. Then A splits.*

Proof. Let $F = \{a \in A: ma \in B$ for some $m \in \mathbf{Z}$ with $(m, p) = 1$ for all $p \in \Lambda\}$.

The proof now follows as in [**2**].

In the above Lemma, the hypothesis that B is Λ-pure can not be weakened to B $\Lambda - \{q\}$-pure where $q \in \Lambda$. Consider the group $A = \mathbf{Q}\hat{}_p \oplus \mathbf{Z}(p)$ where $\mathbf{Q}\hat{}_p$ is formed by adjoining to $\mathbf{Q}_p$ new generators $a_1, a_2, \ldots$ such that $q^i a_i = 1$ for $i = 1, 2, \ldots$ and where q is a prime not p. Let B be a subgroup of A isomorphic to $\mathbf{Q}_p$. Note that $T(A)$ is $\{p, q\}$-primary, B is torsion free p-pure but not q-pure and A/B is torsion. However, A does not split since $A/T(A)$ contains a subgroup isomorphic to $\mathbf{Q}_q$ while A does not.

Lemma 3.1 generalizes the familiar

COROLLARY 3.2. *Let A be a group with a torsion free pure subgroup B such that A/B is torsion. Then A splits.*

We now extend the definition of p-sequence and generalize [**2**, Proposition 3.1] to include groups of arbitrary rank and with arbitrary torsion subgroups.

Definition 3.3. Let A be a group and $x \in A$. The element x is said to have a Λ-sequence if x has a p-sequence for all $p \in \Lambda$.

THEOREM 3.4. *Let A be a group with $T(A)$ Λ-primary. Then A splits and $\bar{A}$ is Λ-divisible if and only if there exists a maximal torsion free independent set M such that every $x \in M$ has a Λ-sequence.*

Proof. Suppose that there exists a maximal torsion free independent set M such that every $x \in M$ has a Λ-sequence. For each $x \in M$ and $p \in \Lambda$ let $x_{p,i}$ be a p-sequence for x. Let B be the subgroup of A generated by $x_{p,i}$ over all $x \in M$, $p \in \Lambda$, and $i \in \mathbf{Z}^+$. Observe that A/B is torsion, since M is maximal, and B is torsion free, by the independence of M. By Lemma 3.1, to show that A splits, it remains to show that B is Λ-pure in A.

Let $q \in \Lambda$ and suppose that $qa = \sum_{x,i,p} n_i x_{p,i}$. It suffices to show that each term $x_{p,i}$ in the sum is divisible by q. If $p = q$, then $qx_{q,i+1} = x_{q,i}$, by the definition of the q-sequence $x_{q,i}$. If $p \neq q$, then there exist l and m such that $lp^i + mq = 1$ and, moreover,

$$\begin{aligned} q(mx_{p,i} + lx_{q,1}) &= qmx_{p,i} + qlx_{q,1} = (1 - lp^i)x_{p,i} + lx \\ &= x_{p,i} - lp^i x_{p,i} + lx \\ &= x_{p,i} - lx + lx = x_{p,i} \end{aligned}$$

as desired. Since $q \in \Lambda$ was arbitrary, B is Λ-pure.

The existence of M with the given properties clearly implies the Λ-divisibility of $\bar{A}$.

The other implication is obvious in light of the assumption that $\bar{A}$ is Λ-divisible.

In Theorem 3.4 it is essential that every $x \in M$ have a p-sequence for every $p \in \Lambda$. The element $(1, 0) \in A = \mathbf{Q}\hat{}_p \oplus \mathbf{Z}(p)$ of our previous example has a p-sequence but no q-sequence, yet, as was shown, A does not split.

One has the familiar result

COROLLARY 3.5. *Let Q be a subgroup of a group A of rank one. Then Q is a summand of A.*

Proof. Let $M = \{1\}$ in Theorem 3.4. The hypothesis are clearly satisfied and so A splits, $A \simeq Q \oplus T(A)$.

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