J. Austral. Math. Soc. (Series A) **54** (1993), 1–19

ON FIVE WELL-KNOWN COMMUTATOR IDENTITIES

GRAHAM J. ELLIS

(Received 25 October 1990; revised 28 February 1991)

Communicated by H. Lausch

Abstract

We conjecture that five well-known identities universally satisfied by commutators in a group generate all such universal commutator identities. We use homological techniques to partially prove the conjecture.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 20 A 05; secondary 08 A 99.

0. Introduction

For elements x, y of a group we write ${}^{x}y = xyx^{-1}$ and $[x, y] = xyx^{-1}y^{-1}$. The following commutator identities are *universal* in the sense that they hold for any elements x, x', y, y', z of an arbitrary group:

(i) $[x, x] = 1$,
(ii) $[x, yy'] = [x, y]{}^{y}[x, y']$,
(iii) $[xx', y] = {}^{x}[x', y][x, y]$.
(iv) $[[y, x], {}^{x}z][[x, z], {}^{z}y][[z, y], {}^{y}x] = 1$,
(v) ${}^{z}[x, y] = [{}^{z}x, {}^{z}y]$.

In this article we ask: is any universal commutator identity a consequence of these five? We obtain a partial answer in the affirmative.

At first sight it might seem that either of the two papers [11] or [13] contain a complete answer to our question. Both these papers claim to give a set of commutator identities from which all universal commutator identities can be deduced. However, both papers use an identity essentially of the form

$$[[x, y], z] = [xyx^{-1}y^{-1}, z]$$

as one of the generating identities. Commutators of weight 3 have thus to be treated as commutators of weight 2. So these two papers should be seen as giving a set of identities which generate all universal identities between commutators of weight 2. In the present paper we interpret our question about identities (i)–(v) as: do these five identities generate all universal identities between commutators of weight n? For $n = 2$ and 3 we prove that they do. [7, Theorem 1.2] would seem to suggest that in fact they do for all n.

For any group P it is known that the quotients $\gamma_n(P)/\gamma_{n+1}(P)$ of the lower central series of P form a Lie algebra. (Here $\gamma_1(P) = P$, $\gamma_n(P) = [\gamma_{n-1}(P), P]$ and all Lie algebras are assumed to be over the ground ring $\mathbb{Z}$.) Another way to obtain a Lie algebra from P is to construct the free Lie algebra $L(P_{ab})$ on the abelian group P_{ab}. A theorem of Magnus and Witt says that if $P = F$ is a free group then these two Lie algebras are isomorphic. This theorem can be seen as saying that any identity modulo $\gamma_{n+1}(F)$ between commutators in $\gamma_n(F)$ (that is between commutators of weight n) is a consequence of the defining relations of a Lie algebra. These defining relations resemble identities (i)–(v).

One way of showing that identities (i)–(v) generate all universal identities between commutators of weight n is to give a non-abelian version of the Magnus-Witt isomorphism in which: the quotient $\gamma_n(F)/\gamma_{n+1}(F)$ is replaced by $\gamma_n(F)$; and the free Lie algebra $L(F_{ab})$ is replaced by the free "multiplicative Lie algebra" on F, where by a multiplicative Lie algebra we mean a (not necessarily abelian) multiplicative group with a binary bracket operation satisfying (i)–(v) (see below for a precise definition). We attempt to obtain such an isomorphism but only succeed in proving injectivity on $\gamma_n(F)$ for $n = 1, 2, 3$. The proof uses a new description (in terms of the non-abelian tensor product defined in [3]) of the triple Pontryagin product $H_1(P_{ab}) \times H_1(P_{ab}) \times H_1(P_{ab}) \to H_3(P_{ab})$ in integral group homology.

Multiplicative Lie algebras are interesting algebraic objects in their own right, and so this paper also contains various results about them which are of purely intrinsic interest.

Our main definitions and results are stated in Section 1. Most of the proofs are deferred to subsequent sections.

I would like to thank Daniel Conduché for helpful conversations about this work.

1. Definitions and main results

A *multiplicative Lie algebra* consists of a multiplicative group G together with a function $\{\ ,\ \}: G \times G \to G$, which we shall call the *lie product*, satisfying the following identities for all x, x', y, y', z in G:

(i)$'$ $\{x, x\} = 1$,

(ii)$'$ $\{x, yy'\} = \{x, y\}^y\{x, y'\}$,

(iii)$'$ $\{xx', y\} = {}^x\{x', y\}\{x, y\}$,

(iv)$'$ $\{\{y, x\}, {}^x z\}\{\{x, z\}, {}^z y\}\{\{z, y\}, {}^y x\} = 1$,

(v)$'$ ${}^z\{x, y\} = \{{}^z x, {}^z y\}$.

From (ii$'$) and (iii)$'$ we easily deduce

(vi) $\{1, x\} = \{x, 1\} = 1$.

We have, using (ii)$'$ and (iii)$'$ to expand $\{xy, xy\}$, and then applying (i)$'$, that

(vii) $\{x, y\} = \{y, x\}^{-1}$.

We have, using (ii)$'$ and (iii)$'$ to expand $\{xx', yy'\}$ in two different ways, that

(viii) ${}^{\{x,y\}}\{x', y'\} = {}^{[x,y]}\{x', y'\}$.

Another consequence of (ii)$'$ and (iii)$'$ is

(ix) $[\{x, y\}, x'] = \{[x, y], x'\}$.

Two consequences of (ii)$'$, (iii)$'$ and (vi) are

(x) $\{x^{-1}, y\} = {}^{x^{-1}}\{x, y\}^{-1}$ and $\{y, x^{-1}\} = {}^{x^{-1}}\{y, x\}^{-1}$

Note that (vi)–(x) do not depend on (iv)$'$; in fact (vi), (viii), (ix) and (x) do not even depend on (i)$'$. As an example let us assume (viii) and prove (ix):

$$\begin{aligned}
[\{x, y\}, x'] &= \{x, y\}^{x'}\{x, y\}^{-1} \\
&= \{x, y\}^{x'}\{xx^{-1}, y\}^{x'}\{x, y\}^{-1} \\
&= \{x, y\}^{x'x}\{x^{-1}, y\} \\
&= \{x, y\}\{x, x'\}^{-1\,xx'}\{x^{-1}, y\}\{x, x'\} \quad \text{by (viii)} \\
&= \{x, y\}\{xx^{-1}, x'\}\{x, x'\}^{-1xx'}\{x^{-1}, y\}\{x, x'\} \\
&= \{x, y\}^x\{x^{-1}, x'\}^{xx'}\{x^{-1}, y\}\{x, x'\} \\
&= \{x, y\}^x\{x^{-1}, x'y\}\{x, x'\} \\
&= \{x, y\}\{xx^{-1}, y\}^{-1\,x}\{x^{-1}, x'y\}\{x, x'\} \\
&= {}^x\{x^{-1}, y\}^{-1\,x}\{x^{-1}, yy^{-1}x'y\}\{x, x'\} \\
&= {}^{xy}\{x^{-1}, y^{-1}x'y\}\{x, x'\} \\
&= {}^x\{{}^y x^{-1}, x'\}\{x, x'\} \\
&= \{[x, y], x'\}.
\end{aligned}$$

https://doi.org/10.1017/S1446788700036934 Published online by Cambridge University Press

This calculation is, modulo notation, the same as that given for the proof of [3, Proposition 2.3(d)].

By imposing conditions on the multiplicative group G certain of the identities (i)$'$–(v)$'$ can be made redundant. As an example we give the following proposition. The proof, which is left to the reader, relies on identity (ix).

