A CLASS OF FUNCTION ALGEBRAS

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Introduction. A problem which has generated considerable interest during the past couple of decades is that of characterizing abstractly systems of real-valued continuous functions with various algebraic or topological-algebraic structures. With few exceptions known characterizations are of systems of bounded continuous functions on compact or locally compact spaces. Only recently have characterizations been given of the systems $C(X)$ of all real-valued continuous functions on an arbitrary completely regular space X **(1)**. One of the main objects of this paper is to provide, by using certain special techniques, a characterization of $C(X)$ for a particular class of (not necessarily compact) completely regular spaces.

Generally speaking, one of the primary difficulties in characterizing all of $C(X)$ is that of obtaining conditions which insure that a subsystem is, in fact, all of $C(X)$. Sets of conditions of two different types have evolved. The first, for X compact, uses the completeness of $C(X)$ in its usual norm and the Stone-Weierstrass Theorem. (For example, see **(10)** and **(13)**.) The second uses the fact that $C(X)$ is, in a sense, maximal in a certain class of algebraic systems (cf. **(1, 6)**. The first of these appears to be applicable only in situations where $C(X)$ possesses a norm or a suitable family of pseudo-norms. The second, although it applies in more general situations and is algebraic in nature, has the slight drawback of the "external" character of the maximality condition.

In this paper we characterize $C(X)$ as a vector lattice, as an l-ring, and as an algebra[1] for the case in which X is a P-space **(7)**. A feature of special interest in these characterizations is that we appeal to neither of the aforementioned methods for obtaining all of $C(X)$; rather we use, for X a P-space, a simple property of certain "fixed" subsets of $C(X)$. En route to obtaining these results we also characterize $M(X, \mathfrak{B})$, the set of all real-valued measurable functions on a total measurable space, as a vector lattice and as an l-ring.

In two recent papers, Brainerd (**(4)** and **(5)**) has also given characterizations of $C(X)$, X a P-space, and $M(X, \mathfrak{B})$ as l-algebras. The characterizations of $C(X)$ by Brainerd as an l-algebra and by us as an l-ring, although obtained independently, use essentially the same techniques.

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[1]For the theory of vector lattices and l-rings see Birkhoff **(2)**, Birkhoff and Pierce **(3)**, and Nakano **(11)**. Our notation will be that of **(2)** except that $\vee$ and $\wedge$ will be used to denote lattice join and meet, respectively. By an algebra we shall always mean an algebra over the real field.

1. Preliminaries. If X is a set, denote by $F(X)$ the set of all real-valued functions on X. If $\mathfrak{B}$ is a Boolean σ-algebra of subsets of X, then we say that the pair $(X, \mathfrak{B})$ is a *total measurable space* and denote by $M(X, \mathfrak{B})$ the set of all $f \in F(X)$ measurable $\mathfrak{B}$. If $\mathfrak{T}$ is a base for a topology on X, then we denote by $C(X, \mathfrak{T})$, or in unambiguous cases simply $C(X)$, the set of all $f \in F(X)$ continuous with respect to $\mathfrak{T}$.

For each $f \in F(X)$ set $Z(f) = \{x \in X; f(x) = 0\}$. A subset $Z \subseteq X$ is a measurable zero set in case $Z \in \mathfrak{B}$, or equivalently, in case $Z = Z(f)$ for some $f \in M(X, \mathfrak{B})$. A subset $Z \subseteq X$ is a *continuous zero set* in case $Z = Z(f)$ for some $f \in C(X)$.

A subset $I \subseteq F(X)$ is *fixed* in case $\cap \{Z(f); f \in I\}$, also written $\cap Z(I)$, is non-empty. Let $A \subseteq F(X)$. A set $I \subseteq A$ is a maximal fixed subset of A if and only if $I = \{f \in A; f(x) = 0\}$ for some $x \in X$. In general, different points in X do not give rise to different maximal fixed subsets of A; if, however, A separates points (that is, $x \neq y$ in X implies $0 = f(x) \neq f(y)$ for some $f \in A$), then the mapping $I \to \cap Z(I)$ is one-one from the maximal fixed subsets of A onto X.

Let $A \subseteq F(X)$. Then for each $I \subseteq A$, set

$$\mathfrak{a}(I) = \{f \in A; X - \cap Z(I) \subseteq Z(f)\}.$$

Thus $f \in \mathfrak{a}(I)$ if and only if for every $x \in X$ and every $g \in I$, $f(x)g(x) = 0$. We say that $I \subseteq A$ is *Z-convex* (in A) provided that

$$I = \{f \in A; \cap Z(I) \subseteq Z(f)\}.$$

It is clear then that for each $I \subseteq A$, if $I = \mathfrak{a}(\mathfrak{a}(I))$, then I is Z-convex. The converse in general is false; for example, every maximal fixed subset I of A is Z-convex, but it need not satisfy $I = \mathfrak{a}(\mathfrak{a}(I))$. If $\mathscr{S}$ is the collection of maximal fixed subsets of A, then it is clear that $I \subseteq A$ is Z-convex if and only if $I = \cap \{N \in \mathscr{S}; I \subseteq N\}$.

