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LENGTH FUNCTIONS AND PREGROUPS

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The idea of a pregroup was introduced by Stallings and provides an axiomatic setting for a well-known argument, due to van der Waerden, used to prove normal form theorems. Details are provided in [7], Section 3.

The normal form theorem for a pregroup ([7], 3.A.4.5) gives a corresponding notion of length on its universal group, which will be described later. In [8], Question B1, p. 372, it was asked whether or not this length satisfied the axioms of Lyndon [4] for a length function, the main point being whether or not Lyndon's axiom A4 is satisfied (we shall list all axioms used later). We show that A4 holds if and only if the pregroup satisfies an extra axiom, called P6. This result was obtained independently and at about the same time by Nesayef ([5], Theorem (2.10)). In Section 2 we given an example of a pregroup not satisfying P6, so that Axiom A4 need not be satisfied by the length associated to a pregroup.

Originally, our argument involved showing that if P6 was satisfied, then one obtained a "normal form structure" in the sense of Hurley [3] and one could then appeal to his results to see that A4 was satisfied. Indeed, this was how we were led to Axiom P6. However, motivated by some of the lemmas in a recent paper by Promislow [6], we have investigated exactly what axioms are satisfied by the length associated to a pregroup, using arguments which are essentially contained in [1]. In the course of this investigation, we give a fairly simple argument, closely related to some of the results in [6], that P6 implies A4. These results are presented in Section 3. Finally, some examples are given to clarify the logical relationships between the new axioms which are introduced.

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2.

We consider sets with a partial multiplication, that is sets P, together with a subset D of $P\times P$ and a mapping $D\to P, (x,y)\mapsto xy$. Instead of saying that $(x,y)\in D$, we shall say that xy is defined.

A pregroup is a set P with a partial multiplication, together with a distinguished element denoted by 1 (and called the identity element of P) and a map $P\to P, x\mapsto x^{-1}$,

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satisfying five axioms:

(P1) for all $x \in P$, $x1$ and $1x$ are defined and equal to x;

(P2) for all $x \in P$, xx^{-1} and $x^{-1}x$ are defined and equal to 1;

(P3) if xy is defined, then so is $y^{-1}x^{-1}$ and $y^{-1}x^{-1} = (xy)^{-1}$;

(P4) suppose xy and yz are defined; then $(xy)z$ is defined if and only if $x(yz)$ is defined, in which case they are equal;

(P5) if wx, xy and yz are all defined, then either $w(xy)$ or $(xy)z$ is defined.

If G is a group and P is a subset of G closed under taking inverses and containing 1, let $D = \{(x, y) \in P \times P \mid xy \in P\}$ the partial multiplication being obtained by restricting multiplication on G to D. Then Axioms (P1) to (P4) are automatically satisfied.

We introduce some additional conditions on a pregroup P:

(P6) if $(x, y) \notin D$, but xa and $a^{-1}y$ are both defined, then au and ua are defined for all $u \in P$.

(P7) ax is defined for all $x \in P$ if and only if xa is defined for all $x \in P$.

Proposition 1. *In a pregroup P, Axiom P7 is equivalent to:*

(P7′) *if ax is defined for all $x \in P$, then xa is defined for all $x \in P$*

and to

(P7″) *if xa is defined for all $x \in P$, then ax is defined for all $x \in P$.*

Proof. It follows easily from P1, P2 and P4 that $(x^{-1})^{-1} = x$ for all $x \in P$. It then follows from P3 that (P7′) and (P7″) are equivalent, so both are equivalent to P7.

Proposition 2. *In a pregroup P, Axiom P6 is equivalent to:*

(P6′) *if $(x, y) \notin D$ but xa and $a^{-1}y$ are both defined, then au is defined for all $u \in P$*

and also to

(P6″) *if $(x, y) \notin D$ but xa and $a^{-1}y$ are both defined, then ua is defined for all $u \in P$. Moreover, P6 implies P7.*

Proof. It suffices to show (P6′) implies (P7′) and (P6″) implies (P7″). Assume (P6′), and suppose ax is defined for all $x \in P$. Assume va is not defined for some $v \in P$. Since av^{-1} is defined, va^{-1} is defined by P3. Also, aa is defined, so by (P6′), $a^{-1}u$ is defined for all $u \in P$, in particular, $a^{-1}v^{-1}$ is defined. By P3, va is defined, a contradiction. Hence xa is defined for all $x \in P$, and (P7′) holds. The proof that (P6″) implies (P7″) is similar.

The following equivalent statement of P6 was given by Nesayef ([5], Theorem (2.7)).