PROPOSITION 1. *If the group G is perfect then* (i)$'$ *and* (iv)$'$ *are consequences of* (ii)$'$, (iii)$'$ *and* (v)$'$.

There are three obvious examples of multiplicative Lie algebras.

EXAMPLE 1. Any group P is a multiplicative Lie algebra with $\{x, y\} = xyx^{-1}y^{-1}$ for all x, y in P.

EXAMPLE 2. Any group P can also be given the structure of a multiplicative Lie algebra by defining $\{x, y\} = 1$ for all x, y in P.

EXAMPLE 3. Any Lie algebra L over $\mathbb{Z}$ is a multiplicative Lie algebra with $\{x, y\}$ the ordinary Lie product for all x, y in L.

Unless otherwise stated, when considering a group P as a multiplicative Lie algebra we take the Lie bracket of Example 1.

A slightly less obvious example is the following.

EXAMPLE 4. Let $E \twoheadrightarrow P$ be a central extension of a group P. Then $x \in P$ acts on $u \in E$ by ${}^{x}u = \overline{x}u\overline{x}^{-1}$ where $\overline{x} \in E$ is any element in the preimage of x. Let $G = E \rtimes P$ be the semi-direct product, and define the Lie product $\{\ ,\ \}: G \times G \to G$ by $\{(u, x), (u', x')\} = ([u\overline{x}, u'\overline{x}'], 1)$.

Let G and G' be multiplicative Lie algebras. By a *map* $\phi: G \to G'$ we mean a group homomorphism such that $\phi\{x, y\} = \{\phi x, \phi y\}$ for all x, y in G. By the *kernel* of a map ϕ we just mean the kernel of ϕ considered as a group homomorphism.

A subgroup N of G will be a *subalgebra* if $\{x, y\} \in N$ for all x, y in N. It will be an *ideal* if it is a normal subgroup and if $\{x, y\} \in N$ for all x in N and y in G. It follows from (vii) that if N is an ideal then $\{y, x\} \in N$ for all x in N and y in G.

Clearly for any map $\phi: G \to G'$, the kernel $\ker(\phi)$ is an ideal of G. Conversely if N is an ideal of G, the quotient group G/N inherits the structure of a multiplicative Lie algebra, and we have a quotient map $G \to G/N$.

For any group P there exists the *free* multiplicative Lie algebra $L(P)$ on P which is characterised (up to isomorphism) by the following two properties:

P is a subgroup of $L(P)$;

any group homomorphism $P \to G$ from P to a multiplicative Lie algebra G extends uniquely to a map $L(P) \to G$.

https://doi.org/10.1017/S1446788700036934 Published online by Cambridge University Press

The existence of $L(P)$ poses no problems. Its construction is a straightforward generalisation of the construction of a free Lie algebra from a magma [15]. The crucial role played in the abelian case by bilinearity of the Lie bracket is played in the non-abelian case by identities $(ii)'$ and $(iii)'$.

PROPOSITION 2. *For any group P the free multiplicative Lie algebra $L(P_{ab})$ on the abelianised group P_{ab} is just the usual free Lie algebra over $\mathbb{Z}$ on P_{ab}.*

PROOF. Identities $(v)'$ and (viii) imply that the underlying group of $L(P_{ab})$ is abelian. Thus identities $(i)'$–$(v)'$ imply that $L(P_{ab})$ is a Lie algebra. The defining universal property of $L(P_{ab})$ therefore coincides with the universal property of the free Lie algebra over $\mathbb{Z}$ on P_{ab}.

In view of Proposition 2 (or in view of Example 3) the identity homomorphism $P_{ab} \to P_{ab}$ extends uniquely to a surjective map $\overline{\theta}$ from $L(P_{ab})$ to the (restricted) direct sum of the quotients $\gamma_n(P)/\gamma_{n+1}(P)$, $n = 1, 2, 3, \dots$,

$$\overline{\theta}\colon L(P_{ab}) \twoheadrightarrow \bigoplus_{n\geq 1} \gamma_n(P)/\gamma_{n+1}(P)$$

since the direct sum is a Lie algebra over $\mathbb{Z}$ with Lie product induced by the functions

$$\gamma_m(P) \times \gamma_n(P) \to \gamma_{m+n}(P), \qquad (x, y) \mapsto [x, y].$$

A theorem of Magnus and Witt [12, Theorem 5.12] states that $\overline{\theta}$ is a Lie isomorphism if $P = F$ is a free group.

With a non-abelian version of this Magnus-Witt isomorphism as our aim (see Theorem 3 and Proposition 4), we let $\Gamma_n(P)$ be the subgroup of $L(P)$ generated by the elements $\{\{\cdots\{\{x_1, x_2\}, x_3\}, \dots\}, x_n\}$ for x_i in P. In particular $\Gamma_1(P) = P$. Then the identity group homomorphism on P induces a surjective map of multiplicative Lie algebras

$$\theta\colon L(P) \twoheadrightarrow P$$

in which P has the structure of Example 1, and which restricts to surjective group homomorphisms

$$\theta_n\colon \Gamma_n(P) \twoheadrightarrow \gamma_n(P)$$

for all $n \geq 1$.

We can now state our main result, in which $H_n(P)$ denotes the nth homology group of P with integral coefficients.

THEOREM 3. (i) *For any group P the homomorphism θ_1 is by definition an isomorphism.*

https://doi.org/10.1017/S1446788700036934 Published online by Cambridge University Press

(ii) *If* $H_2(P) = 0$ (*for instance if* P *is free*) *then* θ_2 *is an isomorphism,* $\theta_2 \colon \Gamma_2(P) \cong \gamma_2(P)$.

(iii) *If* $H_1(P)$ *is torsion-free and if* $H_2(P) = 0$ (*for instance if* P *is free*) *then* θ_3 *is an isomorphism* $\theta_3 \colon \Gamma_3(P) \cong \gamma_3(P)$.

We prove Theorem 3 in Section 3.

PROPOSITION 4. *Any element* y *of* $L(P)$ *can be expressed as a finite product* $y = y_1 y_2 \cdots y_n$ *with* y_i *in* $\Gamma_i(P)$.

In order to prove Proposition 4 we need to define subgroups $\beta\{P\}$ of $L(P)$ for any "bracketing" β. We call the bracket arrangement $\beta = *$ involving no brackets, the *bracketing of weight* 1, and for this bracketing we set $\beta\{P\} = P$. We say that β is a *bracketing of weight* $n \geq 2$ if it is of the form $\beta = (\beta_1, \beta_2)$ with β_i a bracketing of weight n_i such that $n_1 + n_2 = n$. For a particular bracketing β of weight n and elements $x_1, \dots, x_n$ of P the expression $\beta\{x_1, \dots, x_n\}$ denotes in an obvious way an element of $L(P)$. For instance, given the bracketing $\beta = (((*, *), *), (*, *))$ of weight 5, then $\beta\{x_1, x_2, x_3, x_4, x_5\}$ denotes the element $\{\{\{x_1, x_2\}, x_3\}, \{x_4, x_5\}\}$. For a particular bracketing β of weight n we take $\beta\{P\}$ to be the subgroup of $L(P)$ generated by the elements $\beta\{x_1, \dots, x_n\}$ for x_i in P. Thus for example if $\beta = ((\cdots((*, *), *), \dots), *)$ is the left normed bracketing of weight n, then $\beta\{P\} = \Gamma_n(P)$.