A topological space X $(= (X, \mathfrak{T}))$ is a *P-space* **(7)** provided that X is completely regular and that every G_δ-set in X is open. In such a space X the family of continuous zero sets of X is an open base for the topology, is a Boolean σ-algebra of subsets of X, and coincides with the family of closed-open subsets of X. Conversely, if $(X, \mathfrak{B})$ is a total measurable space which separates points of X (that is, $x \neq y$ in X implies $x \in E$ and $y \notin E$ for some $E \in \mathfrak{B}$), then $\mathfrak{B}$ is an open base for a topology on X relative to which X is a P-space. Moreover, it is clear that in this case $M(X, \mathfrak{B}) \subseteq C(X, \mathfrak{B})$. We now prove a test for equality.[2]

LEMMA 1.1. *Let X be a P-space, let $\mathfrak{B}$, a Boolean σ-algebra of subsets of X, be an open base for the topology of X, and let $\mathscr{S}$ be the set of maximal fixed subsets of $M(X, \mathfrak{B})$. Then $M(X, \mathfrak{B}) = C(X, \mathfrak{B})$ if and only if for every Z-convex set $I \subseteq M(X, \mathfrak{B})$, if $N \in \mathscr{S}$ implies $I \not\subseteq N$ or $\mathfrak{a}(I) \not\subseteq N$, then $I = \mathfrak{a}(f)$ for some $f \in M(X, \mathfrak{B})$.*

[2]See also **(5**, Theorem 1**)** for a variation of this result.

Proof. Let $I \subseteq M(X, \mathfrak{B})$ be Z-zoncvx. We shall prove first that the two conditions

(1) $I \not\subseteq N$ or $\mathfrak{a}(I) \not\subseteq N$ for all $N \in \mathcal{S}$;

(2) $\bigcap Z(I)$ is a continuous zero set;

are equivalent. Assume (1). Then $F_1 = \bigcap Z(I)$ and $F_2 = \bigcap Z(\mathfrak{a}(I))$ are disjoint. For let $x \in X$ and let

$$N_x = \{f \in M(X, \mathfrak{B}); f(x) = 0\}.$$

Since N_x is Z-convex, we have $x \in F_1$ if and only if $I \subseteq N_x$, and $x \in F_2$ if and only if $\mathfrak{a}(I) \subseteq N_x$. Thus, by (1), $F_1 \cap F_2 = \phi$. Since $M(X, \mathfrak{B}) \subseteq C(X, \mathfrak{B})$, we conclude that F_1 and F_2 are closed. Since $\mathfrak{B}$ is an open base, if $x \in X$, then $\{x\} = \bigcap Z(N_x)$. Therefore if $x \notin F_1$ and if $f \in M(X, \mathfrak{B})$, then $\{x\} = \bigcap Z(N_x) \subseteq X - F_1 \subseteq Z(f)$ implies $f \in N_x$. That is, $\mathfrak{a}(I) \subseteq N_x$, so that $x \in F_2$. Hence $X = F_1 \cup F_2$. We have then that F_1 is both closed and open, and therefore $F_1 = \bigcap Z(I)$ is a continuous zero set. Conversely, assume (2). Then since X is a P-space, $F = \bigcap Z(I)$ is closed and open. Since $\mathfrak{B}$ is an open base, if $x \in F$, then there is an $f \in M(X, \mathfrak{B})$ such that $f(x) \neq 0$ and $X - F \subseteq Z(f)$; that is, $f \in \mathfrak{a}(I)$ and $f \notin N_x$. Hence $I \subseteq N_x$ implies $\mathfrak{a}(I) \not\subseteq N_x$. Thus (1) and (2) are equivalent.

We now easily prove the "only if" portion of the lemma. For suppose that $M(X, \mathfrak{B}) = C(X, \mathfrak{B})$ and that $I \subseteq M(X, \mathfrak{B})$ satisfies (1). Then $F = \bigcap Z(I)$ is closed and open so that the characteristic function f of F is in $M(X, \mathfrak{B})$. It is evident then that $I = \mathfrak{a}(1 - f)$.