In a pregroup, P6 holds if and only if: if $(x, y) \notin D$ and $(ax)y$ is defined, then $(ax)z$ and $z(ax)$ are defined for all $z \in P$.

The reason for this is that $x(ax)^{-1}$ is always defined, while if xa and $a^{-1}y$ are defined, then $b = a^{-1}x^{-1}$ is defined by P3 and $(bx)y$ is defined. In view of Proposition 2, we can obtain two further equivalent statements of this axiom, replacing the conclusion by "then $(ax)z$ is defined for all $z \in P$" and by "then $z(ax)$ is defined for all $z \in P$".

We now exhibit a pregroup not satisfying P7, so not satisfying P6. Let G be a free group of rank 2 with basis $\{a, b\}$. Define

$$P = \{a^m b^{\pm 1} a^n \in G \mid m \leqq 0 \text{ and } n \geqq 0\} \cup \{a^m \mid m \in \mathbb{Z}\}$$

$$D = \{(x, y) \in P \times P \mid xy \in P\}.$$

As we observed after listing the axioms for a pregroup, (P1)–(P4) are satisfied since $1 \in P$ and P is closed under taking inverses. Note that

$$\begin{aligned} D = &\{(a^m b^{\pm 1} a^n, a^k) \mid m \leqq 0, n \geqq 0 \text{ and } n + k \geqq 0\} \\ &\cup \{(a^k, a^m b^{\pm 1} a^n) \mid m \leqq 0, n \geqq 0 \text{ and } k + m \leqq 0\} \\ &\cup \{(a^m b^{\pm 1} a^n, a^{-n} b^{\mp 1} a^k) \mid m \leqq 0,\ n \geqq 0 \text{ and } k \geqq 0\} \\ &\cup \{(a^m, a^n) \mid m, n \in \mathbb{Z}\}. \end{aligned}$$

Lemma 1. *Suppose uv and vw are defined. Then $u(vw)$ is defined if and only if $(uv)w$ is defined. They are both undefined if and only if $v = a^n$ for some $n \in \mathbb{Z}$ and one of the following holds:*

(1) $u = a^m, w = a^l b^{\pm 1} a^k$ *with* $m + n + l > 0$,

(2) $w = a^m, u = a^l b^{\pm 1} a^k$ *with* $k + n + m < 0$,

(3) $u = a^m b^{\pm 1} a^k, w = a^l b^{\pm 1} a^p$, *where either b occurs with the same sign in u and w, or* $k + n + l \neq 0$.

Proof. This is a routine verification and is left to the reader.

It follows easily from the lemma that P is a pregroup. To establish (P5), the only case not taken care of by the lemma is where the sequence w, x, y, z, has the form $a^m b^{\pm 1} a^n, a^k, a^l, a^p b^{\pm 1} a^q$ with $n + k + l < 0$ and $k + l + p > 0$. But $n \geqq 0$, so $k + l < 0$ and $p \leqq 0$, so $k + l > 0$, and this case cannot occur. However, (P7) is not satisfied; for xa is defined for all $x \in P$ but, for instance, $(a, b) \notin D$.

The results in the next section give many examples of pregroups satisfying P6. To complete the picture, we give an example of a pregroup which satisfies P7 but not P6. First, let $G = \langle u \rangle$ be an infinite cyclic group, let $P_1 = \{1, u, u^{-1}\}$ and define $D_1 = \{x, y) \in P \times P \mid xy \in P\}$. As has been noted, (P1)–(P4) are automatically satisfied and it is easy to see that P5 holds, so (P_1, D_1) is a pregroup. Let (P_2, D_2) be a pregroup in which P6 does not hold (we have just shown that such pregroups exist). We can assume that P_1 and P_2 have the same identity element and $P_1 \cap P_2 = \{1\}$. Let $P = P_1 \cup P_2$, $D = D_1 \cup D_2$. It is easily checked that the mappings $D_i \to P_i$ $(i = 1, 2)$ giving multiplication on P_i extend to a mapping $D \to P$, and (P, D) becomes a pregroup. In P, if xa is defined for all x, then $a = 1$, so P7 holds. However, P6 does not hold in P since it does not hold in P_2.

We note that P is the coproduct of P_1 and P_2 in the category of pregroups and this construction of coproduct works for any pair of pregroups P_1 and P_2. (See [7], 3.A.4.2, for the definition of morphism in this category.)

3.

We shall use the terminology of [6] and call a function $p: G \to \mathbb{Z}$, where G is a group, a *$\mathbb{Z}$-semigauge* on G if the following three axioms hold (with the original numbering of [4]);

A1′. $p(1) = 0$;

A2. $p(x) = p(x^{-1})$ for all $x \in G$;

A3. $p(xy) \leqq p(x) + p(y)$ for all x and y in G.