With this notation we can give a

PROOF OF PROPOSITION 4. We first show that if β is any bracketing of weight i then $\beta\{P\} \subseteq \Gamma_i(P)$. This inclusion can be proved by induction on i. It is trivially true for $i = 1, 2, 3$. Suppose it is true for all bracketings of weight less then i. Assume that $i \geq 4$ and that β is of the form $\beta = (\beta', \beta'')$ with β' of weight i', β'' of weight i'' and $i' + i'' = i$. By the inductive hypothesis any generator g of $\beta\{P\}$ can be written in the form $g = \{u_1 \cdots u_{k'},\ v_1 \cdots v_{k'}\}$ with each $u_j^{\pm 1}$ a generator of $\Gamma_{i'}(P)$ and each $v_j^{\pm 1}$ a generator of $\Gamma_{i''}(P)$. It follows from identities (ii)$'$, (iii)$'$, (v)$'$ and (viii) that g is a product of elements of the form $\{u, v\}$ with $u^{\pm 1}$ a generator of $\Gamma_{i'}(P)$ and $V^{\pm 1}$ a generator of $\Gamma_{i''}(P)$. Identities (x) and (v)$'$ imply that g is a product of elements of the form $\{u, v\}^{\pm 1}$ with u a generator of $\Gamma_{i'}(P)$ and v a generator of $\Gamma_{i''}(P)$. If $i'' = 1$ then any such $\{u, v\}$ is a generator of $\Gamma_i(P)$. Hence g is in $\Gamma_i(P)$ and consequently $\beta\{P\} \subseteq \Gamma_i(P)$. If $i'' \neq 1$ then by (vii) and (iv)$'$ we have

$$\{u, v\} = \{u, \{v', x\}\} \quad (\text{with } x \in P,\ v' \text{ a generator of } \Gamma_{i''-1}(P))$$

$$= \{\{v', x\}, {}^{x}u'\}^{-1} \quad (\text{with } u' = {}^{x^{-1}}u \text{ a generator of } \Gamma_{i'}(P))$$
$$= \{\{x, u'\}, {}^{u'}v'\}\{\{u', v'\}, {}^{v'}x\}$$
$$= \{\{u, u'\}, {}^{u'}v'\}\{\{u', v'\}, x'\} \quad (\text{with } x' \in P).$$

Since ${}^{u'}v'$ can be replaced by a generator of $\Gamma_{i''-1}(P)$ it follows by a subsidiary induction on i'' that $\{u, v\}$ is in $\Gamma_i(P)$. Hence the generator g is in $\Gamma_i(P)$, and consequently $\beta\{P\} \subseteq \Gamma_i(P)$ for all $i \geq 1$. As a group $L(P)$ is generated by the elements $\beta\{x_1, \dots, x_i\}$ for $x_j \in P$ and β any bracketing of weight $i = 1, 2, 3 \dots$. It follows that $L(P)$ is generated as a group by the subgroups $\Gamma_i(P)$ for $i = 1, 2, \dots$. Identities (v)$'$ and (viii) show that $\Gamma_i(P)$ is a normal subgroup of $L(P)$ for $i \geq 2$. This proves the proposition.

So far we have completely determined the structure of the free multiplicative Lie algebra $L(P)$ for P a free abelian group, and we have partially determined the structure for P a free group. For purely intrinsic interest we now investigate the structure of $L(P)$ for other classes of groups P.

For N a normal subgroup of a group P we let $\gamma_1(N, P) = N$ and $\gamma_{n+1}(N, P) = [\gamma_n(N, P), P]$.

Theorem 5. *Let R be a normal subgroup of a free group F, and let $P \cong F/R$. Then for $n \geq 1$ there is an isomorphism of abelian groups*

$$\Gamma_n(P_{ab}) \cong \frac{\gamma_n(F)}{\gamma_{n+1}(F)\gamma_n(R, F)}.$$

Consequently there is a Lie algebra isomorphism

$$L(P_{ab}) \cong \bigoplus_{n\geq 1} \frac{\gamma_n(F)}{\gamma_{n+1}(F)\gamma_n(R, F)}.$$

We prove Theorem 5 in Section 4. An immediate consequence of this theorem is that the kernel of the canonical homomorphism

$$\Gamma_n(P_{ab}) \to \gamma_n(P)/\gamma_{n+1}(P),$$

which we shall denote by $B(P, n)$, is given by the formula

$$B(P, n) = \frac{\{R\gamma_{n+1}(F)\} \cap \gamma_n(F)}{\gamma_{n+1}(F)\gamma_n(R, F)}$$

when $P \cong F/R$ with F a free group. Clearly $B(P, n)$ is an invariant of P. In fact $B(P, n)$ is one of the invariants of P_{ab} due to R. Baer (see [1] or [10]). Theorem 5 has the following obvious corollary.

COROLLARY 6. *The Lie homomorphism of Magnus and Witt*

$$\bar{\theta}: L(P_{ab}) \to \bigoplus_{n\geq 1} \gamma_n(P)/\gamma_{n+1}(P)$$

is an isomorphism if and only if $B(P, n) = 0$ *for* $n \geq 2$.

Note that any finite cyclic group P is an example of a non-free group satisfying $B(P, n) = 0$ for all $n \geq 2$. It would be interesting to know if there exist other non-free groups P satisfying $B(P, n) = 0$ for all $n \geq 2$. It would also be interesting to investigate the relationship between Corollary 6 and the main result in [8].

THEOREM 7. *Let* $G = L(P)$ *be the free multiplicative Lie algebra on a group* P.

(i) *If* P *is a* p*-group then* G *is a* p*-group.*

(ii) *If* P *is finite then* $\Gamma_n(P)$ *is finite for* $n \geq 1$.

(iii) *If* P *is nilpotent of class* c *then* $\Gamma_n(P)$ *is abelian for* $n \geq c+1$.

(iv) *If* $\gamma_2(P)$ *is nilpotent of class* c' *then* $\Gamma_2(P)$ *is nilpotent of class* c' *or* $c'+1$.

(v) *If* P *is prefect then* $G \cong U \rtimes P$ *where* U *is the universal central extension of* P (*cf. Example* 4).

We prove Theorem 7 in Section 4. Part (iv) is presumably a genuine dichotomy, although we have no example to prove this. (For any group P we recall below the construction of its non-abelian tensor square $P \otimes P$. There is a surjection $P \otimes P \twoheadrightarrow \Gamma_2(P)$ whose kernel is generated by the elements $x \otimes x$ with $x \in P$. There are a couple of examples in [2] of groups P with $\gamma_2(P)$ abelian and $P \otimes P$ non-abelian. Such examples might yield a group P with $\gamma_2(P)$ of class 1 and $\Gamma_2(P)$ of class two.)

The following result shows that the underlying group of $L(F)$ is not free when F is a free group of rank ≥ 2.

PROPOSITION 8. *Let* G *be a free group. Then there are only two possible Lie products under which* G *becomes a multiplicative Lie algebra*: $\{x, y\} = [x, y]$ *for all* $x, y \in G$; *or* $\{x, y\} = 1$ *for all* $x, y \in G$.

PROOF. If G is of rank 1 then the result follows from the easily deduced identity

$$\{x^i, x^j\} = 1, \quad \text{for } x \in G, \ i, j \in \mathbb{Z}.$$

So suppose that G is free on more than two generators. For any x, y in G identity (viii) implies $[[x, y], \{x, y\}] = 1$. Thus $\{x, y\}$ and $[x, y]$ are both powers of some element in G. Since $[x, y] = z^i$ implies $i = -1, 0, 1$ (see [14]), we must have $\{x, y\} = [x, y]^n$ for some $n \in \mathbb{Z}$. Let a, b, c be free generators of G with $c \neq a, b$. Then there are integers l, m, n such that

$$\{ab, c\} = [ab, c]^n = ({}^a[b, c])^l [a, c]^m = {}^a\{b, c\}\{a, c\}.$$

For $n \neq 0, 1$ this last identity is false. Hence $n = 0$ or 1, and consequently $\{a, c\} = 1$ or $\{a, c\} = [a, c]$. This is the case for all pairs a, c of distinct free generators. But the Lie product is determined by its effect on the free generators. This proves the proposition.

