Can. J. Math., Vol. XXXIV, No. 4, 1982, pp. 769–796

# DEGREE ONE MAPS AND A REALIZATION THEOREM

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**0. Introduction.** In [8] we classified degree one maps defined on $S^p \times S^q \times S^r$. In this paper we shall study degree one maps defined on the $n$-dimensional torus $T = S^1 \times S^1 \times \ldots \times S^1$ as well as certain general properties of degree one maps. Theorem 1.1 must be known to experts; however we could not find it in the literature. Theorem 1.5 b) says that a Poincaré complex is nilpotent if it admits a degree one map from another nilpotent Poincaré complex. Theorem 1.5 a) means that certain stable properties are preserved by degree one maps and we use it later in Section 2.

Given a degree one map $f : T \rightarrow X, f^* : H^*(X; \mathbf{Z}) \rightarrow H^*(T; \mathbf{Z})$ is split injective in each dimension and this defines $H^*(X; \mathbf{Z})$ as a 'split' subalgebra of the exterior algebra $\Lambda^*(n, \mathbf{Z}) \simeq H^*(T; \mathbf{Z})$. In Section 5 we deal with the following realization problem: Given a $PD^n$-algebra $P^*$ over $\mathbf{Z}$ and a split injection $\alpha^* : P^* \rightarrow \Lambda^*(n, \mathbf{Z})$ does there exist a Poincare complex $X$ and a degree one map $f : T \rightarrow X$ such that $f^* : H^*(X; \mathbf{Z}) \rightarrow H^*(T; \mathbf{Z})$ will correspond to $\alpha^* : P^* \rightarrow \Lambda^*(n, \mathbf{Z})$ under some algebra isomorphism $P^* \xrightarrow{\sim} H^*(X; \mathbf{Z})$? Since $\Sigma T \simeq$ 'a wedge of spheres', Theorem 1.5 a) yields that if such a complex $X$ exists then $\Sigma X \simeq$ 'a wedge of spheres'. In Section 2 we define a quadratic operation $\Psi_g$ on $H^*(X; \mathbf{Z}_2)$ with respect to a homotopy equivalence $g$ : "a wedge of spheres" $\xrightarrow{\sim} \Sigma X$. If $X$ and $X'$ are two such spaces and $f : X \rightarrow X'$ is a degree one map we show that $f^*(H^*(X'; \mathbf{Z}_2))$ is invariant under $\Psi_g$ for a suitable $g$. In the case of the torus, taking $g$ to be the obvious homotopy equivalence with respect to a product structure of the torus, we obtain a simple algebraic description of $\Psi_g$. Assuming that $P^*$ is invariant under $\Psi_g$ and that $P^*$ is highly connected we solve the problem positively in Section 5 (Theorem 5.14). The method of proof is, first, to convert the problem to that of realization of split subalgebras of $H^*(T/T^{(k-1)})$ using the connectivity of $P^*$. Then the structure of stunted tori are determined up to carefully chosen homotopy in Section 4. Using this, Proposition 5.3 solves this converted problem. However, the proof of 5.3 involves certain computational work which we have conveniently put at the end of the paper, in

Received February 16, 1979 and in revised form November 20, 1981. This research was done while the author was on leave from T.I.F.R. and spending a year at the University of Calgary. Financial support from NRC grants A8225 and A7693 is gratefully acknowledged.

Section 6. Section 3 deals with recalling certain results concerning the structure of homotopy groups of wedges of spheres from [1] and deriving a few useful corollaries. At the end of Section 5 we also give an example to illustrate the fact that not all split subalgebras, are $\Psi_g$-invariant.

The author is thankful to the referee for his suggestions (particularly 1.2, 1.3 and 4.2) leading to the present improved presentation of this paper.

In the sequel, all spaces are assumed to be connected CW-complexes, with each cell oriented. All Poincaré complexes are oriented. Homology and cohomology modules are taken with trivial $\mathbf{Z}$-coefficients unless specifically mentioned otherwise. $T$ will always denote the $n$-dimensional torus $S^1 \times \ldots \times S^1$ ($n$-copies).

## 1. Some general results.

1.1. THEOREM. *Let $f : X \rightarrow Y$ be any map of $n$-dimensional Poincaré complexes $X$ and $Y$. Then $f_\#(\pi_1 X)$ has finite index $r$ in $\pi_1 Y$ if* $\deg f \neq 0$; *moreover if $r < \infty$ then $r$ divides* $\deg f$. *In particular if* $\deg f = 1$ *then $f_\# : \pi_1 X \rightarrow \pi_1 Y$ is onto.*