It is easy to see that $\mathrm{Ker}(p) = \{x \in G \mid p(x) = 0\}$ is a subgroup of G.

The term *length function* will be used to mean a normalised integer-valued length function, that is, a $\mathbb{Z}$-semigauge p satisfying:

A4. $d(x, y) \geqq m$ and $d(y, z) \geqq m$ implies that $d(x, z) \geqq m$,

for any $m \in \mathbb{R}$ and x, y, z in G, where $d(x, y) = \frac{1}{2}(p(x) + p(y) - p(xy^{-1}))$.

An alternative way of stating A4 is:

$$d(x, y) > d(x, z) \text{ implies that } d(x, z) = d(y, z), \text{ for all } x, y \text{ and } z \text{ in } G.$$

Thus, of the three numbers $d(x, y), d(x, z), d(y, z)$, at least two of them are equal, and not greater than the third. Note that A3 is equivalent to:

$$d(x, y) \geqq 0 \text{ for all } x, y \in G,$$

and it is easy to see that this follows from A1′, A2 and A4.

From the identity $d(xy, y) + d(x, y^{-1}) = p(y)$ and A2, it follows that $d(x, y) \leqq p(y)$ for any $x, y, \in G$, where p is any semigauge. Since $d(x, y) = d(y, x)$, we have $0 \leqq d(x, y) \leqq \min\{p(x), p(y)\}$. Hence, if either x or y is in $\mathrm{Ker}(p), d(x, y) = 0$, from which we obtain the following simple but useful lemma (see also [6], Lemma (2.1) (c)).

Lemma 2. *If $p: G \to \mathbb{Z}$ is a semigauge and $a \in \mathrm{Ker}(p)$, then $p(au) = p(u) = p(ua)$ for all $u \in G$.*

We shall use the notation $\langle a, b \rangle$ to mean $d(a, b^{-1})$, that is,

$$\tfrac{1}{2}(p(a) + p(b) - p(ab)).$$

We shall also need an axiom introduced in [3]:

N1*. G is generated by $\{x \in G \mid p(x) \leqq 1\}$.

The universal group $U(P)$ of a pregroup P is defined in [7], 3.A.4.2, and we shall view P as embedded in $U(P)$ in accordance with [7], 3.A.4.6. It is then an easy consequence of the universal property of $U(P)$ that P generates $U(P)$. Suppose P is a pregroup, define

$$B = \{a \in P \mid ua \text{ and } au \text{ are defined for all } u \in P\}$$

and let $X = P \backslash B$. If $(a_1, \ldots, a_n)$ is a reduced word in the sense of [7], 3.A.1.2, define:

$$\mu(a_1, \ldots, a_n) = \begin{cases} n \text{ if } n > 1, \\ 1 \text{ if } a_1 \in X \text{ and } n = 1, \\ 0 \text{ if } a_1 \in B \text{ and } n = 1. \end{cases}$$

Then by [7], Theorem 3.A.4.5, μ induces a mapping $p: U(P) \to \mathbb{Z}$ which is clearly a $\mathbb{Z}$-semigauge on $U(P)$. Since P generates $U(P)$, p satisfies N1*. We call p the *semigauge associated with the pregroup P*. Another condition satisfied by p is:

A6. $\langle a, b \rangle = \langle b, c \rangle = 0$ implies that either $p(b) = 0$ or $\langle ab, c \rangle = 0$.

For $\langle a, b \rangle = 0$ means that either $p(a) = 0$, or $p(b) = 0$, or else, if $(a_1, \ldots, a_m)$ and $(b_1, \ldots, b_n)$ are reduced words representing a and b respectively, then $(a_1, \ldots, a_m, b_1, \ldots, b_n)$ is also reduced, and A6 follows easily. These conditions N1* and A6 are sufficient to ensure that a semigauge is associated with some pregroup.

Proposition 3. *Let p be a $\mathbb{Z}$-semigauge on a group G, satisfying A6 and N1*.*

Let $P = \{g \in G \mid p(g) \leqq 1\}$, $D = \{(g, h) \in P \times P \mid p(gh) \leqq 1\}$.

Then (P, D) is a pregroup, with G isomorphic to $U(P)$.