The proofs of Theorems 3 and 7 depend on the non-abelian tensor and exterior products introduced in [3] (see also [2]). For convenience we recall these constructions here. They depend on the notion of a crossed module.

A *crossed P-module* is a group homomorphism $\partial \colon M \to P$ with a group action $(x, y) \mapsto {}^x y$ of $x \in P$ on $y \in M$ satisfying $\partial({}^x y) = x(\partial y)x^{-1}$ and ${}^{(\partial y)}y' = yy'y^{-1}$ for all $x \in P$, $y, y' \in M$. Thus for example any normal subgroup M of P is a crossed P-module with ∂ the inclusion, and action given by conjugation in P.

Given two crossed P-modules $\partial \colon M \to P$ and $\partial \colon N \to P$, the *tensor product* $M \otimes N$ is the group generated by the symbols $x \otimes y$ with $x \in M$, $y \in N$ subject to the following relations for $x, x' \in M$, $y, y' \in N$ (in which M and N are assumed to act on each other via the actions of P):

(T1) $x \otimes yy' = (x \otimes y)({}^y x \otimes {}^y y')$;

(T2) $xx' \otimes y = ({}^x x' \otimes {}^x y)(x \otimes y)$.

There is a homomorphism $\partial \colon M \otimes N \to P$ given by $\partial(x \otimes y) = [\partial x, \partial y]$. There is an action of $z \in P$ on $M \otimes N$ given by ${}^z(x \otimes y) = ({}^z x \otimes {}^z y)$. The homomorphism and action satisfy the conditions of a crossed module.

For any group P the crossed P-module $\partial \colon P \otimes P \to P$ is constructed from the identity homomorphism $P \to P$ (which of course is itself a crossed module). Note that if P is abelian then $P \otimes P$ is the usual tensor product of abelian groups.

The *exterior product* $M \wedge N$ is obtained from the tensor product $M \otimes N$ by imposing the extra relation:

(T3) $x \otimes y = 1$ whenever $\partial x = \partial y$.

The canonical image of $x \otimes y$ in $M \wedge N$ is denoted $x \wedge y$. The crossed P-module $\partial \colon M \otimes N \to P$ induces a crossed P-module $\partial \colon M \wedge N \to P$.

For any group P the crossed P-module $\partial \colon P \wedge P \to P$ is constructed from the identity homomorphism. Note that if P is abelian than $P \wedge P$ is the usual (associative) exterior product of abelian groups.

THEOREM 9. *If P is a group with normal subgroup N such that P/N is abelian, then there is a homomorphism*

$$\psi\colon (P/N)\wedge(P/N)\wedge(P/N)\to N\wedge P,$$
$$(\bar{x},\bar{y},\bar{z})\mapsto([y,x]\wedge {}^{x}z)([x,z]\wedge {}^{z}y)([z,y]\wedge {}^{y}x)$$

where $x\in P$ is a representative of $\bar{x}\in P/N$. If $H_1(P)$ is torsion-free and if $H_2(P)=0$ (for instance if P is free), then there is an exact sequence

$$P_{ab}\wedge P_{ab}\wedge P_{ab}\xrightarrow{\psi}[P,P]\wedge P\xrightarrow{\partial}\gamma_3(P)\to 1.$$

If in addition $H_3(P)=0$ then ψ is injective.

We prove Theorem 9 in Section 2.

In [3] (and also in [5]) it is shown that for any group P with normal subgroup N such that $H_2(P)=H_3(P)=0$, there is an isomorphism

$$H_3(P/N)\cong\ker(N\wedge P\to[N,P]).$$

Under this isomorphism the homomorphism ψ of Theorem 10 is such that $-\psi$ is induced by the Pontryagin product $H_1(P/N)\times H_1(P/N)\times H_1(P/N)\to H_3(P/N)$. The details are given in Lemma 11.

The exact sequence in Theorem 9 gives us an interesting presentation of $\gamma_3(F)$ for F a free group. We can generalise this presentation to a presentation of $[R,F]$ for any subgroup R containing $[F,F]$. To do this we let $M\bar{\wedge}N$ be the quotient of $M\otimes N$ obtained by imposing the relation

(T4) $x\otimes y=1$ whenever ∂x is a power of ∂y.

THEOREM 10. *Let F be a free group with normal subgroup R such that F/R is abelian. Then the homomorphisms ψ and ∂ of Theorem 9 induce an exact sequence*

$$F_{ab}\wedge F_{ab}\wedge F_{ab}\to R\bar{\wedge}F\to[R,F]\to 1.$$

We prove Theorem 10 in Section 2.

REMARK 1. One possible method of proving that $\theta_n\colon\Gamma_n(F)\to\gamma_n(F)$ is an isomorphism for all n when F is free is to use the free generating set of the free group $\gamma_n(F)$ given in [16, Lemma 8]. This lemma says that $\gamma_n(F)$ is freely generated by all invertators (= commutators sprinkled with inverses) of the form $[c,b_1^{\beta_1},b_2^{\beta_2},\dots,b_q^{\beta_q}]$ where $q\geq 1$, c and the b_i are basic commutators of weight $<n$, weight of $[c,b_1^{\beta_1}]\geq n$, $c=[c_1,c_2]$ implies $c_2\leq b_1$, $\beta_i=\pm 1$, $c\geq b_1\leq b_2\leq\cdots\leq b_q$, and $b_i=b_j$ implies $\beta_i=\beta_j$. There is thus a homomorphism $\theta_n'\colon\gamma_n(F)\to\Gamma_n(F)$ which sends each free generator to the element of $\Gamma_n(F)$ obtained by replacing commutator

brackets by Lie product brackets. Certainly θ'_n is injective, since $\theta_n\theta'_n$ is the identity. It may be possible to show that θ'_n is also surjective.

REMARK 2. For N and P two subgroups of some group we define

$$\gamma_1(N, P) = N \quad \text{and } \gamma_{m+1}(N, P) = [\gamma_m(N, P), P].$$

With this notation there is, for any free group F and integers, $m, n \geq 1$, an isomorphism

$$\theta_{m,n}: \frac{\Gamma_n(F)}{\gamma_{m+1}(\Gamma_n(F), F)} \xrightarrow{\cong} \frac{\gamma_n(F)}{\gamma_{m+n}(F)}.$$

The existence of such a surjection $\theta_{m,n}$ is clear. Injectivity is proved as follows.