*Proof*. We first claim that if $\bar{Y} \xrightarrow{p} Y$ is an infinite covering of $Y$, then $H_n(\bar{Y}) = 0$. For let $\pi' = \pi_1 \bar{Y}$, $\pi = \pi_1 Y$ and $A = \mathbf{Z}\pi \otimes_{\mathbf{Z}\pi'} \mathbf{Z}$ where $\pi'$ acts trivially on $\mathbf{Z}$. Let $Y$ be the universal covering space of $Y$ (and hence that of $\bar{Y}$ also). Then

$$H_i(\bar{Y}) = H_i(C_*(\tilde{Y}) \otimes_{\mathbf{Z}\pi'} \mathbf{Z}) \quad \text{and}$$

$$H_i(Y; A) = H_i(C_*(\tilde{Y}) \otimes_{\mathbf{Z}\pi} A)$$

by definition. Since

$$C_*(\tilde{Y}) \otimes_{\mathbf{Z}\pi'} \mathbf{Z} \simeq C_*(\tilde{Y}) \otimes_{\mathbf{Z}\pi} \mathbf{Z}\pi \otimes_{\mathbf{Z}\pi'} \mathbf{Z} = C_*(\tilde{Y}) \otimes_{\mathbf{Z}\pi} A,$$

we have $H_i(\bar{Y}) \simeq H_i(Y; A)$. Now by duality,

$$H_n(Y; A) \simeq H^0(Y; A) \simeq H^0(\pi; A) = A^\pi,$$

the submodule of invariant elements in $A$. If $\pi'$ has infinite index in $\pi$ then $A^\pi = 0$ and hence $H_n(\bar{Y}) = 0$.

We now let $\pi' = f_\#(\pi_1 X)$. Clearly there is a lift $\bar{f} : X \rightarrow \bar{Y}$ of $f$, i.e., $p \circ \bar{f} = f$. If $\pi'$ has infinite index in $\pi$, then since $H_n(\bar{Y}) = 0$ we have $f_* = p_* \circ \bar{f}_* = 0$ on $H_n(X)$ i.e., $\deg f = 0$. Further if $\pi'$ has finite index $r$ in $\pi$ it is known that $\bar{Y}$ is an $n$-dimensional Poincaré complex and $p : \bar{Y} \rightarrow Y$ is of degree $r$ (see [4]). Since

$$\deg f = \deg (p \circ \bar{f}) = (\deg p) \cdot (\deg \bar{f})$$

it follows that $r | \deg f$. In particular if $\deg f = 1$ then $r = 1$ and hence $f_\# : \pi_1 X \rightarrow \pi_1 Y$ is onto.

1.2. LEMMA. *A simply connected* CW-*complex $X$ has the homotopy type of a wedge of spheres if and only if the following conditions hold:*

1) *$H_*(X)$ is free*

2) *The Hurewicz homomorphism $h : \pi_*(X) \to H_*(X)$ is surjective.*

*Proof.* Let $\alpha_i : S^{n_i} \to X$ be such that $\{h([\alpha_i])\}_i$ forms a basis for $H_*(X)$. Then $\vee \alpha_i : \vee S^{n_i} \to X$ induces a homology isomorphism. Since $X$ is simply connected, $n_i \geqq 2$. Hence $\vee S^{n_i}$ is also simply connected. Hence $\vee \alpha_i$ is a homotopy equivalence. The converse is clear.

1.3. PROPOSITION. *Let $f : X \to Y$ be such that $f_* : H_*(X) \to H_*(Y)$ is split surjective. If $\Sigma^k X$, the k-fold suspension of $X$, $k \geqq 1$, has the homotopy type of a wedge of spheres, then so does $\Sigma^k Y$.*

*Proof.* Apply the above lemma.

1.4. PROPOSITION. *Let $X$ be a nilpotent space, $K \rightarrowtail \pi_1 X \twoheadrightarrow G$ be an exact sequence of groups and $\bar{X} \to X$ be the covering corresponding to the normal subgroup $K$ of $\pi_1 X$. Then $G$ acts nilpotently on $H_*(\bar{X})$.*

The proof of this proposition can be found in [**7**] page 406–7 except for change of notations.

1.5. THEOREM. *Let $f : X \to Y$ be a degree one map of Poincare complexes $X$ and $Y$.*

a) *If for some $k \geqq 1$, $\Sigma^k X$ has the homotopy type of a wedge of spheres so does $\Sigma^k Y$.*

b) *If $X$ is a nilpotent space so is $Y$.*

*Proof.* a) Since $f$ is a degree one map $f_* : H_*(X) \to H_*(Y)$ is split surjective. Hence we can apply Proposition 1.3.