Proof. As noted in Section 2, P1–P4 are automatically satisfied and only P5 needs verification. Suppose w, x, y, z are in P and wx, xy, yz are all in P. Let $X = \{g \in G \mid p(g) = 1\}$. By Lemma 2, to verify P5, we may assume w, z, wx, xy, yz are all in X. The argument is then like that of Lemma 7 in [1], but is short enough to be repeated here. Suppose that $wxy \notin P$ and $xyz \notin P$. By A3, $p(wxy) \leqq p(w) + p(xy) = 2$, and since $wxy \notin P, p(wxy) = 2$, hence $\langle w, xy \rangle = 0$.

Similarly, $\langle xy, z \rangle = 0$. By A6 (with $a = w, b = xy, c = z$), $\langle wxy, z \rangle = 0$, that is, $p(wxyz) = 3$. Hence, $\langle wx, yz \rangle = -\frac{1}{2}$, contradicting A3, and so P5 holds.

The inclusion map $P \to G$ induces a group homomorphism $\Phi: U(P) \to G$ by the universal property of $U(P)$; if $u \in U(P)$ is represented by a reduced word $(u_1, \ldots, u_n)$, then $\Phi(u) = u_1 \ldots u_n$ (see [7], 3.A.4.2 to 3.A.4.6). Since p satisfies N1*, Φ is onto. If $(u_1, \ldots, u_n)$

is a reduced word with $n>1$, then $p(u_i)=1$ for $1\leqq i\leqq n$, and $p(u_iu_{i+1})\geqq 2$, so $\langle u_i,u_{i+1}\rangle=0$ for $1\leqq i\leqq n-1$ by A3. Now if $\langle u_1\dots u_{i-1},u_i\rangle=0$, where $2\leqq i\leqq n-1$, it follows from A6 that $\langle u_1\dots u_{i-1}u_i,u_{i+1}\rangle=0$. By induction on i, $\langle u_1\dots u_{i-1},u_i\rangle=0$ for $2\leqq i\leqq n$, and $p(u_1\dots u_i)=i$ for $1\leqq i\leqq n$. Thus $p(u_1\dots u_n)=n$, and it follows that Φ is one-to-one.

Note. If $\tilde{p}$ is the semigauge associated with P, the proof shows that $p\Phi=\tilde{p}$, so we may identify (G,p) with $(U(P),\tilde{p})$, justifying the assertion immediately before Proposition 3.

Next, we consider when the pregroup P in Proposition 3 satisfies P6. This involves a new axiom:

A7. if $\langle a,b\rangle=0$ and $p(c)\leqq p(b)$, then $\langle a,bc\rangle=0$.

We begin by showing that A7 and N1* imply A6. To do this we state a simple lemma which will also be useful later.

Lemma 3. *Let p be a semigauge on group G, satisfying N1* and A7. Then*

(a) *if $(u_1,\dots,u_n)$ is a sequence of elements of G with $p(u_i)=1$ for $1\leqq i\leqq n$ then $p(u_1\dots u_n)=n$ if and only if $\langle u_i,u_{i+1}\rangle=0$ for $1\leqq i\leqq n$;*

(b) *if $u\in G$ and $p(u)>0$, there exist elements $u_1,\dots,u_n$ in G such that $u=u_1\dots u_n$, $p(u_i)=1$ for $1\leqq i\leqq n$, and $n=p(u)$.*

Proof. The proof is omitted; part (a) is similar to the proof of [**1**], Lemma 4 using A7 and the identity of [**1**], Lemma 2, in place of [**1**], Lemma 3. Likewise, the proof of (b) is very similar to that of [**1**], Lemma 5.

We shall call a decomposition $u=u_1\dots u_n$ as in part (b) of Lemma 3 a *reduced decomposition* of u.

Proposition 4. *If $p\colon G\to\mathbb{Z}$ is a semigauge satisfying A7 and N1*, then p satisfies A6.*

Proof. Assume A7 and N1*, suppose $\langle a,b\rangle=0$, $\langle b,c\rangle=0$ but $p(b)\neq 0$. If $p(a)=0$ or $p(c)=0$ then $\langle ab,c\rangle=0$ by Lemma 2, so we may assume $p(a)>0$ and $p(c)>0$. Then a,b,c have reduced decompositions:

$$a=a_1\dots a_m,\quad b=b_1\dots b_n,\quad c=c_1\dots c_k.$$

Since $\langle a,b\rangle=0$, $p(ab)=p(a_1\dots a_mb_1\dots b_n)=m+n$, and by Lemma 3(a), $\langle a_m,b_1\rangle=0$. Similarly $\langle b_n,c_1\rangle=0$, and again by Lemma 3(a), $p(abc)=p(a_1\dots a_mb_1\dots b_nc_1\dots c_k)=m+n+k$, so $\langle ab,c\rangle=0$, as required.