The group $\gamma_n(F)/\gamma_{n+1}(F)$ is known to be a free abelian group with a basis consisting of all basic commutators of weight n (see [12]). Identity (viii) implies that the group $\Gamma_n(F)/\gamma_2(\Gamma_n(F), F)$ is abelian. The surjection $\theta_{1,n}$ is thus split by a homomorphism which sends each basic commutator of weight n to the element obtained by replacing commutator brackets by Lie product brackets. Since only identities (i)–(v) are needed to show that $\gamma_n(F)/\gamma_{n+1}(F)$ is generated by the basic commutators of weight n, it follows by analogous arguments that $\Gamma_n(F)/\gamma_2(\Gamma_n(F), F)$ is generated by the images of these basic commutators. Thus the splitting is actually surjective, and consequently $\theta_{1,n}$ is an isomorphism for all $n \geq 1$. The following commutative diagram in which the columns are short exact shows, by induction on m, that $\theta_{m,n}$ is an isomorphism for all $m \geq 2$.

$$\begin{tikzcd}
1 \arrow[d] & 1 \arrow[d] \\
\frac{\Gamma_{m+n-1}(F)}{\gamma_2(\Gamma_{m+n-1}(F), F)} \arrow[r, "\theta_{1,m+n-1}"] \arrow[d, "\omega"] & \frac{\gamma_{m+n-1}(F)}{\gamma_{1+(m+n-1)}(F)} \arrow[d, "\iota"] \\
\frac{\Gamma_n(F)}{\gamma_{m+1}(\Gamma_n(F), F)} \arrow[r, "\theta_{m,n}"] \arrow[d] & \frac{\gamma_n(F)}{\gamma_{m+n}(F)} \arrow[d] \\
\frac{\Gamma_n(F)}{\gamma_m(\Gamma_n(F), F)} \arrow[r, "\theta_{m-1,n}", "\cong"'] \arrow[d] & \frac{\gamma_n(F)}{\gamma_{(m-1)+n}(F)} \arrow[d] \\
1 & 1
\end{tikzcd}$$

The homomorphism ω is induced by replacing $m-1$ pairs of Lie product brackets by commutator brackets. For instance if $m = 3$ and $n = 2$ then

https://doi.org/10.1017/S1446788700036934 Published online by Cambridge University Press

$\{\{\{\{t, u\}, v\}, w\}x\} \mapsto \{\{[[t, u], v], w\}, x\}$. Of course, there is a certain amount of choice as to which pairs of brackets are replaced, and we leave to the reader the details of the proof (which uses identity (ix)) that ω is well defined. The injectivity of ω follows from the injectivity of $\theta_{m,n}\omega = \iota\theta_{1,m+n-1}$ where ι is the canonical inclusion.

Since free groups are residually nilpotent the isomorphisms $\theta_{m,n}$ show that for each n there is an isomorphism

$$\frac{\Gamma_n(F)}{\bigcap_{m=1}^{\infty} \gamma_m(\Gamma_n(F), F)} \cong \gamma_n(F).$$

There seems to be no easy argument showing $\bigcap_{m=1}^{\infty} \gamma_m(\Gamma_n(F), F) = 1$.

REMARK 3. Theorem 3 can be generalised. For any bracketing β and free group F the map $L(F) \to F$ which extends the identity on F (thinking of F as a multiplicative Lie algebra as in Example 1) restricts to a homomorphism $\beta\{F\} \to F$. We denote by $\beta[F]$ the image of this restricted homomorphism. We call any bracketing β of weight 1, 2, 3 *torsion free abelian* or tfa. More generally a bracketing $\beta = (\beta_1, \beta_2)$ is tfa if both β_1 and β_2 are tfa, and if for any free group F either $\beta_1[F] \subseteq \beta_2[F]$ with $\beta_2[F]/\beta_1[F]$ a torsion-free abelian group, or $\beta_2[F] \subseteq \beta_1[F]$ with $\beta_1[F]/\beta_2[F]$ torsion-free abelian. In general it seems a difficult problem to decide whether a particular bracketing is tfa. The following is an example of a tfa bracketing of weight 5: $(((*, *), *), (*, *))$. It can be shown that for any free group F and tfa bracketing β the surjection $\beta\{F\} \twoheadrightarrow \beta[F]$ is in fact an isomorphism. The details are sketched at the end of Section 3.

2. Proof of Theorems 9 and 10

Let P be a group with normal subgroup N. Then $H_1(P/N) \cong (P/N)_{ab}$. It is shown by topological methods in [3] (and algebraically in [5]) that if $H_2(P) = H_3(P) = 0$ then there is an isomorphism $H_3(P/N) \cong \ker(N \wedge P \to P)$.

LEMMA 11. *Let P be a group with normal subgroup N such that P/N is abelian and $H_2(P) = H_3(P) = 0$. For $x \in P$ let $\overline{x} = xN$. Then, under the isomorphisms $H_1(P/N) \cong P/N$ and $H_3(P/N) \cong \ker(N \wedge P \to P)$, the Pontryagin product*

$$H_1(P/N) \times H_1(P/N) \times H_1(P/N) \to H_3(P/N)$$

corresponds to the function

$$P/N \times P/N \times P/N \to N \wedge P$$
$$(\overline{x}^{-1}, \overline{y}^{-1}, \overline{z}^{-1}) \mapsto ([y, x] \wedge {}^{x}z)([x, z] \wedge {}^{z}y)([z, y] \wedge {}^{y}x).$$

PROOF. Let $Q = P/N$. Let $\{B_n(Q), d_n\}$ be the Bar resolution over Q. Thus $B_n(Q)$ is the free $\mathbb{Z}Q$-module on the symbols $[\overline{x}_1|\overline{x}_2|\cdots|\overline{x}_n]$ with $\overline{x}_i \in Q$ for $n \geq 0$; in particular $B_0(Q) = \mathbb{Z}Q$. In low dimensions the boundary homomorphisms are:

$$d_1 \colon B_1(Q) \to B_0(Q), \ [\overline{x}] \mapsto \overline{x} - 1;$$
$$d_2 \colon B_2(Q) \to B_1(Q), \ [\overline{x}|\overline{y}] \mapsto \overline{x}[\overline{y}] - [\overline{x}\,\overline{y}] + [\overline{x}];$$
$$d_3 \colon B_3(Q) \to B_2(Q), \ [\overline{x}|\overline{y}|\overline{z}] \mapsto \overline{x}[\overline{y}|\overline{z}] - [\overline{x}\,\overline{y}|\overline{z}] + [\overline{x}|\overline{y}\,\overline{z}] - [\overline{x}|\overline{y}].$$

The Pontryagin product $H_1(P/N) \times H_1(P/N) \times H_1(P/N) \to H_3(P/N)$ is induced by the function $B_1(Q) \times B_1(Q) \times B_1(Q) \to B_3(Q)$, $([\overline{x}], [\overline{y}], [\overline{z}]) \mapsto [\overline{x}] \cdot [\overline{y}] \cdot [\overline{z}] = [\overline{x}|\overline{y}|\overline{z}] - [\overline{x}|\overline{z}|\overline{y}] + [\overline{z}|\overline{x}|\overline{y}] - [\overline{y}|\overline{x}|\overline{z}] + [\overline{y}|\overline{z}|\overline{x}] - [\overline{z}|\overline{y}|\overline{x}]$. (See for instance [4] for further details.)

The isomorphism $H_3(Q) \cong \ker(N \wedge P \to P)$ was obtained in [5] as a sequence of isomorphisms:

$$H_3(Q) \cong^1 H_2(Q, IQ) \cong^2 H_1(Q, N_{ab}) \cong^3 \ker(N_{ab} \otimes_{ZQ} IQ \to N_{ab})$$
$$\cong^4 \ker((N \wedge P)/\iota(N \wedge N) \to P/[N, N]) \cong^5 \ker(N \wedge P \to P).$$