b) Since $X$ is a nilpotent space $\pi_1 X$ is nilpotent. From 1.1 $f_* : \pi_1 X \to \pi_1 Y$ is surjective and hence $\pi_1 Y$ is also nilpotent. We shall now show that $\pi_1 Y$ acts nilpotently on $H_*(Y)$, where $\tilde{Y}$ is the universal covering space of $Y$, and then appeal to II.2.19 of [**5**] to conclude that $Y$ is a nilpotent space. Let $K = \ker f_\#$. Then from 1.4 $\pi_1 Y$ acts nilpotently on $H_*(\bar{X})$, where $\bar{X}$ is the covering space of $X$ corresponding to the subgroup $K$ of $\pi_1 X$. Treating $\mathbf{Z}\pi_1 Y$ as a $\pi_1 X$-module via $f_\# : \pi_1 X \to \pi_1 Y$ we have

$$\mathbf{Z}\pi_1 Y \simeq \mathbf{Z}\pi_1 X \otimes_{\mathbf{Z}K} \mathbf{Z}$$

and hence

$$H_*(\bar{X}) \simeq H_*(X; \mathbf{Z}\pi_1 Y).$$

Since $f : X \to Y$ is of degree one, from Lemma 2.2 of part I in [**9**]

$$f_* : H_*(X; \mathbf{Z}\pi_1 Y) \to H_*(Y; \mathbf{Z}\pi_1 Y)$$

is surjective. Hence, by the compatibility of $\pi_1$-actions it follows that

$\pi_1 Y$ acts nilpotently on $H_*(Y; \mathbf{Z}\pi_1 Y) = H_*(\tilde{Y})$. This completes the proof of the theorem.

**2. The quadratic $\psi_\theta$.** Let $X$ be any space and $\theta : S^{2k+r} \to \Sigma^r X$ be any map, $r \geqq 1$. We shall first define $\Psi_\theta : H^k(X; \mathbf{Z}_2) \to \mathbf{Z}_2$ as follows: for any $x \in H^k(X; \mathbf{Z}_2)$ let $\varphi_x : X \to K$ be the unique homotopy class such that $\varphi_x^*(\iota_k) = x$ where $K = K(\mathbf{Z}_2; k)$ is the Eilengerg-Maclane space and $\iota_k \in H^k(K; \mathbf{Z}_2)$ is the fundamental class. Put $\beta = (\Sigma^r \varphi_x) \circ \theta$. Then the functional Steenrod square,

$$Sq_\beta^{k+r}(\Sigma^{r*}\iota_k) \in H^{2k+r}(S^{2k+r}; \mathbf{Z}_2) = \mathbf{Z}_2,$$

which we denote by $\Psi_\theta(x)$, is defined and depends only on the homotopy class of $\theta$. Note that $\Sigma^{r*}$ denotes the suspension isomorphism in cohomology.

2.1. LEMMA. *If $\Sigma^r \varphi_x \circ \theta \simeq 0$, then $\Psi_\theta(x) = 0$.*

2.2. LEMMA. *For any map $f : X \to Y$ and any $y \in H^k(Y; \mathbf{Z}_2)$ we have*

$$\Psi_\theta(f^*(y)) = \Psi_{\Sigma^r f \circ \theta}(y).$$

2.3. LEMMA.

$$\Psi_\theta(x_1 + x_2) = \Psi_\theta(x_1) + \Psi_\theta(x_2) + (x_1 \cup x_2)((\Sigma_*{}^r)^{-1}(\theta))$$

*where $(\theta)$ is the element represented by $\theta$ in $H_{2k+r}(\Sigma^r X; \mathbf{Z}_2)$, and $\Sigma_*{}^r$ is the suspension isomorphism in homology.*

2.4. LEMMA. *$\Psi_{\theta_1+\theta_2}(x) = \Psi_{\theta_1}(x) + \Psi_{\theta_2}(x)$ for any two maps $\theta_i : S^{2k+r} \to \Sigma^r X$, $i = 1, 2$, and for any $x \in H^k(X; \mathbf{Z}_2)$. (Here the sum $\theta_1 + \theta_2$ is defined using the "folding map"*

$$\Delta' : \Sigma^r X \vee \Sigma^r X \to \Sigma^r X.$$

*(See* [**2**]).)

Lemmas 2.1 and 2.2 are easily seen using functoriality of the functional Steenrod Squares. The proofs of 2.3 and 2.4 are similar to (in fact simpler than) those of 1.4 and 1.6 in [**2**].

2.5. For any space $W$ which is a wedge of oriented spheres,

$$W = \bigvee_{i \in I} S_i^{r_i}$$

indexed over some (finite) set $I$ we shall denote by $\eta_i{}^{r_i}$ (or sometimes simply by $\eta_i$) the $i$th inclusion map $S^{r_i} \hookrightarrow W$; $\eta_i{}^{r_i}$ (or $\eta_i$) will also denote the image of the positive generator of $\pi_{r_i}(S^{r_i})$ under this inclusion induced homomorphism. The Hurewicz-image of this element in $H_{r_i}(W; \mathbf{Z})$ (and the corresponding element in $H_{r_i}(W; \mathbf{Z}_2)$) will be denoted by $\underline{\eta}_i{}^{r_i}$ (or $\underline{\eta}_i$). Similarly $p_i{}^{r_i} : W \to S^{r_i}$ denotes the $i$th-projection map and