To prove our result on when the pregroup (P,D) of Proposition 3 satisfies P6, we shall use the notion of a regular semigauge ([**6**], Section 4). A semigauge on a group G is regular if $\langle x,y\rangle=0$ implies that, for all $z\in G$, either $\langle x,z\rangle=0$ or $\langle y^{-1},z\rangle=0$. A simple argument shows that a regular semigauge satisfies A7 (see [**6**], Lemma 4.3(b)).

Remark. If p is the semigauge associated with a pregroup P, then it is an easy consequence of the definition of p that, if $x, y \in P$ then xy is undefined if and only if $p(x) = p(y) = 1$ and $\langle x, y\rangle = 0$, also that if $(x_1, \ldots, x_m)$, $(y_1, \ldots, y_n)$ are reduced words (in the sense of [7], 3.A.1.2) representing elements x, y, respectively, of $U(P)$ then $\langle x, y\rangle$ is greater than 0 if and only if $p(x) > 0$, $p(y) > 0$ and $x_m y_1$ is defined.

Proposition 5. *If p is the semigauge associated with a pregroup P, then P satisfies P6 if and only if p satisfies A7.*

Proof. Assume P satisfies P6; it is enough to show that p is regular. Suppose $\langle x, y\rangle = 0$. We have to show that, for all $z \in U(P)$, either $\langle x, z\rangle = 0$ or $\langle y^{-1}, z\rangle = 0$. In view of Lemma 2, we may assume $p(x), p(y)$ and $p(z)$ are all non-zero. Then x, y and z are represented by reduced words $(x_1, \ldots, x_m)$, $(y_1, \ldots, y_n)$ and $(z_1, \ldots, z_k)$ where each x_i, y_i and z_i is in X, and $X = P \backslash \mathrm{Ker}\,(p) = \{u \in U(P) \mid p(u) = 1\}$.

If $\langle x, z\rangle \neq 0$, then by the preceding remark, $x_m z_1$ is defined. If also $\langle y^{-1}, z\rangle \neq 0$, then $y_1^{-1} z_1$ is defined, so $z_1^{-1} y_1$ is defined (using the axioms for a pregroup). Since $\langle x, y\rangle = 0$, $x_m y_1$ is not defined, so by P6, $p(z_1) = 0$, a contradiction since $z_1 \in X$.

Conversely, assume p satisfies A7. Suppose $x, y \in P$ and xy is undefined, so $\langle x, y\rangle = 0$, and suppose $a \in P$ and $(ax)y$ is defined. Then $\langle y^{-1}, x^{-1}\rangle = 0$ and $p(a^{-1}) \leqq 1 = p(x^{-1})$ so by A7, $\langle y^{-1}, x^{-1}a^{-1}\rangle = 0$, that is, $\langle ax, y\rangle = 0$. Thus $p((ax)y) = p(ax) + p(y) \leqq 1$, and since $p(y) = 1, p(ax) = 0$, so $ax \in B = \{b \in P \mid bu$ and ub are defined for all $u \in P\}$. By the observations after Proposition 2, P6 holds.

Remark. It follows from Propositions 3, 4 and 5, that a $\mathbb{Z}$-semigauge p is the semigauge associated to some pregroup satisfying P6 if and only if p satisfies N1* and A7.

Lemma 4. *Let p be a length function on a group G. Then p satisfies A6 and A7.*

Proof. It is clear that p is a regular semigauge on G and so this follows from [6], Lemma (4.3) (a) and (b).

Suppose $x = x_n x_{n-1} \ldots x_2 x_1$, where $p(x_i) = 1$ for $1 \leqq i \leqq n$, the x_i being elements of a group G equipped with a semigauge p. Then, using A3 and induction on $n, p(x) \leqq n$. Recall that this is a reduced decomposition if $p(x) = n$. The next lemma could be deduced from the proof of [6], Lemma (4.7)(a), but we include a direct argument. The proof of Proposition 6 below is likewise related to [6], Lemma 6.1.