The first three isomorphisms are well known and follow from the long exact Tor-sequences induced by a short exact sequence of modules (see for instance [9]). The fourth isomorphism is got from the the exact sequence $N \wedge N \xrightarrow{\iota} N \wedge P \xrightarrow{\pi} N_{ab} \otimes_{ZQ} IQ \to 1$ where $\pi(w \wedge x) = w[N, N] \otimes (\overline{x}^{-1} - 1)$ for $w \in N$, $x \in P$. Note that $\overline{x} \in Q$ acts on N_{ab} on the right by

$$(w[N, N]) \cdot \overline{x} = x^{-1}wx[N, N].$$

The fifth isomorphism is got by lifting. The element $[\overline{x}] \cdot [\overline{y}] \cdot [\overline{z}] \in B_3(Q)$ corresponds "under" $\cong^1$ to the element

$$(\overline{x} - 1) \otimes ([\overline{y}|\overline{z}] - [\overline{z}|\overline{y}]) + (\overline{y} - 1) \otimes ([\overline{z}|\overline{x}] - [\overline{x}|\overline{z}]) + (\overline{z} - 1) \otimes ([\overline{x}|\overline{y}] - [\overline{y}|\overline{x}])$$

in $IQ \otimes_{ZQ} B_2(Q)$. Under $\cong^2$ this element corresponds to

$$[x, y][N, N] \otimes \overline{y}\,\overline{x}[\overline{z}] + [y, z][N, N] \otimes \overline{z}\,\overline{y}[\overline{x}] + [z, x][N, N] \otimes \overline{x}\,\overline{z}[\overline{y}]$$

in $N_{ab} \otimes_{ZQ} B_1(Q)$. Under $\cong^3$ this in turn corresponds to $[x^{-1}, y^{-1}][N, N] \otimes (\overline{z} - 1) + [y^{-1}, z^{-1}][N, N] \otimes (\overline{x} - 1) + [z^{-1}, x^{-1}][N, N] \otimes (\overline{y} - 1)$ in $N_{ab} \otimes_{ZQ} IQ$. Under $\cong^4$ this corresponds to

$$\{([x^{-1}, y^{-1}] \wedge z^{-1})([z^{-1}, x^{-1}] \wedge y^{-1})([y^{-1}, z^{-1}] \wedge x^{-1})\}\iota(R \wedge R)$$

https://doi.org/10.1017/S1446788700036934 Published online by Cambridge University Press

in $N \wedge P/\iota(N \wedge N)$. This lifts to

$$([y^{-1}, x^{-1}] \wedge^{x^{-1}} z^{-1})([x^{-1}, z^{-1}] \wedge^{z^{-1}} y^{-1})([z^{-1}, y^{-1}] \wedge^{y^{-1}} x^{-1})$$

in $N \wedge P$. This proves the lemma.

Suppose now that P is an arbitrary group with normal subgroup N such that P/N is abelian. We can always choose a free group P' with normal subgroup N' such that $P/N \cong P'/N'$ and such that there exists a surjection $P' \twoheadrightarrow P$. Now the Pontryagin product on the homology of P'/N' induces a homomorphism (see [4]) $\overline{\psi}\colon (P'/N') \wedge (P'/N') \wedge (P'/N') \to H_3(P'/N')$. Also, the surjection $P' \twoheadrightarrow P$ induces a surjective homomorphism $N' \wedge P' \twoheadrightarrow N \wedge P$. We let ψ' be the following composite homomorphism

$$\psi'\colon (P/N) \wedge (P/N) \wedge (P/N) \xrightarrow{\cong} (P'/N') \wedge (P'/N') \wedge (P'/N') \xrightarrow{\overline{\psi}} N' \wedge P' \twoheadrightarrow N \wedge P.$$

Lemma 11 enables us to set $\psi = -\psi'$ in Theorem 9.

By [4, Theorem V6.4 (ii)] if P_{ab} is torsion-free then the Pontryagin product induces an isomorphism $P_{ab} \wedge P_{ab} \wedge P_{ab} \cong H_3(P_{ab})$. In [3] and also in [5] it is shown that for any group P with normal subgroup N there is a homomorphism $H_3(P/N) \to \ker(N \wedge P \to P)$, and that this homomorphism is: surjective when $H_2(P) = 0$; an isomorphism when $H_2(P) = H_3(P) = 0$. The exact sequence of Theorem 9, and the injectivity of ψ, thus follows from the composite homomorphism $P_{ab} \wedge P_{ab} \wedge P_{ab} \cong H_3(P_{ab}) \to \ker([P, P] \wedge P \to P)$.

We now turn our attention to Theorem 10.

Let A be an abelian group and for some integer $n \geq 2$ consider the canonical short exact sequence $n\mathbb{Z} \rightarrowtail A \oplus \mathbb{Z} \twoheadrightarrow A \oplus (\mathbb{Z}/n\mathbb{Z})$. Associated to this sequence is a natural long exact homology sequence (see [3, Corollary 4.6] or sequence 1 in [5]) part of which is:

$$H_3(A \oplus \mathbb{Z}) \to H_3(A \oplus \mathbb{Z}/n\mathbb{Z}) \to n\mathbb{Z} \wedge (A \oplus \mathbb{Z}) \to (A \oplus \mathbb{Z}) \wedge (A \oplus \mathbb{Z}).$$

From this we can deduce an exact sequence

$$(*) \qquad H_3(A \oplus \mathbb{Z}) \to H_3(A \oplus \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\xi} n\mathbb{Z} \wedge \mathbb{Z} \to 0.$$

Suppose we have a free presentation $S \rightarrowtail F \twoheadrightarrow A \oplus (\mathbb{Z}/n\mathbb{Z})$. Then we can identify $H_3(A \oplus \mathbb{Z}/n\mathbb{Z})$ with the kernel of $\partial\colon S \wedge F \to F$. For any $x \in F$ such that $x^n \in S$ we clearly have $x^n \wedge x \in H_3(A \oplus \mathbb{Z}/n\mathbb{Z})$. Since the elements $\xi(x^n \wedge x)$ for $x^n \in S$ generate $n\mathbb{Z} \wedge \mathbb{Z}$ (in fact $n\mathbb{Z} \wedge \mathbb{Z}$ is cyclic), the exact sequence $(*)$ implies the following lemma.

LEMMA 12. *With the above notation $H_3(A \oplus \mathbb{Z}/n\mathbb{Z})$ is generated by the image of $H_3(A \oplus \mathbb{Z})$ together with the element $x^n \wedge x$ where $x^n \in S$.*

Let us now suppose that $\widehat{R}$ is a normal subgroup of a free group $\widehat{F}$ such that $\widehat{F}/\widehat{R}$ is a finitely generated abelian group. Thus

$$\widehat{F}/\widehat{R} \cong \mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_k\mathbb{Z}.$$

There is a family $S_1 \leq S_2 \leq \cdots \leq S_k = \widehat{R}$ of normal subgroups of $\widehat{F}$ such that $\widehat{F}/S_i$ is isomorphic to

$$\mathbb{Z}/n_1 \oplus \cdots \oplus \mathbb{Z}/n_i\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}.$$

The following lemma follows by induction on i, from Lemma 12.

LEMMA 13. *With the above notation $H_3(\widehat{F}/\widehat{R})$ is generated by the image of $H_3(\widehat{F}/[\widehat{F}, \widehat{F}])$ together with those elements $x^n \wedge x$ where $x \in \widehat{F}$ is such that $x^n \in \widehat{R}$ for some integer n.*

We can generalise Lemma 13 to infinitely generated abelian groups. Suppose that R is a normal subgroup of a free group F such that F/R is abelian. Suppose that z is an element in $H_3(F/R) = \ker(R \wedge F \to F)$. Then we can find a finitely generated subgroup $\widehat{F}$ of F and a normal subgroup $\widehat{R}$ of $\widehat{R}$ with $\widehat{R}$ contained in R such that z lies in the image of the canonical homomorphism $\widehat{R} \wedge \widehat{F} \to R \wedge F$. Let $\hat{z} \in \widehat{R} \wedge \widehat{F}$ be an element in the preimage of z. Then by Lemma 14, $\hat{z}$ is in the subgroup of $H_3(\widehat{F}/\widehat{R})$ generated by the image of $H_3(\widehat{F}_{ab})$ together with those elements $x^n \wedge x$ with $x \in \widehat{R}$ for some n. It follows that z lies in the subgroup of $H_3(F/R)$ generated by the image of the composite homomorphism $H_3(\widehat{F}_{ab}) \to H_3(F_{ab}) \to H_3(F/R)$ together with those elements $x^n \wedge x$ where $x^n \in \widehat{R}$ for some n. This proves the following lemma.