$\bar{p}_i^{r_i}$ will be the element carried by $p_i^{r_i}$, in the cohomology. Now given a homotopy equivalence $g : W \rightarrow \Sigma^r X$, $(r \geqq 1)$, let

$$\underline{g}_i = (\Sigma_*{}^r)^{-1} g_*(\underline{\eta}_i) \quad \text{and} \quad \bar{g}_i = (\Sigma^{r*})^{-1} g^{*-1}(\bar{p}_i).$$

Then clearly $\{\underline{g}_i\}_i$ and $\{\bar{g}_i\}_i$ form bases for $H_*(X; \mathbf{Z}_2)$ and $H^*(X; \mathbf{Z}_2)$ respectively dual to each other, viz.:

$$\bar{g}_i(\underline{g}_j) = \bar{g}_i \cap \underline{g}_j = \delta_{ij}$$

where $i$ and $j$ are such that $r_i = r_j$. Define

$$\Psi_g : H^k(X; \mathbf{Z}_2) \rightarrow H^{2k}(X; \mathbf{Z}_2)$$

by

$$(2.5)^7 \quad \Psi_g(x) = \sum_{i, r_i = 2k+r} \Psi_{g \circ \eta_i}(x) \bar{g}_i.$$

2.6. THEOREM. *$\Psi_g$ is quadratic, i.e., for every $x$, $y \in H^k(X; \mathbf{Z}_2)$ we have*

$$\Psi_g(x + y) = \Psi_g(x) + \Psi_g(y) + x \cup y.$$

*Proof.* This follows directly from 2.3 and evaluating both the sides on each $\underline{g}_j$ for $j$ such that $r_j = 2k + r$.

We shall now need the following two results about "nice" maps between wedges of spheres. Let $V$ and $W$ denote any two wedges of spheres $V = \bigvee_{i \in I} S_i^{n_i}$; $W = \bigvee_{j \in J} S_j^{n_j}$. A map $f : V \rightarrow W$ is called *nice* if the following condition holds for every $i \in I$:

$$f_*(\underline{\eta}_i) = \sum_j \lambda_{ij} \underline{\eta}_j \text{ in } H_*(W; \mathbf{Z}) \text{ if and only if}$$

$$f_\#(\eta_i) = \sum_j \lambda_{ij} \eta_j \text{ in } \pi_\#(W).$$

The following lemma is easily proved:

2.7. LEMMA. *Let $f : V \rightarrow W$ be a map with a right homotopy inverse. Then there exists a homotopy equivalence $h : V \rightarrow V$ inducing identity on the homology groups and such that $f \circ h$ is "nice".*

2.8. PROPOSITION. *Let $f : X \rightarrow X'$ be any map $g : V \rightarrow \Sigma^r X$, $g' : V' \rightarrow \Sigma^r X'$ be homotopy equivalences where $V$ and $V'$ are certain wedges of spheres and let $\rho : V \rightarrow V'$ be a "nice" map. Suppose $(\Sigma^r f) \circ g \simeq g' \circ \rho$. Then $\Psi_g \circ f^* = f^* \circ \Psi_{g'}$.*

*Proof.* Let $f_*(\underline{g}_i) = \Sigma \lambda_{ij} \underline{g}_j'$ where $\underline{g}_i$, $\underline{g}_j'$ etc. have meanings as in 2.5. Then it follows that

$$g_*' \rho_*(\underline{\eta}_i) = (\Sigma^r f)_* \circ g_*(\underline{\eta}_i) = (\Sigma^r f)_*(\Sigma_*{}^r(\underline{g}_i))$$
$$= \sum_j \lambda_{ij} \Sigma_*{}^r(\underline{g}_j') = g_*' \Big( \sum_j \lambda_{ij} \underline{\eta}_j' \Big).$$

Hence

$$\rho_*(\underline{\eta}_i) = \sum_j \lambda_{ij}\underline{\eta}_j'.$$

Since $\rho$ is a "nice" map it follows that

$$\rho \circ \eta_i \simeq \sum_j \lambda_{ij}\eta_j'.$$

The rest of the proof is easy, using 2.4 and the definition of $\Psi_g$, (2.5)$'$.

2.9. THEOREM. *Let $f : X \to Y$ be a degree one map of Poincaré complexes. Suppose that $\Sigma^r X \simeq V$, a wedge of spheres. Then there exists a homotopy equivalence $g : V \to \Sigma^r X$ such that $f^*(H^*(Y; \mathbf{Z}_2))$ is $\Psi_g$-invariant.*

*Proof.* From 1.5 a), $\Sigma^r Y$ is also homotopic to a wedge of spheres. Since $(\Sigma^r f)_*$ has a right homotopy inverse (because $f$ is a degree one map) it follows that $\Sigma^r f$ has a right homotopy inverse. Now by 2.7 it follows that we have homotopy equivalences $g : V \to \Sigma^r X$, $g' : W \to \Sigma^r Y$, such that $(g')^{-1} \circ (\Sigma^r f) \circ g$ is a "nice" map. Applying 2.8 with $\rho = (g')^{-1} \circ (\Sigma^r f) \circ g$, now, completes the proof of the theorem.