Conversely, let $g \in C(X, \mathfrak{B})$ and let α be a real number. Set $Z = \{x \in X; g(x) \geqq \alpha\}$. Then $Z = Z((\alpha - g) \vee 0)$ is a continuous zero set. Let

$$I = \{f \in M(X, \mathfrak{B}); Z \subseteq Z(f)\}.$$

Then I is Z-convex and $\bigcap Z(I) = Z$. Therefore I satisfies (2) and hence (1). Thus, if $M(X, \mathfrak{B})$ satisfies the condition of the lemma, $I = \mathfrak{a}(f)$ for some $f \in M(X, \mathfrak{B})$. We claim that $Z = X - Z(f)$. Certainly $X - Z(f) \subseteq Z$. Suppose then that $x \in Z(f)$. Since $f \in M(X, \mathfrak{B}) \subseteq C(X, \mathfrak{B})$, $Z(f)$ is a continuous zero set, and therefore, since X is a P-space, $Z(f)$ is open. Now $\mathfrak{B}$ is an open base, so there is an $h \in M(X, \mathfrak{B})$ such that $h(x) \neq 0$ and $X - Z(f) \subseteq Z(h)$. Then $h \in I$ and $Z = \bigcap Z(I) \subseteq Z(h)$. Hence $x \notin Z$, and we have the desired reverse inclusion $X - Z(f) \supseteq Z$. Now $Z(f)$ is measurable since $f \in M(X, \mathfrak{B})$, and therefore its complement Z is measurable. Consequently, since α was arbitrary, we conclude that g is measurable $\mathfrak{B}$ and hence that $g \in M(X, \mathfrak{B})$. Thus $M(X, \mathfrak{B}) = C(X, \mathfrak{B})$ and the lemma is proved.

2. Vector lattices of functions. In this section we characterize $M(X, \mathfrak{B})$ and $C(X, \mathfrak{T})$ abstractly as vector lattices where $(X, \mathfrak{B})$ is a total measurable space and $(X, \mathfrak{T})$ is a P-space.

Let A be a vector lattice. For $f, g \in A$ we write $f \perp g$ in case $|f| \wedge |g| = 0$. A countable set $\{f_n\}$ of elements of A is a $\sigma^{\perp}$-*set* in case $f_n \geqq 0$ $(n = 1, 2 \ldots)$

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and for each $n \neq m$, $f_n \perp f_m$. We say that A is *$\sigma^{\perp}$-complete* in case every $\sigma^{\perp}$-set $\{f_n\}$ in A has a least upper bound, $\vee_n f_n$, in A.[3]

LEMMA 2.1. *Let A be a vector sublattice of $F(X)$ which separates points of X and contains the constant function 1. Then $A = M(X, \mathfrak{B})$ for some point separating σ-algebra $\mathfrak{B}$ of subsets of X if and only if A is $\sigma^{\perp}$-complete and σ-complete.*

Proof. The necessity of these conditions follows readily from the fact that if $A = M(X, \mathfrak{B})$, then the desired countable spurema are simply the "pointwise" suprema.

Conversely, let A satisfy the stated conditions. If $\{f_n\} \subseteq A$ with $f = \vee_n f_n \in A$, then we claim that $f(x) = \vee_n [f_n(x)]$ for each $x \in X$. For suppose, on the contrary, that there is an $x \in X$ with $f(x) > \vee_n [f_n(x)]$. Without loss of generality, we may assume that, for all n, $0 < f_n \leqq f_{n+1} < 1$ and $f_n(x) = 0$, and that $f(x) = 1$. Now define sequences $\{g_n\}$, $\{h_n\}$, and $\{e_n\}$ in A by

$$g_1 = 2f_1 \wedge 1 \quad \text{and} \quad g_n = 2(f_n \vee g_{n-1}) \wedge 1 \quad \text{for} \quad n > 1;$$
$$h_n = (2g_n - g_{n+1})^+;$$

and

$$e_1 = h_1, \quad e_2 = 2h_2, \quad \text{and} \quad e_n = n(h_n - h_{n-2}) \quad \text{for} \quad n > 2.$$

Also, for each n, set

$$Y_n = \{y \in X;\, g_n(y) = 1\}.$$

Then one easily shows that, for each n, $0 \leqq h_n \leqq h_{n+1} \leqq 1$, $h_n(Y_n) = 1$, and $h_n(X - Y_{n+1}) = 0$. From these it follows that $0 \leqq e_n \leqq n$, $e_n(x) = 0$, $e_n(Y_n - Y_{n-1}) = n$, and

$$X - Z(e_n) \subseteq Y_{n+1} - Y_{n-2},$$

where $Y_{-1} = Y_0 = \phi$. This implies that if $|m - n| > 2$, then $e_m \perp e_n$; hence each of the sets $\{e_{3n}\}$, $\{e_{3n-1}\}$, and $\{e_{3n-2}\}$ is a $\sigma^{\perp}$-set in A. Therefore, since A is $\sigma^{\perp}$-complete,