LEMMA 14. *For any free group F with normal subgroup R such that F/R is abelian, $H_3(F/R)$ is generated by the image of $H_3(F_{ab})$ together with those elements $x^n \wedge x$ where $x \in F$ is such that $x^n \in R$ for some integer n.*

By Theorem 9 the image of $H_3(F_{ab})$ in $H_3(F/R)$ is just the image of the homomorphism $\psi\colon F_{ab} \wedge F_{ab} \wedge F_{ab} \to H_3(F/R)$. Thus Theorem 10 follows immediately from Lemma 14.

3. Proof of Theorem 3

When $H_2(P) = 0$ the injectivity of θ_2 follows from the exact sequence $H_2(P) \to P \wedge P \xrightarrow{\partial} \gamma_2(P) \to 1$ and the fact that ∂ factors as a surjection $P \wedge P \twoheadrightarrow \Gamma_2(P)$, $x \wedge y \mapsto \{x, y\}$ followed by $\theta_2\colon \Gamma_2(P) \to \gamma_2(P)$.

https://doi.org/10.1017/S1446788700036934 Published online by Cambridge University Press

If $H_2(P) = 0$ and $H_1(P)$ is torsion-free then we have a commutative diagram of group homomorphisms:

$$\begin{tikzcd} P_{ab} \wedge P_{ab} \wedge P_{ab} \arrow[r, "\psi"] & {[P, P] \wedge P} \arrow[r] \arrow[d, "\partial^{-1}\wedge \mathrm{id}"', "\cong"] & \gamma_3(P) \arrow[r] & 1 \\ & (P \wedge P) \wedge P \arrow[r, "\zeta_3'"] & \Gamma_3(P) \arrow[r] \arrow[u, "\theta_3"] & 1 \end{tikzcd}.$$

The homomorphism ζ_3' is the restriction of the composite homomorphism $P\wedge P\wedge P \xrightarrow{\tau} L(P)\wedge L(P)\wedge L(P) \xrightarrow{\zeta_3} L(P)$ where τ is induced by the inclusion $P \hookrightarrow L(P)$, and where $\zeta_3((x\wedge y)\wedge z) = \{\{x, y\}, z\}$ for all $x, y \in, z \in L(P)$. So ζ_3' is surjective. The top row is exact by Theorem 9. The isomorphism $\partial: P \wedge P \cong \gamma_2(P)$ allows us to construct the homomorphism $\partial^{-1} \wedge \mathrm{id}$, and this homomorphism is an isomorphism since it is inverse to $\partial \wedge \mathrm{id}$. By identity (iv)$'$ we see that $\zeta_3(\partial^{-1} \wedge \mathrm{id})$ maps the image of ψ to the trivial element in $\Gamma_3(P)$. Hence θ_3 is an isomorphism. This proves Theorem 3.

We now sketch a proof of the result mentioned in Remark 3. For any bracketing β and free group F there is a canonical surjection $\theta_\beta: \beta\{F\} \twoheadrightarrow \beta[F]$, and we can construct a crossed module $\bigwedge^\beta F \to F$. For instance if $\beta = (((*, *), *), (*, *))$ then $\bigwedge^\beta F = ((F \wedge F) \wedge F) \wedge (F \wedge F)$. The construction of the above homomorphism $\zeta_3: \bigwedge^3 F \twoheadrightarrow \Gamma_3(F)$ can be mimicked to yield a surjective homomorphism $\zeta_\beta': \bigwedge^\beta F \twoheadrightarrow \beta\{F\}$. So suppose that θ_β has been shown to be an isomorphism for all tfa bracketings of weight $\le n-1$. Let $\beta = (\beta', \beta'')$ be a tfa bracketing of weight n with say $\beta'[F] \subseteq \beta''[F]$. The following commutative diagram can be constructed in which the rows are exact and in which the image of $H_3(\beta''[F]/\beta'[F])$ is trivial in $\beta\{F\}$. Hence θ_β is an isomorphism.

$$\begin{tikzcd} 1 \arrow[r] & H_3(\beta''[F]/\beta'[F]) \arrow[r] & \beta'[F] \wedge \beta''[F] \arrow[r] \arrow[d, "\cong"] & \beta[F] \arrow[r] & 1 \\ & & (\wedge^{\beta'} F) \wedge (\wedge^{\beta''} F) \arrow[r, "\zeta_\beta'"] & \beta\{F\} \arrow[r] \arrow[u, "\theta_\beta"] & 1. \end{tikzcd}$$

4. Proof of Theorems 5 and 7

For N a normal subgroup of P and $n \ge 2$ we denote by $\Gamma_n(N, P)$ the subgroup of $L(P)$ generated by the elements $\{\{\cdots\{\{x_1, x_2\}, x_3\}, \ldots, \}, x_n\}$ with $x_i \in P$ at at least one of the x_i in N. It follows from (v)$'$ and (viii) that $\Gamma_n(N, P)$ is a normal subgroup of $L(P)$. We let $\Gamma_1(N, P)$ be the

https://doi.org/10.1017/S1446788700036934 Published online by Cambridge University Press

normal closure of the group N in (the underlying group of) $L(P)$. We let $L(N, P)$ denote the subgroup of $L(P)$ generated by all the subgroups $\Gamma_n(N, P)$, $n = 1, 2\dots$. Certainly $L(N, P)$ is a normal subgroup. It is in fact an ideal. For if $w \in L(P)$ and if g is a generator of $\Gamma_n(N, P)$ then $\{w, g\}$ can be written, by Proposition 4, as a product $\{w, g\} = y_{n+1}y_{n+2}\cdots y_m$ with $y_i \in \Gamma_i(P)$. A careful analysis of the proof of Proposition 4 shows that we can take each y_i to be in $\Gamma_i(N, P)$. Thus $\{w, g\} \in L(N, P)$. Now use identities (ii)$'$ and (iii)$'$.

LEMMA 15. *If P is a group with normal subgroup N, then there is a short exact sequence $L(N, P) \rightarrowtail L(P) \overset{\pi}{\twoheadrightarrow} L(P/N)$.*

PROOF. The map π is induced by the quotient homomorphism $P \twoheadrightarrow P/N$ and is clearly surjective. Identity (vi) implies that π maps $L(N, P)$ to the trivial element. The inclusion $P \hookrightarrow L(P)$ induces a group homomorphism $P/N \to L(P)/L(N, P)$ which in turn induces a map $\phi\colon L(P/N) \to L(P)/L(N, P)$. Since ϕ has a section induced by π it follows that $\ker(\pi) = L(P, N)$.

For F a free group we have the Magnus-Witt isomorphism $L(F_{ab}) \cong \bigoplus_{n\geq 1} \gamma_n(F)/\gamma_{n+1}(F)$. If $P \cong F/R$ then Lemma 15 applied to the exact sequence $R[F, F]/[F, F] \rightarrowtail F_{ab} \twoheadrightarrow P_{ab}$ yields Theorem 5.

We now turn our attention to Theorem 7.

For any group P we define the crossed module $\bigwedge^n P \to P$ inductively from the crossed modules $\bigwedge^{n-1} P \to P$ and $P \to P$ by setting $\bigwedge^1 P = P$ and $\bigwedge^n P = (\bigwedge^{n-1} P) \wedge P$.