We shall now give a simple algebraic description of $\Psi_g$ on $H^*(T; \mathbf{Z}_2)$ for suitably chosen $g$ with respect to the product structure of $T$. Let $\Lambda^*(n, \mathbf{Z}_2)$ denote the exterior algebra on $n$ generators $\{\bar{e}_i\}_{1 \leqq i \leqq n}$ over $\mathbf{Z}_2$. Let $N = \{1, 2, \ldots, n\}$. The exterior product $\Lambda$ gives rise to a $\mathbf{Z}_2$-basis $\mathscr{E} = \{\bar{e}_A\}_{A \subseteq N}$ for $\Lambda^*(n, \mathbf{Z}_2)$ where $\bar{e}_A = \bar{e}_{i_1} \wedge \ldots \wedge \bar{e}_{i_k}$ for $A = \{i_1, \ldots, i_k\}$. We shall refer to such a basis $\mathscr{E}$ for $\Lambda^*(n, \mathbf{Z}_2)$ as a natural basis. With respect to such a natural basis $\mathscr{E}$ we define

$$\Psi_{\mathscr{E}} : \Lambda^k(n, \mathbf{Z}_2) \to \Lambda^{2k}(n, \mathbf{Z}_2), \quad k \geqq 1,$$

by the formula

$$\Psi_{\mathscr{E}}\left(\sum_{\#A=k} t_A \bar{e}_A\right) = \sum_{\{A,B\}} t_A t_B \bar{e}_A \wedge \bar{e}_B, \tag{2.10}$$

$t_A$, $t_B \in \mathbf{Z}_2$, where on the right hand side the summation is taken over all unordered pairs $\{A, B\}$, with $\#A = \#B = k$, $A, B \subseteq N$. The following proposition is easily seen.

2.11. PROPOSITION. *$\Psi_{\mathscr{E}}$ is characterised by the following two properties:*

(i) $\Psi_{\mathscr{E}}(\bar{e}_A) = 0$ *for every* $A \subseteq N$
(ii) $\Psi_{\mathscr{E}}(x + y) = \Psi_{\mathscr{E}}(x) + \Psi_{\mathscr{E}}(y) + x \wedge y$.

2.12. Let $T = S^1 \times \ldots \times S^1$ ($n$-copies) and fix a product cell structure on $T$. Let $R$ denote either $\mathbf{Z}$ or $\mathbf{Z}_2$ throughout. The various projections $T \to S^1$ give rise to a basis $\{\bar{e}_i\}$ of $H^*(T; R) \simeq \mathrm{Hom}\,(H_1(T; R); R)$. This defines an obvious isomorphism of $H^*(T; R)$ with the exterior algebra $\Lambda^*(n; R)$ of Hom $(H_1(T; R); R)$ over $R$, under which we shall identify

$H^*(T; R)$ with $\Lambda^*(n; R)$. Note that the cup product corresponds to the exterior product under this identification. Given $A \subseteq N$, let

$$T(A) = \{(x_1, x_2, \ldots, x_n) \in T/x_i = *, \text{ the base point, if } i \notin A\}.$$

Let $p_A : T \to T(A)$ denote the projection map and let $q_A : T(A) \to S_A^{\#A}$ denote the quotient map obtained by collapsing all but the top dimensional cell to a point. We now have:

2.13. PROPOSITION. *Let*

$$V = \bigvee_{\emptyset \neq A \subseteq N} S_A^{\#A+1}.$$

*Then there is a homotopy equivalence* $g : V \to \Sigma T$ *such that*

(i) $\bar{g}_A = \bar{e}_A$ *for every* $\emptyset \neq A \subseteq N$

(ii) $\Sigma q_A \circ \Sigma p_A \circ g : V \to S^{\#A+1}$ *is "nice".*

*Proof.* The existence of $g$ satisfying condition (i) is obvious. Let $g' : V \to \Sigma T$ be one such. Let $\alpha : V \to V$ be defined by

$$\alpha = \sum_A (\Sigma q_A \circ \Sigma p_A \circ g').$$

Then it is easily verified that $\alpha$ is a homotopy equivalence and taking $g = \alpha^{-1} \circ g'$ then

$$\Sigma q_A \circ \Sigma p_A \circ g : V \to S^{\#A+1}$$

is actually homotopic to the projection map and hence "nice".

2.14. THEOREM. *If* $g$ *satisfies the conditions in* 2.13, *then* $\Psi_g = \Psi_{\mathfrak{g}}$.

*Proof.* By 2.11 it suffices to show that for every $A \subseteq N$, $\Psi_g(\bar{g}_A) = 0$. If $\nu_k : S^k \to K(\mathbf{Z}_2, k)$ represents the generator of $H^k(S^k; \mathbf{Z}_2)$, then clearly $\bar{g}_A$ is represented by

$$\nu_k \circ q_A \circ p_A : T \to K(\mathbf{Z}_2, k), \ (\#A = k).$$

Hence for any $B$, with $\#B = 2k$,

$$\Sigma\varphi_{\bar{g}_A} \circ g_B \simeq \Sigma(\nu_k \circ q_A \circ p_A) \circ g_B = (\Sigma\nu_k) \circ (\Sigma q_A) \circ (\Sigma p_A) \circ g \circ \eta_B \simeq 0$$

by 2.13. Hence $\Psi_{g_B}(\bar{g}_A) = 0$ for every $B$ with $\#B = 2k$ (by 2.1). Hence $\Psi_g(\bar{g}_A) = 0$. This completes the proof of 2.14.