$$e = \bigvee_{i=0}^{2} \left(\bigvee_{n=1}^{\infty} e_{3n-i} \right) = \bigvee_{n=1}^{\infty} e_n$$

is in A. Now if $f_n(y) > 0$, then $2^k f_n(y) \geqq 1$ for some k; therefore, since

$$g_{n+k}(y) \geqq [2^k f_n(y)] \wedge 1 = 1,$$

we have $y \in Y_{n+k}$. That is, if

$$P = \bigcup_{n=1}^{\infty} (X - Z(f_n)),$$

then

$$P \subseteq \bigcup_{n=1}^{\infty} Y_n = \bigcup_{n=0}^{\infty} (Y_n - Y_{n-1}).$$

[3]Other, possibly less descriptive, terminology for this notion includes σ-full **(2)** and complete **(11)**.

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Thus we have that $e \geqq 1$ on P, and consequently, that $f \leqq e$ on X. Hence there is an integer $k \geqq 2$ such that $e(x) \leqq k - 1$. Set

$$e' = \left(\bigvee_{i=1}^{k-1} ke_i\right) \vee e.$$

Then $e'(y) \geqq k$ for all $y \in P$ and $e'(x) \leqq e(x) \leqq k$. Therefore $(e' - k + 1)^+ \geqq 1$ on P and, as a result, $f \leqq (e' - k + 1)^+$. This is a contradiction since $(e' - k + 1)^+(x) = 0$. We conclude then that $f(x) = 0$, and therefore countable suprema in A, when defined, are defined pointwise.

For each $f \in A$, set $e_f = \vee_n (|nf| \wedge 1)$; then, by the result of the preceding paragraph, e_f is the characteristic function of $X - Z(f)$. Thus A contains e_f and $1 - e_z$ the characteristic functions of $X - Z(f)$ and $Z(f)$, respectively. Now let $\mathfrak{B} = \{Z(f); f \in A\}$. Then $\mathfrak{B}$ is an algebra of subsets of X; for $Z(f) \cup Z(g) = Z(|f| \wedge |g|)$ and $X - Z(f) = Z(1 - e_f)$. Since A is point separating, it is clear that $\mathfrak{B}$ also is point separating. Moreover, $\mathfrak{B}$ is a σ-algebra; for, using the result of the first paragraph and the σ-completeness of A, we have

$$\cap_n Z(f_n) = \cap_n Z(e_{f_n}) = Z(\vee_n e_{f_n}) \in \mathfrak{B}.$$

We show next that $A \subseteq M(X, \mathfrak{B})$. Let $f \in A$ and let α be real. Then

$$\{x \in X; f(x) \geqq \alpha\} = Z((\alpha - f)^+) \in \mathfrak{B},$$

so that $f \in M(X, \mathfrak{B})$. On the other hand, A contains all measurable characteristic functions, and so, since A is σ-complete, A contains all bounded $f \in M(X, \mathfrak{B})$. (Cf. (**8**, Theorem 20.B).) To complete the proof we need only show that A contains all non-negative $f \in M(X, \mathfrak{B})$. So let $f \geqq 0$ in $M(X, \mathfrak{B})$. For each $n = 1, 2, \ldots$, set

$$E_n = \{x \in X; n - 1 \leqq f(x) < n\}$$

and let $f_n \in F(X)$ be defined by $f_n = f$ on E_n and $f_n = 0$ on $X - E_n$. Then obviously $f_n \in M(X, \mathfrak{B})$ and is bounded; hence $f_n \in A$ for all n. But $\{f_n\}$ is a $\sigma^\perp$-set, so that $f = \vee_n f_n \in A$. Thus the proof of the lemma is complete.

It is interesting to note that neither $\sigma^\perp$-completeness nor σ-completeness alone is adequate to insure that $A = M(X, \mathfrak{B})$. For example, if X is uncountable, then the set of all $f \in F(X)$ with $f(X)$ countable is a vector sublattice of $F(X)$ which is $\sigma^\perp$-complete but not σ-complete. Next let X be the Stone-Čech compactification of an infinite discrete space and let $A = C(X)$. Then A is a vector sublattice of $F(X)$ which is σ-complete but not $\sigma^\perp$-complete; in fact, there exist bounded sequences $\{f_n\}$ in A such that $Z(\vee_n f_n) \neq \cap_n Z(f_n)$.

Let A be a vector lattice. An element $e \in A$ is a *weak order unit* in case for all $f \in A$, $|f| \wedge |e| = 0$ implies $f = 0$. A subset $I \subseteq A$ is an *ideal* of A in case I is a linear subspace such that $f \in I$ and $|g| \leqq |f|$ implies that $g \in I$.