We have already obtained homomorphisms $\zeta_2\colon \bigwedge^2 L(P) \to L(P)$, $(x \wedge y) \mapsto \{x, y\}$ and $\zeta_3\colon \bigwedge^3 L(P) \to L(P)$, $((x \wedge y) \wedge z) \mapsto \{\{x, y\}, z\}$. The homomorphism ζ_3 was obtained from ζ_2. A similar argument shows that we can construct a homomorphism $\zeta_n\colon \bigwedge^n L(P) \to L(P)((\cdots(x_1 \wedge x_2) \wedge x_3 \cdots) \wedge x_n) \mapsto \{\{\cdots\{\{x_1, x_2\}, x_3, \dots\}x_n\}$ inductively from ζ_{n-1}.

We take $\zeta'_n\colon \bigwedge^n P \twoheadrightarrow \Gamma_n(P)$ to be the restriction of the homomorphism got by composing ζ_n with the homomorphism $\bigwedge^n P \to \bigwedge^n L(P)$. (This last homomorphism is induced by the inclusion $P \hookrightarrow L(P)$.) Clearly ζ'_n is surjective.

In [6] it is shown that if $M \to P$, $N \to P$ are crossed modules such that M and N are p-groups, then the exterior product $M \wedge N$ is a p-group. By induction this result shows that if P is a p-group then $\bigwedge^n P$ is also a p-group. It follows from the surjection ζ'_n that $\Gamma_n(P)$ is a p-group. Since $L(P)$ is generated by the subgroups $\Gamma_n(P)$, which are normal for $n \geq 2$, the following easy lemma from group theory shows that $L(P)$ is a p-group.

LEMMA. *Let Q, Q' be subgroups of some group and suppose that Q' is normal. Suppose also that both Q and Q' are p-groups. Then the group QQ' generated by Q and Q' is a p-group.*

It is shown in [6] that $M \wedge N$ is finite whenever M and N are finite. Hence for any finite group P we see by induction that $\bigwedge^n P$ is finite, and hence that $\Gamma_n(P)$ is finite.

Let P be a nilpotent group of class c. An arbitrary commutator in $\Gamma_n(P)$ can be written, using (ii), (iii) and (viii), as a product of commutators $[w_1, w_2]$ with $w_i = \{\{\cdots\{\{x_1, x_2\}, x_3\}, \dots, \}, x_n\}$ for $x_i \in P$. From (viii) we see that $[w_1, w_2] = [w_1, \theta w_2]$ where θ just replaces curly brackets by commutator brackets. If $n \geq c+1$ then $\theta w_2 = 1$ and thus $\Gamma_n(P)$ is abelian.

Suppose that $\gamma_2(P)$ is nilpotent of class c'. An arbitrary element in $\gamma_{c'+2}(\Gamma_2(P))$ can be written, from (ii), (iii) and (viii), as a product of elements $[\cdots[[w_1, w_2], w_3], \dots], w_{c+2}]$ with $w_i = \{x_1, x_2\}$ for $x_i \in P$. From (viii) and (ix) the term $[w_1, w_2]$ can be replaced by $\{\theta w_1, \theta w_2\}$. By continuing this argument we find that

$$\{[[\cdots[[\theta w_1, \theta w_2], \theta w_3], \dots], \theta w_{c'+1}], w_{c'+2}\} = \{1, w_{c'+2}\} = 1.$$

Hence $\Gamma_2(P)$ is nilpotent of class $\leq c'+1$. But the surjection $\theta_2 \colon \Gamma_2(P) \to \gamma_2(P)$ implies that $\Gamma_2(P)$ is of class $\leq c'$.

If P is a perfect then it is shown in [3] that $P \wedge P \twoheadrightarrow P$ is the universal central extension of P. As in Example 4 we can form the multiplicative Lie algebra $(P \wedge P) \rtimes P$. The homomorphisms $P \hookrightarrow L(P)$ and $\zeta_2' \colon P \wedge P \to \Gamma_2(P)$ combine to form a homomorphism $\tau \colon (P \wedge P) \rtimes P \to L(P)$, $(w, x) \mapsto (\zeta_2' w)x$. By the universal property of $L(P)$ the inclusion $P \hookrightarrow (P \wedge P) \rtimes P$ extends to a map $\tau' \colon L(P) \to (P \wedge P) \rtimes P$. Since $\tau'\tau$ is the identity it follows that τ is injective. To see that τ is surjective let $g = \{\cdots\{x_1, x_2\} \dots, x_n\}$ with $x_i \in P$ be an arbitrary generator of $\Gamma_n(P)$ for $n \geq 3$. Since P is perfect we can use identities (ii)$'$, (iii)$'$, (viii) and (ix) to write g as a product of elements

$$\begin{aligned}
&\{\{\cdots\{[a_1, b_1], [a_2, b_2]\}\cdots\}, [a_n, b_n]\}^{\pm 1} \\
&\quad = [\{\cdots\{[a_1, b_1], [a_2, b_2], \dots\}, \{a_n, b_n\}]^{\pm 1} \\
&\quad = (\{\cdots\{[a_1, b_1], [a_2, b_2], \dots\}^{[a_n, b_n]}\{\cdots\{[a_1, b_1], [a_2, b_2], \dots\}^{-1})^{\pm 1}
\end{aligned}$$

with $a_i, b_i \in P$ This shows that $\Gamma_n(P) \subseteq \Gamma_{n-1}(P)$. By induction $\Gamma_n(P) \subseteq \Gamma_2(P)$. It follows that τ is surjective and hence an isomorphism.

References

[1] R. Baer, 'Representations of groups as quotient groups I–III', *Trans. Amer. Math. Soc.* **58** (1945), 295–419.

[2] R. Brown, D. L. Johnson and E. F. Robertson, 'Some computations of non-abelian tensor products of groups', *J. Algebra* **111** (1987), 177–202.

[3] R. Brown and J.-L. Loday, 'Van Kampen theorems for diagrams of spaces', *Topology* **26** (1987), 311–335.

[4] K. S. Brown, *Cohomology of groups*, Graduate Texts in Math. 87 (Springer-Verlag 1982).

[5] G. J. Ellis, 'Non-abelian exterior products of groups and exact sequences in the homology of groups', *Glasgow Math. J.* **29** (1987), 13–19.

[6] G. J. Ellis, 'The non-abelian tensor product of finite groups is finite', *J. Algebra* **111** (1987), 203–205.

[7] G. J. Ellis and R. J. Steiner, 'Higher dimensional crossed modules and the homotopy groups of $(n+1)$-ads', *J. Pure Appl. Algebra* **46** (1987), 117–136.

[8] P. J. Higgins, 'Baer invariants and the Birkoff-Witt theorem,' *J. Algebra* **11** (1969), 469–482.

[9] P. J. Hilton and U. Stammbach, *A course in homological algebra*, Graduate Texts in Math. 4 (Springer-Verlag 1971).

[10] J. MacDonald, 'Group derived functors', *J. Algebra* **10** (1968), 448–477.

[11] C. Miller, ' The second homology group of a group', *Proc. Amer Math. Soc.* **3** (1952), 588–595.

[12] W. Magnus, A. Karass and D. Solitar, *Combinatorial group theory* (Interscience Publishers 1966).

[13] I. D. Macdonald and B. H. Neumann, 'On commutator laws in groups 2', in: Proc. Amer. Math. Soc. Special Session on Combinatorial Group Theory at Univ. Maryland, to appear.

[14] H. Neumann, *Varieties of groups* (Springer-Verlag Berlin 1967).

[15] J.-P. Serre, *Lie algebras and Lie groups*, Lecture Notes, Harvard University (Benjamin, New York, 1965).

[16] M. Ward, 'Bases for polynilpotent groups', *Proc. London Math. Soc.* (3) **24** (1972), 409–431.

University College Galway
National University of Ireland
Galway, Ireland