We conclude this section with the following proposition. Let $T_k$ denote the quotient space $T/T^{(k)}$ and $q : T \to T_k$ the quotient map, for any $k \geqq 1$.

2.15. PROPOSITION. *There is a homotopy equivalence* $g' : W \to \Sigma T_k$ such *that* $q^* \circ \Psi_{g'} = \Psi_g \circ q^*$ $(= \Psi_{\mathfrak{g}} \circ q^*)$ *and hence* $\Psi_{g'}$ *vanishes on the natural basis of* $H^*(T_k; \mathbf{Z}_2)$ *obtained from that of* $H^*(T, \mathbf{Z}_2)$ *via* $q^*$.

*Proof.* Let $W$ be the subspace of $V$ consisting of all spheres of dimension $>k$, and $g' = (\Sigma q) \circ g \circ \iota$ where $\iota$ is the inclusion map $W \hookrightarrow V$. If $\rho : V \to W$ is the obvious retraction map then $g' \circ \rho = \Sigma q \circ g$. Clearly $g'$ is a homotopy equivalence and from (2.8) we have $q^* \circ \Psi_{g'} = \Psi_g \circ q^*$.

**3. $\pi_\#$ of a wedge of spheres.** In this section we shall recall the results from Section 3 of [**1**] concerning the structure of the homotopy groups of a wedge of spheres and derive two useful corollaries. So let $\mathcal{S}_m{}^r = S_1{}^r \vee \ldots \vee S_m{}^r$, $r > 2$. Let $\eta_i{}^r \in \pi_r(\mathcal{E}_m{}^r)$ be as introduced in 2.5. For any $p$, the homomorphisms

$$\varphi_i : \pi_p(S^r) \to \pi_p(\mathcal{S}_m{}^r)$$
$$\varphi_{ij} : \pi_p(S^{2r-1}) \to \pi_p(\mathcal{S}_m{}^r)$$
$$\varphi_{ijk} : \pi_p(S^{3r-2}) \to \pi_p(\mathcal{S}_m{}^r)$$

are defined by

$$\varphi_i(\alpha) = \eta_i{}^r \circ \alpha,\ \alpha \in \pi_p(S^r)$$
$$\varphi_{ij}(\alpha) = [\eta_i{}^r, \eta_j{}^r] \circ \alpha,\ \alpha \in \pi_p(S^{2r-1})$$
$$\varphi_{ijk}(\alpha) = [[\eta_i{}^r, \eta_j{}^r], \eta_k{}^r] \circ \alpha,\ \alpha \in \pi_p(S^{3r-2}).$$

For $p < 3r - 2$, $\varphi_i$ and $\varphi_{ij}$ are injective and one has

$$\pi_p(\mathcal{S}_m{}^r) = \left(\bigoplus_{1 \leqq i \leqq m} \operatorname{Im} \varphi_i\right) \oplus \left(\bigoplus_{1 \leqq i < j \leqq m} \operatorname{Im} \varphi_{ij}\right)$$

and for $p = 3r - 2$, $\varphi_{ijk}$ are also injective and one has

$$\pi_{3r-2}(\mathcal{S}_m{}^r) = \left(\bigoplus_{1 \leqq i \leqq m} \operatorname{Im} \varphi_i\right) \oplus \left(\bigoplus_{1 \leqq i < j \leqq m} \operatorname{Im} \varphi_{ij}\right) \oplus \left(\bigoplus_{\substack{1 \leqq i < j \leqq m \\ 1 \leqq k \leqq j \leqq m}} \operatorname{Im} \varphi_{ijk}\right).$$

There is an obvious generalization of these results giving direct sum decompositions of homotopy groups (in the appropriate range) of any finite wedge of spheres of arbitrary dimensions $\geqq 2$. As an immediate corollary we obtain:

3.1. PROPOSITION. *Let $V = \bigvee_{i \in I} S_i{}^{r_i}$ be a wedge of spheres where $k \leqq r_i \leqq 2k - 1$ for some integer $k \geqq 2$. Let $J \subset I$ be any (nonempty) subset and $W = \bigvee_{i \in J} S_i{}^{r_i}$. Let $\rho : V \to W$ be the quotient map. Then for any $i, j, k \in J$ and any $\alpha \in \pi_p(V)$ for $p \leqq 3k - 2$, the component of $\alpha$ in* $\operatorname{Im} \varphi_i$ *(respectively* $\operatorname{Im} \varphi_{ij}$ *or* $\operatorname{Im} \varphi_{ijk}$*) is the same as that of $\rho_*(\alpha)$ in* $\operatorname{Im} \varphi_i$ *(respectively* $\operatorname{Im} \varphi_{ij}$ *or* $\operatorname{Im} \varphi_{ijk}$*).*