THEOREM 2.2. *A vector lattice A is isomorphic to the vector lattice $M(X, \mathfrak{B})$ for some total measurable space $(X, \mathfrak{B})$ if and only if A is $\sigma^\perp$-complete, σ-complete,* *has a weak order unit, and* $\bigcap \mathscr{S} = 0$ *where* $\mathscr{S}$ *is the set of maximal ideals of* A. *In fact, when* A *satisfies the stated conditions,* A *is isomorphic to* $M(\mathscr{S}, \mathfrak{B})$, *where* $\mathfrak{B}$ *is a point separating* σ*-algebra of subsets of* $\mathscr{S}$.

Proof. Since the family $\mathscr{F}$ of fixed maximal ideals (= maximal fixed ideals) of $M(X, \mathfrak{B})$ satisfies $\bigcap \mathscr{F} = 0$, the necessity of the conditions is obvious.

Conversely, let A satisfy the stated conditions. Let $e \in A$ be a weak order unit for A; we may assume that $e \geqq 0$. We claim that if $\mathscr{T} = \{N \in \mathscr{S}; e \notin N\}$, then $\bigcap \mathscr{T} = 0$. For if $f \in \bigcap \mathscr{T}$, then, for every $N \in \mathscr{S}$, either $f \in N$ or $e \in N$. Thus $(|f| \wedge e) \in \bigcap \mathscr{S}$ so that $|f| \wedge e = 0$. Since e is a weak order unit, this implies $f = 0$. That is, $\bigcap \mathscr{T} = 0$. By a familiar technique **(1)** we can define an isomorphism of A onto a point-separating vector sublattice A^* of $F(\mathscr{T})$ such that e is mapped onto the constant function 1. Appealing to Lemma 2.1 we have that $A^* = M(\mathscr{T}, \mathfrak{B})$ for some σ-algebra $\mathfrak{B}$ of subsets of $\mathscr{T}$.

To complete the proof it will suffice to show that $\mathscr{S} = \mathscr{T}$, and for this it will suffice to show that if $(X, \mathfrak{B})$ is a total measurable space, then no maximal ideal of $M(X, \mathfrak{B})$ contains 1. Suppose, on the contrary, that N is a maximal ideal of $M(X, \mathfrak{B})$ and that $1 \in N$. Then since N is proper, there is an $f \geqq 0$ with $f \notin N$. Since N is maximal and since $f^2 > f$ is in $M(X, \mathfrak{B})$, there is a real number α such that $f^2 - \alpha f \in N$. Let $\beta = \frac{1}{4}(\alpha + 1)^2$. Since $1 \in N$, it follows that β, and hence $f^2 - \alpha f + \beta$, belongs to N. But

$$f^2 - \alpha f + \beta = [f - \tfrac{1}{2}(\alpha + 1)]^2 + f \geqq f,$$

contrary to $f \notin N$. Thus the assumption $1 \in N$ is untenable and the proof is complete.

Let A be a vector lattice and let $I \subseteq A$. We set

$$I^{\perp} = \{f \in A; f \perp g \text{ for all } g \in I\}.$$

Then clearly, $I \subseteq I^{\perp\perp}$. If $\mathscr{S}$ is a family of ideals of A, then an ideal I of A is $\mathscr{S}$*-complemented* in case $I = I^{\perp\perp}$, and for each $N \in \mathscr{S}$, either $I \not\subseteq N$ or $I^{\perp} \not\subseteq N$.

THEOREM 2.3. *Let* A *be a vector lattice and let* $\mathscr{S}$ *be the set of all maximal ideals of* A. *Then* A *is isomorphic to the vector lattice* $C(X)$ *for some completely regular* P*-space* X *if and only if* A *is* $\sigma^{\perp}$*-complete,* σ*-complete,* $\bigcap \mathscr{S} = 0$, *and for each* $\mathscr{S}$*-complemented ideal* I *of* A, $I = \{f\}^{\perp}$ *for some* $f \in A$.

Proof. To prove the necessity we may assume that X is a Q-space **(9)**; for if X is a P-space, then so is vX, and, of course, $C(X)$ and $C(vX)$ are isomorphic. With this assumption the maximal ideals of $C(X)$ coincide with the maximal fixed subsets of $C(X)$. Moreover, if $I \subseteq C(X)$, then $I^{\perp}$ coincides with the set $\mathfrak{a}(I)$ defined in § 1. These observations combine with Lemma 1.1 and Theorem 2.2 to establish the necessity of the conditions in the present theorem.