Lemma 5. *Suppose p is a $\mathbb{Z}$-semigauge on a group G, satisfying N1* and A7, and let x, y be elements of $G \backslash \mathrm{Ker}(p)$. Suppose that k is a non-negative integer, with $d(x, y) \geqq k$, and $x = x_n \ldots x_1$ is a reduced decomposition of x (so $n = p(x)$), and let $m = p(y)$. Then if $m > k$, y has a reduced decomposition $y = y_m \ldots y_{k+1} x_k \ldots x_1$ (the y_i depending on k), while if $m = k$, $y = a x_k \ldots x_1$ for some $a \in \mathrm{Ker}(p)$.*

Proof. We use induction on k, the result being trivial if $k = 0$. Assume true for k, and suppose $d(x, y) \geqq k+1$. Inductively we can write a reduced decomposition $y =$

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$y_m \ldots y_{k+1}x_k \ldots x_1$, so that $xy^{-1}=x_n \ldots x_{k+1}y_{k+1}^{-1} \ldots y_m^{-1}$. If $\langle x_{k+1}, y_{k+1}^{-1}\rangle = 0$, then by Lemma 3(a), $p(xy^{-1})=m+n-2k$, so $d(x,y)=k$, a contradiction. Therefore $\langle x_{k+1}, y_{k+1}^{-1}\rangle$ is greater than zero, that is, $p(x_{k+1}y_{k+1}^{-1}) \leqq 1$. If $n>k+1$, then $\langle x_{k+2}, x_{k+1}\rangle = 0$ by Lemma 3(a), so by A7, $\langle x_{k+2}, x_{k+1}y_{k+1}^{-1}\rangle = 0$. Similarly, if $m>k+1$, then $\langle x_{k+1}y_{k+1}^{-1}, y_{k+2}\rangle = 0$. It follows from Lemma 3(a) that, if $p(x_{k+1}y_{k+1}^{-1})=1$, then $p(xy^{-1})=n+m-2k-1$, so $d(x,y)=k+\frac{1}{2}$, a contradiction. Hence $p(x_{k+1}y_{k+1}^{-1})=0$. If $m=k+1$, define $a=x_{k+1}y_{k+1}^{-1}$, so $y=ax_{k+1}\ldots x_1$, and if $m>k+1$, define $y'_m=y_m$, $y'_{m-1}=y_{m-1}, \ldots, y'_{k+2}=y_{k+2}(y_{k+1}x_{k+1}^{-1})$. Then $p(y'_{k+2})=1$ by Lemma 2, clearly $y=y'_m \ldots y'_{k+2}x_{k+1}\ldots x_1$, and this is a reduced decomposition since $p(y)=m$.

Proposition 6. *If p is the semigauge associated to a pregroup P, the following are equivalent:*

(1) P satisfies *P6;*

(2) *p satisfies A7;*

(3) *p satisfies A4.*

Proof. The equivalence of (1) and (2) is asserted in Proposition 5, and (3) implies (2) by Lemma 4, so it remains to show that (2) implies (3). Assume A7, and take x,y,z in $U(P)$. We have to show that $d(x,y)\geqq m$ and $d(x,z)\geqq m$ implies that $d(y,z)\geqq m$. Since $d(x,y)$ is always a non-negative integer or half-integer, we can assume m is also a non-negative integer or half-integer. If any of x,y,z are in Ker(p) then the desired conclusion follows from Lemma 2 (m must be zero) so we assume $p(x), p(y), p(z)$ are all non-zero.

Case 1: $m=k\in\mathbb{Z}$. Replacing y by cy for some $c\in\text{Ker}(p)$ if necessary, and similarly changing z (which by Lemma 2 does not change $d(x,y)$, $d(x,z)$ or $d(y,z)$), we can use Lemma 5 to write reduced decompositions

$$x=x_r\ldots x_1$$

$$y=y_s\ldots y_{k+1}\,x_k\ldots x_1$$

$$z=z_t\ldots z_{k+1}\,x_k\ldots x_1$$

where $0\leqq k\leqq s,t$. Then $p(yz^{-1})=p(y_s\ldots y_{k+1}z_{k+1}^{-1}\ldots z_t)\leqq s+t-2k$, so $d(y,z)\geqq k$.

Case 2: $m=k+\frac{1}{2}, k\in\mathbb{Z}$. Again we may write reduced decompositions as in Case 1. (Notice that, in this case, $s=p(y)\geqq d(x,y)\geqq m$, so $s\geqq k+1$, and similarly $t\geqq k+1$). Now $xy^{-1}=x_r\ldots x_{k+1}\,y_{k+1}^{-1}\ldots y_s^{-1}$, and if $d(x_{k+1},y_{k+1})=0$ then $p(xy^{-1})=r+s-2k$ by Lemma 3(a), so $d(x,y)=k$, a contradiction. Hence $d(x_{k+1},y_{k+1})>0$, and similarly $d(x_{k+1},z_{k+1})>0$.