3.2. PROPOSITION. *Let $X = \bigvee_{i \in I} S_i{}^{r_i}$, $Y = \bigvee_{j \in J} S_j{}^{r_j}$ such that $k \leqq r_j \leqq 2k - 1$ for every $j \in J$ and $2k - 1 \leqq r_i \leqq 3k - 3$, for every*

https://doi.org/10.4153/CJM-1982-054-6 Published online by Cambridge University Press

*$i \in I$, $k \geqq 2$ and $I$ and $J$ be finite. Let $\theta : X \to Y$ be any map, $\theta_i = \theta_\#(\eta_i^{r_i}) \in \pi_{r_i}(Y)$. Let $\Theta_i : \pi_t(Y) \to \pi_{t+r_i-1}(Y)$ be the homomorphism defined by*

$$\Theta_i(x) = [\theta_i, x].$$

*Let $c(\theta)$ denote the mapping cone and $\iota : Y \to c(\theta)$ be the inclusion. Then $\iota_\# : \pi_s(Y) \to \pi_s(c(\theta))$ for $2k - 1 \leqq s \leqq 3k - 2$ has the property*

$$\operatorname{Ker} \iota_\# = \operatorname{Im} \theta_\# + \sum_{i \in I} \operatorname{Im} \Theta_i.$$

*Proof.* Since $X$ is $(2k - 2)$-connected and $Y$ is $(k - 1)$-connected, the Blakers-Massey theorem gives an exact sequence of homotopy groups

$$\ldots \pi_s(X) \xrightarrow{\theta_\#} \pi_s(Y) \xrightarrow{\iota_\#} \pi_s(c(\theta)) \ldots$$

for $s \leqq 3k - 3$, and hence the conclusion of the proposition holds for $s \leqq 3k - 3$, since in this case the $\Theta_i$ are trivial. Now consider the case $s = 3k - 2$. Let

$$X_a = \bigvee_{r_i = 2k-1} S_i^{r_i}, \quad X_b = \bigvee_{r_i > 2k-1} S_i^{r_i}$$

so that $X = X_a \vee X_b$. Let $\theta_a = \theta/X_a$ and $\iota_a : Y \to c(\theta_a)$ be the inclusion. Let $\theta_b$ denote the composite map:

$$X_b \hookrightarrow X \xrightarrow{\theta} Y \hookrightarrow c(\theta_a) \quad \text{and} \quad \iota_b : c(\theta_a) \hookrightarrow c(\theta_b)$$

be the inclusion. Clearly there is a homeomorphism $c(\theta) \to c(\theta_b)$ such that the following diagram is commutative:

$$\begin{tikzcd}
X \arrow[r, "\theta"] & Y \arrow[r, hook, "\iota"] \arrow[d, hook, "\iota_a"] & c(\theta) \arrow[d] \\
X_b \arrow[u, hook, "j_b"] \arrow[r, "\theta_b"] & c(\theta_a) \arrow[r, hook, "\iota_b"] & c(\theta_b)
\end{tikzcd}$$

Hence $\operatorname{Ker} \iota_\# = (\iota_a)_\#^{-1}(\operatorname{Ker} (\iota_b)_\#)$. Again, the Blakers-Massey theorem applied to $\iota_b$ yields

$$\operatorname{Ker} (\iota_b)_\# = \operatorname{Im} (\theta_b)_\# = (\iota_a)_\#(\operatorname{Im} (\theta \circ j_b)_\#).$$

On the other hand the homotopy exact sequence of the pair $(c(\theta_a), Y)$ yields that

$$\operatorname{Ker} (\iota_a)_\# = \operatorname{Im} \partial$$

where

$$\partial : \pi_{3k-1}(c(\theta_a), Y) \to \pi_{3k-2}(Y)$$

https://doi.org/10.4153/CJM-1982-054-6 Published online by Cambridge University Press

is the boundary homomorphism. From Theorem II of [**1**] it follows that

$$\mathrm{Im}\ \partial = \mathrm{Im}\ (\theta_a)_{\#} + \sum_{r_i=2k-1} \mathrm{Im}\ \Theta_i.$$

Hence

$$\begin{aligned}\mathrm{Ker}\ \iota_{\#} &= \mathrm{Im}\ (\theta_a)_{\#} + \sum_{r_i=2k-1} \mathrm{Im}\ \Theta_i + \mathrm{Im}\ (\theta \circ j_b)_{\#} \\ &= \mathrm{Im}\ \theta_{\#} + \sum_{i\in I} \mathrm{Im}\ \Theta_i.\end{aligned}$$

3.3. LEMMA. *Let $f : X \to Y$ be any map where $X$ and $Y$ are simply connected complexes of dimension $< r$. Then $f$ extends to a map*

$$\bar{f} : X \cup_\alpha e^r \to Y \cup_\beta e^r$$

*such that*

$$\bar{f}_* : H_r(X \cup_\alpha e^r, X) \to H_r(Y \cup_\beta e^r, Y)$$

*is an isomorphism if and only if $f_{\#}[\alpha] = \pm[\beta]$, in $\pi_{r-1}(Y)$.*

*Proof.* This is straightforward.