Conversely, let A satisfy the stated conditions. Since the zero ideal of A is clearly $\mathscr{S}$-complemented, it follows that A has a weak order unit. Therefore, by Theorem 2.2, A is isomorphic to $M(X, \mathfrak{B})$ for some total measurable space $(X, \mathfrak{B})$ where, in fact, the maximal ideals $\mathscr{S}$ correspond to the maximal fixed ideals of $M(X, \mathfrak{B})$. A Z-convex set I^* of $M(X, \mathfrak{B})$ is then the image of some $I = \bigcap \{N \in \mathscr{S}; I \subseteq N\}$ in A, and therefore is an ideal of $M(X, \mathfrak{B})$. Since we clearly have $\mathfrak{a}(I^*) = (I^*)^{\perp}$, it follows from Lemma 1.1 that $M(X, \mathfrak{B}) = C(X, \mathfrak{B})$, and the proof is complete.

3. f-rings of functions. In this section we characterize $M(X, \mathfrak{B})$ and $C(X, \mathfrak{T})$, $(X, \mathfrak{B})$ and $(X, \mathfrak{T})$ as before, as f-rings. Although these characterizations still require σ-completeness, we are able to dispense with the full force of the $\sigma^{\perp}$-completeness requirement. In its place we use ring regularity and a condition of countbable character on certain ideals. These characterizations are slightly sharpened versions of those given in **(4** and **5)**.

Recall that an f-ring **(3)** is a lattice-ordered ring A with the property that for all $f, g, h \in A$, $f \wedge g = 0$ and $h \geqq 0$ together imply $hf \wedge g = fh \wedge g = 0$. Clearly $M(X, \mathfrak{B})$ and $C(X, \mathfrak{T})$ are f-rings.

A ring A is *regular* **(12)** in case for each $f \in A$, there is an $f' \in A$ such that $ff'f = f$. It is known **(7)** that a completely regular space X is a P-space if and only if $C(X)$, as a ring, is regular. Concerning regular f-rings we prove the following result which may be of independent interest.

LEMMA 3.1. *Let A be a regular f-ring. Then*

(1) *For all $f, g \in A$, $|f| \wedge |g| = 0$ if and only if $fg = 0$.*

(2) *If A has a weak order unit, A has an identity.*

Proof. Since A has no non-zero nilpotent elements, the l-radical of A is zero **(3)**. Therefore (1) follows from **(3**, Corollary 1, p. 57**)** and **(3**, Corollary 2, p. 63**)**. Next let $e \geqq 0$ be a weak order unit for A and let $ee'e = e$. Then, by (1), $f \wedge ee' = 0$ implies $fee'e = fe = 0$ which implies $|f| \wedge e = 0$ and thus $f = 0$. That is, the idempotent $e'' = ee'$ is also a weak order unit. Let $f \in A$; then $(fe'' - f)e'' = 0$ implies $|fe'' - f| \wedge e'' = 0$. Therefore, since e'' is a weak order unit, $fe'' = f$. Similarly, $e''f = f$, which establishes (2).

An ideal I of a ring A is σ-closed in case for every countable set $\{f_n\} \subseteq I$ there is an $f \in A$ with $ff_n = f_nf = f_n$ for all n.

THEOREM 3.2. *Let A be an f-ring and let $\mathscr{S}$ be the set of σ-closed maximal ring ideals of A. Then A is isomorphic to the f-ring $M(X, \mathfrak{B})$ for some total measurable space $(X, \mathfrak{B})$ if and only if A is regular, σ-complete, has a weak order unit, and $\bigcap \mathscr{S} = 0$. Moreover, if A satisfies these conditions, the space $(X, \mathfrak{B})$ and the isomorphism of A onto $M(X, \mathfrak{B})$ may be so chosen that the set $\mathscr{S}$ is mapped one-one onto the maximal fixed subsets of $M(X, \mathfrak{B})$.*

Proof. The necessity of the conditions is easily proved; we omit the details. Conversely, let A satisfy the stated conditions. Then, by Lemma 3.1, A has a ring identity e. Moreover, since A is σ-complete, it is Archimedean (**2**, p. 229), and therefore A is commutative (**3**, Theorem 13). Since the regular σ-complete subring of A generated by e is isomorphic to the ordered field R of real numbers, we may regard A as a regular f-algebra over R (that is, A is a regular F-ring in the sense of (**4**)). Now let $N \in \mathscr{S}$ and $\{f_n\} \subseteq N$ such that $\vee_n f_n \in A$. Since N is σ-closed, there is an $f \in A$ with $ff_n = f_n$ for all n. By the regularity of A we may assume that f is idempotent. Then (**11**, Theorem 25.1), $\vee_n f_n = \vee_n ff_n = f(\vee_n f_n) \in N$. Therefore A satisfies the conditions required in Brainerd's characterization (**4**, p. 682). Thus there exist a total measurable space $(X, \mathfrak{B})$ and an isomorphism of A onto $M(X, \mathfrak{B})$ with the desired properties.