Suppose that $d(y_{k+1},z_{k+1})=0$, that is, $\langle y_{k+1}, z_{k+1}^{-1}\rangle=0$. Since $d(x_{k+1},z_{k+1})>0$, $p(z_{k+1}\,x_{k+1}^{-1})\leqq 1$. By A7 (with $a=y_{k+1}, b=z_{k+1}^{-1}$, $c=z_{k+1}x_{k+1}^{-1}$), $\langle y_{k+1}, x_{k+1}^{-1}\rangle=0$, that is, $d(y_{k+1},x_{k+1})=d(x_{k+1},y_{k+1})=0$, a contradiction. Hence $d(y_{k+1},z_{k+1})>0$, and so

$p(y_{k+1}z_{k+1}^{-1}) \leqq 1$. Since $yz^{-1} = y_s \dots y_{k+2}(y_{k+1}z_{k+1}^{-1})z_{k+2}^{-1} \dots z_t^{-1}$, we have $p(yz^{-1}) \leqq s+t-2k-1$ (see the remark after Lemma 4), hence $d(y,z) \geqq k+\frac{1}{2}$.

Proposition 6 gives a negative answer to Question B1 (p. 372) in [8], since we have shown that there exist pregroups not satisfying P6. On the other hand, given a group with a length function satisfying N1*, there is, by Propositions 3, 4, 5 and Lemma 4, a corresponding pregroup satisfying P6. Examples of such length functions are the usual length functions on free groups, free products and HNN extensions (see [1] and [2]), giving examples of pregroups satisfying P6.

We conclude by giving some examples to show there is no further logical dependence between axioms A6, A7 and A4 for a semigauge.

First, note that, if X is a metric space with an integer-valued metric δ, and G is a group acting on X as isometries, then choosing $v \in X$ and defining (for $g \in G$):

$$p(g) = \delta(v, gv)$$

gives a $\mathbb{Z}$-semigauge on G. This applies when X is the vertex set of a connected graph Γ, G is acting as graph automorphisms on Γ and distance in X is given by:

$$\delta(x, y) = \text{length of a shortest path from } x \text{ to } y \text{ in } \Gamma.$$

(We view a path as a sequence of edges, $(e_1, \dots, e_n)$ and the length of such a path is n). Note that $\langle g,h \rangle = 0$ means that gv lies on a shortest path from v to ghv in Γ (where $g, h \in G$).

We give an example of a $\mathbb{Z}$-semigauge which, by contrast with Proposition 4, satisfies A7 but not A6. Let Γ be the graph indicated in Figure 1, choose a vertex v as indicated and take $G = \mathrm{Aut}(\Gamma)$. Thus, for $g \in G$,

$$p(g) = \text{graph distance from } v \text{ to } gv.$$

Let a, b, c be elements of G taking v to the indicated vertices (with the embedding of Γ in the plane suggested by Figure 1, a, b, c can be accomplished by obvious Euclidean transformations, in particular, b by reflection in the vertical dotted line and a by a translation).

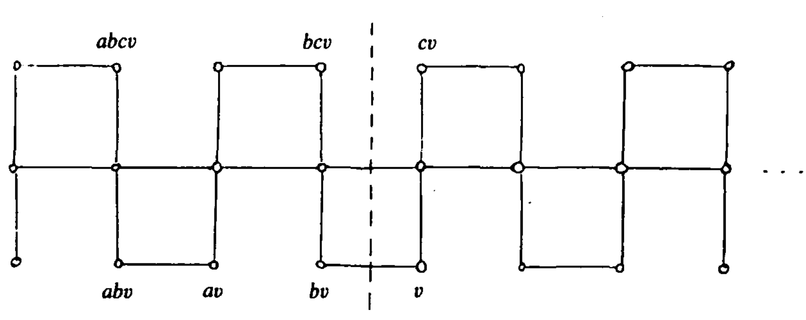

FIGURE 1

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Clearly $p(a)=4$, $p(b)=1$, $p(c)=2$, $p(bc)=3$, $p(ab)=5$ and $p(abc)=5$. Hence $\langle a,b\rangle=\langle b,c\rangle=0$ but $\langle ab,c\rangle=1$, so A6 fails. Now the stabilizer of v is trivial (by comparing degrees of vertices adjacent to v) from which one can deduce that the following relations hold in G:

$$b^2=c^2=1 \quad \text{and} \quad a=(bc)^2.$$

It follows that G is the infinite dihedral group $\langle b\rangle * \langle c\rangle$ with p given by:

$$p(b)=1$$

$$p((bc)^n)=p((cb)^n)=n+2 \quad (n>0)$$

$$p(c(bc)^n)=n+2 \quad (n\geqq 0)$$

$$p(b(cb)^n)=n+3 \quad (n>0)$$

and, of course, $p(1)=0$. One can take these equations as a definition of p on the infinite dihedral group and check directly that p is a semigauge on G. If one does so, the following facts emerge:

(1) if g,h are in $G\backslash\{1\}$ and $\langle g,h\rangle=0$, then either $g=b$ or $h=b$;

(2) $\langle b,g\rangle=0$ if and only if either $g=(cb)^n$ for some $n\geqq 0$ or $g=c(bc)^n$ for some $n\geqq 0$.

