ON THE INDEX OF A QUADRATIC FORM

Jonathan Wild, Prince Albert, Sask.

Given a vector space $V = \{x, y, \ldots\}$ over an arbitrary field. In V a symmetric bilinear form (x,y) is given. A subspace W is called totally isotropic [t.i.] if $(x,y) = 0$ for every pair x W, y W.

Let V_n and V_m be two t.i. subspaces of V; $n < m$. Lower indices always indicate dimensions . It is a well known and fundamental fact of analytic geometry that there exists a t.i. subspace W_m of V containing V_n [cf. Dieudonné: Les Groupes classiques, P. 18]. As no simple direct proof seems to be available, we propose to supply one.

We first consider the case that $V_n \cap V_m = 0$. Thus V_n and V_m span a subspace V_{n+m}. The vectors of V_{n+m} orthogonal to V_n form a subspace W.

Every vector of V_{n+m}, in particular every vector $x \in W$ permits a decomposition $x = y + z$; $y \in V_n$, $z \in V_m$. Suppose also $x' = y' + z' \in W$; $y' \in V_n$, $z' \in V_m$. Since V_n and V_m are t.i., we have $(y,y') = (z,z') = 0$. By the definition of W, $0 = (y,x') = (y,y' + z') = (y,y') + (y,z') = (y,z')$. Similarly $(y',z) = 0$. Hence

$$(x,x') = (y + z, y' + z') = (y,y') + (y,z') + (z,y') + (z,z')$$
$$= 0 + 0 + 0 + 0 = 0.$$

Thus W is t.i. As $\dim W \geqq m$ and $V_n \subset W$, this disposes of our special case.

Assume now $V_n \cap V_m = V_d$. Thus V_n and V_m permit direct decompositions $V_n = V_d \dotplus V_{n-d}$, $V_m = V_d \dotplus V_{m-d}$. From the above, there exists a t.i. subspace W_{m-d} satisfying

(1) $\quad V_{n-d} \subset W_{m-d} \subset V_{n-d} \dotplus V_{m-d}$.

Since $V_d \cap (V_{n-d} \dotplus V_{m-d}) = 0$, we also have $V_d \cap W_{m-d} = 0$ and

(2) $\quad V_n = V_d \dotplus V_{n-d} \subset V_d \dotplus W_{m-d} = W_m$.

Let $y \in V_d$, $z \in W_{m-d}$. By (1), $z = r + s$ where $r \in V_{n-d}$, $s \in V_{m-d}$. Since y and r [y and s] lie in the t.i. subspace V_n [V_m], we have $(y,z) = (y,r) + (y,s) = 0 + 0 = 0$. Thus V_d and W_{m-d} are orthogonal.

By (2), any two vectors x, x' of W_m permit decompositions $x = y + z$, $x' = y' + z'$ where $y, y' \in V_d$; $z, z' \in W_{m-d}$. From the above $(y,z') = (z,y') = 0$. Since V_d and W_{m-d} are t.i., we also have $(y,y') = (z,z') = 0$. Thus $(x,x') = (y,y') + (y,z') + (z,y') + (z,z') = 0$ and W_m is t.i.

https://doi.org/10.4153/CMB-1958-019-8 Published online by Cambridge University Press