Let A be a ring. For $I \subseteq A$, set $\mathfrak{a}(I) = \{f \in A; fg = 0 \text{ for all } g \in I\}$. In general $\mathfrak{a}(I)$ is a left ideal of A; if A is commutative or if A is a regular f-ring, then $\mathfrak{a}(I)$ is a two-sided ideal.[4] A left ideal I of A is $\mathfrak{a}$-*principal* in case $I = \mathfrak{a}(f)$ for some $f \in A$.

If $\mathscr{S}$ is a family of ideals of a ring A, then an ideal I of A is $\mathscr{S}$-*complemented* in case $I = \mathfrak{a}(\mathfrak{a}(I))$ and for each $N \in \mathscr{S}$, either $I \not\subseteq N$ or $\mathfrak{a}(I) \not\subseteq N$.

THEOREM 3.3. *Let A be an f-ring and let $\mathscr{S}$ be the set of σ-closed maximal ring ideals of A. Then A is isomorphic to the f-ring $C(X)$ for some P-space X if and only if A is regular, σ-complete, $\bigcap \mathscr{S} = 0$, and every $\mathscr{S}$-complemented ideal of A is $\mathfrak{a}$-principal.*

Proof. To prove the necessity, we may assume that X is a Q-space. Then an application of Theorem 3.2 and Lemma 1.1 completes this portion of the proof.

Conversely, since the zero ideal of A is $\mathscr{S}$-complemented, it is $\mathfrak{a}$-principal. But from $\{0\} = \mathfrak{a}(f)$ and Lemma 3.1 we conclude that $|f|$ is a weak order unit for A. Therefore, by Theorem 3.2 and Lemma 1.1, we have that A is isomorphic to $M(X, \mathfrak{B})$ and that $M(X, \mathfrak{B}) = C(X, \mathfrak{B})$ where $(X, \mathfrak{B})$ is a P-space.

4. The algebra $C(X)$.

With no assumptions concerning order properties it seems to be difficult to obtain a reasonably simple characterization of the algebras $M(X, \mathfrak{B})$. It is possible, however, to characterize the algebra $C(X)$, X a P-space, and it is the object of this section to present such a characterization.

Let A be a ring and let $\mathscr{S}$ be a family of ideals of A. A set $\{f_\alpha\} \subseteq A$ is a *discrete $\mathscr{S}$-cover* in case $\alpha \neq \beta$ implies $f_\alpha f_\beta = 0$ and the set $\{f_\alpha\}$ is contained in no member of $\mathscr{S}$. We say that A is *$\mathscr{S}$-regular* in case for each discrete $\mathscr{S}$-cover $\{f_\alpha\}$ in A there is an $f \in A$ such that $f_\alpha f f_\alpha = f_\alpha$ for all α.

The condition of $\mathscr{S}$-regularity provides the means by which we avoid order assumptions in the characterizations of $C(X)$. In general, however, it is not

[4] If A is a subring of $F(X)$, then $\mathfrak{a}(I)$ as defined here coincides with $\mathfrak{a}(I)$ as defined in §1.

suitable for a characterization of $M(X, \mathfrak{B})$. For example, let X be uncountable and let $\mathfrak{B}$ be the algebra of countable sets and their complements. If $\mathscr{S}$ is the set of maximal fixed ideals of the algebra $M(X, \mathfrak{B})$, then $M(X, \mathfrak{B})$ is not $\mathscr{S}$-regular. In fact, there is no algebra $M(X, \mathfrak{B}) \subseteq A \subseteq F(X)$ other than $F(X)$ itself which is $\mathscr{S}$-regular relative to its set $\mathscr{S}$ of maximal fixed ideals.

THEOREM 4.1. *Let A be an algebra and let $\mathscr{S}$ be the family of σ-closed real ideals*[5] *of A. Then A is isomorphic to the algebra $C(X)$ for some P-space X if and only if A is $\mathscr{S}$-regular, $\bigcap \mathscr{S} = 0$, and each $\mathscr{S}$-complemented ideal of A is $\mathfrak{a}$-principal.*

Proof. Let X be a P-space. Again we may assume that X is also a Q-space; hence every real ideal of $C(X)$ is fixed. As before one easily proves that each such ideal is σ-closed. Thus, clearly, $\bigcap \mathscr{S} = 0$. If $\{f_\alpha\} \subseteq C(X)$ is a discrete $\mathscr{S}$-cover, then the family $\{X - Z(f_\alpha)\}$ is a disjoint open cover of X; hence $f = \Sigma_\alpha f_\alpha$ is in $C(X)$. Since $C(X)$ is regular, there is an $f' \in C(X)$ with $f'f^2 = f$. Now an obvious pointwise argument shows that $f'f_\alpha^2 = f_\alpha$ for each α; therefore $C(X)$ is $\mathscr{S}$-regular. That $C(X)$ satisfies the final condition follows from Lemma 1.1.