We leave the geometric interpretation of (1) and (2) to the reader; using (1) and (2), it is routine to verify that A7 holds.

By Proposition 6, any semigauge associated with a pregroup not satisfying P6 gives an example of a $\mathbb{Z}$-semigauge satisfying A6 and N1*, but not A7. For an example of a $\mathbb{Z}$-semigauge satisfying neither A6 nor A7, take $G=\mathbb{Z}\times\mathbb{Z}$ and Γ to be the Cayley diagram of G with respect to the obvious generators $(1,0)$ and $(0,1)$ for G, to obtain the semigauge given as an example in [6], Example 4.2 (3), namely $p(m,n)=|m|+|n|$ $(m,n\in\mathbb{Z})$. To see that A6 and A7 fail, take $a=(1,0)$, $b=(0,1)$, and $c=(-1,0)$.

Finally, in part 2 of Example 4.2 in [6], there is an example of a $\mathbb{Z}$-semigauge satisfying A6 and A7, but not A4. Essentially, we may take $G=\mathbb{Z}$ and define p by $p(x)=2$ if x is odd, $p(x)=3$ if x is even, and $p(0)=0$. Then p is a trivial semigauge in the sense of [6], Example 2.4 (1), so is regular, hence satisfies A6 and A7 by [6], Lemma (4.3) (a) and (b). However, $d(1,3)=\frac{1}{2}$, $d(1,2)=d(2,3)=\frac{3}{2}$, so A4 fails. We give yet another example of a trivial semigauge not satisfying A4.

Let Γ be the graph indicated in Figure 2, and let $G=\text{Aut}(\Gamma)$. The element a interchanges v and av and leaves all other vertices fixed, c is the obvious translation, and $b=c^{-2}a$.

Let p be the semigauge determined by the basepoint v. Then it is easy to calculate that $d(a,b)=d(a,c)=1$, while $d(b,c)=\frac{1}{2}$, so A4 fails. If $g\in G$, any shortest path from v to gv passes through just two vertices in the G-orbit of v, namely v and gv. Thus if $\langle g,h\rangle=0$, which means gv is on a shortest path from v to ghv, either $gv=v$ or $gv=ghv$, that is, either $p(g)=0$ or $p(h)=0$. It follows that p is trivial, as claimed.

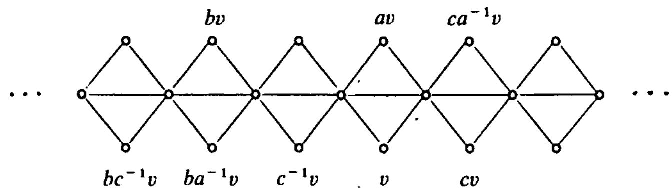

FIGURE 2

To obtain non-trivial semigauges satisfying A6 and A7 but not A4, we can take free products of the examples just given, using [**6**], Theorem 4.8 and the fact that a regular semigauge satisfies A6 and A7 ([**6**], Lemma (4.3)(a) and (b)).

REFERENCES

1. I. M. Chiswell, Embedding theorems for groups with an integer-valued length function, *Math. Proc. Cambridge Philos. Soc.* **85** (1979), 417–429.

2. I. M. Chiswell, Length functions and free products with amalgamation of groups, *Proc. London Math. Soc.* **42** (1981), 42–58.

3. B. Hurley, On length functions and normal forms in groups, *Math. Proc. Cambridge Philos. Soc.* **84** (1978), 455–464.

4. R. C. Lyndon, Length functions in groups, *Math. Scand.* **12** (1963), 209–234.

5. F. H. Nesayef, *Groups generated by elements of length zero and one* (Ph.D. thesis, University of Birmingham , 1983).

6. D. Promislow, Equivalence classes of length functions on groups, *Proc. London Math. Soc.* **51** (1985), 449–477.

7. J. R. Stallings, *Group theory and three-dimensional manifolds* (New Haven and London, Yale University Press, 1971).

8. C. T. C. Wall (ed.), *Homological group theory* (London Mathematical Society Lecture Note Series **36**, Cambridge University Press, 1979).

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