Conversely, let A satisfy the stated conditions. Then, as a subdirect sum of fields, A is commutative. Since the zero ideal of A is $\mathscr{S}$-complemented, there is an $f \in A$ such that $\{0\} = \mathfrak{a}(f)$. If $f \in N$ for some $N \in \mathscr{S}$, then, since N is σ-closed, there is a $g \in N$ with $fg = f$. Let $h \in A$; then $f(h - gh) = 0$. But $\{0\} = \mathfrak{a}(f)$, so we have $h = gh \in N$; that is, $A = N$. This contradiction shows that $\{f\}$ is a discrete $\mathscr{S}$-cover. Then since A is $\mathscr{S}$-regular, $f'f^2 = f$ for some $f' \in A$. Thus $\{0\} = \mathfrak{a}(e)$ for some idempotent e $(= f'f)$ in A; in fact, e is easily seen to be an identity for A. Therefore (cf. **(1)**) we may assume that A is (isomorphic to) a subalgebra of $C(X)$ for some completely regular space X and that (i) the maximal fixed subsets of A are the members of $\mathscr{S}$, and (ii) for each $x \in X$ and each neighbourhood U of x, there is an $f \in A$ such that $f(x) = 0$ and $f(y) \geqq 1$ for all $y \notin U$. It therefore remains to prove that X is a P-space and that $A = C(X)$. So let $U = \bigcap_n U_n$ be a G_δ-set in X, let $x \in U$, and let

$$N_x = \{f \in A; f(x) = 0\} \in \mathscr{S}.$$

By (ii), there is, for each n, an $f_n \in N_x$ such that $f_n(y) \geqq 1$ for all $y \notin U_n$. Since N_x is σ-closed, there is an $f \in N_x$ such that $ff_n = f_n$ for all n. It is clear that $f(x) = 0$ and that $f(y) = 1$ for all $y \notin U$. Consequently U is a neighbourhood of x. This establishes that X is a P-space.

Now let $Z \subseteq X$ be a continuous zero set; that is, Z is closed and open in X. For each $x \in X - Z$, there is an $f \in N_x$ such that $f(Z) = 1$. Therefore $g = 1 - f \in A$ and $g(x) = 1$ and $g(Z) = 0$. We have from this that $I = \{f \in A;$ $Z \subseteq Z(f)\}$ is Z-convex and $\bigcap Z(I) = Z$. Then with essentially the same argu-

[5]An ideal N of A is *real* if A/N is isomorphic to the real field.

ment as that used in the proof of Lemma 1.1, we conclude that I is $\mathscr{S}$-complemented. Therefore $I = \mathfrak{a}(f)$ for some $f \in A$; thus, using the fact that X is a P-space, it follows that $Z = Z(f)$ for some $f \in A$. Since $X - Z$ is also a continuous zero set, $X - Z = Z(g)$ for some $g \in A$. This clearly implies that $\{f, g\}$ is a discrete $\mathscr{S}$-cover; hence $f'g^2 = g$ for some $f' \in A$. Thus $(f'g)(Z) = 1$ and $(f'g)(X - Z) = 0$. We have proved then that A contains the characteristic function of each continuous zero set of X.

To complete the proof it will suffice to prove that A contains every strictly positive function in $C(X)$, for if $f \in C(X)$, then $f = [(f \vee 0) + 1] - [-(f \wedge 0) + 1]$. So let $f \in C(X)$ be strictly positive and for each positive real number α, set $Z_\alpha = \{x \in X; f(x) = \alpha\}$. Then each Z_α is a continuous zero set; let $e_\alpha \in A$ be the characteristic function of Z_α. Since $\{Z_\alpha\}$ is a disjoint cover of X, it follows that $\{\alpha e_\alpha\}$ is a discrete $\mathscr{S}$-cover in A. Therefore there is an $f' \in A$ such that $(\alpha e_\alpha)^2 f' = \alpha e_\alpha$ for each α. Then for each $x \in Z_\alpha$,

$$f'(x) = \alpha^{-1} = [f(x)]^{-1}.$$

Since $\{Z_\alpha\}$ covers X, it follows that $\{f'\}$ is a discrete $\mathscr{S}$-cover in A. Thus there is an $f'' \in A$ with $(f')^2 f'' = f'$. Clearly then $f'' = (f')^{-1} = f$ and $f \in A$ as desired.

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