3.4. LEMMA. *Let $X = (S_1{}^t \vee S_2{}^r) \cup_\alpha e_3{}^{t+r}$ where $[\alpha] = \delta[\eta_1, \eta_2]$ for some integer $\delta$. Let $\bar{X}_i$ denote the integral cohomology element carried by the oriented sphere $S_i$ $(i = 1, 2)$ and let $X_3$ denote the homology element represented by the oriented cell $e_3$. Then $(\bar{X}_1 \cup \bar{X}_2) \cap X_3 = \delta$.*

*Proof.* If $\delta = 1$, then $X \simeq S_1{}^t \times S_2{}^r$ and hence the assertion follows. Now let

$$X' = S_1'^t \times S_2'^r = (S_1'^t \vee S_2'^r) \cup_{\alpha'} e_3'^{t+r}$$

with $[\alpha'] = [\eta_1', \eta_2']$. Let $f : X' \to X$ be a map such that

$$f/S_1'^t : S_1'^t \to S_1{}^t$$

is an orientation preserving homeomorphism and

$$f/S_2'^r : S_2'^r \to S_2{}^r$$

has degree $\delta$. Then it follows that $f_*(e_3') = e_3$ and hence

$$\begin{aligned}\delta &= f_*((\bar{X}_1' \cup \delta\bar{X}_2') \cap X_3') = f_*((f^*(\bar{X}_1) \cup f^*(\bar{X}_2)) \cap X_3') \\ &= (\bar{X}_1 \cup \bar{X}_2) \cap f_*(X_3') = (\bar{X}_1 \cup \bar{X}_2) \cap X_3.\end{aligned}$$

## 4. Cell structure of stunted tori.

4.1. As in 2.12 fix a product structure on $T = S^1 \times S^1 \times \ldots \times S^1$. For any complex $X$ let $X_t$ denote the quotient space $X/X^{(t)}$. The aim of this section is to describe the cell structure of $T_t$ (for $n \leqq 3t + 2$) obtained from the cell structure of $T$. Let $\bar{p}_A : T_t \to T(A)_t$ be the map induced by $p_A : T \to T(A)$. In what follows $V$ will denote the wedge of spheres

$$V = \bigvee_C S_C{}^m$$

where $C$ ranges over all subsets of $N$ such that $k \leqq \#C = m \leqq 2k - 1$, for a fixed integer $k \geqq 2$. For any $A \subseteq N$ let $V(A)$ denote the subspace of $V$ consisting of all spheres $S_C{}^m$ such that $C \subseteq A$. Let $L$ be the wedge of spheres

$$L = \bigvee_D S_D{}^{r-1}$$

where $D$ ranges over subsets of $N$ such that $2k - 1 < \#B = r \leqq 3k - 1$. For any $A \subseteq N$ let $L(A)$ be defined likewise. Various inclusion maps will be denoted by $\iota$, the precise meaning being clear from the context.

4.2. LEMMA. *If $K$ is a $(t - 1)$-connected complex of dimension $<2t$ and if $\Sigma^r K$ has the homotopy type of a wedge of spheres, then $K$ has the homotopy type of a wedge of spheres.*

*Proof.* The attaching maps of the cells of $K$ are all in the stable range and stably trivial and hence trivial.

4.3. LEMMA. *For any $t \geqq 1$, $T_{t-1}^{(2t-1)}$, $T_{2t-1}^{(3t-1)}$ and $\Sigma T_t$ all have the homotopy type of certain wedges of spheres.*

*Proof.* Apply 1.3 and 4.2.

4.4. LEMMA. *There is a family $\{h(r)\}_{k \leqq r \leqq 2k-1}$ of homotopy equivalences $h(r) : V^{(r)} \to T_{k-1}{}^{(r)}$ such that*

(i) *$h(r)/V^{(s)} = h(s)$ for $s \leqq r$.*

(ii) *If $h(r)(A) = h(r)/V^{(r)}(A)$ then*

$$h(r)(A) : V^{(r)}(A) \to T(A)_{k-1}^{(r)}$$

*is a homotopy equivalence.*

(iii) *If $\rho_A : V^{(r)} \to V^{(r)}(A)$ are the projection maps then*

$$h(r)(A) \circ \rho_A \simeq \bar{p}_A \circ h(r).$$

*Proof.* We construct $\{h(r)\}$ cellularly, by induction on $r$. Consider the first stage $r = k$. Let $A \subset N$ be any subset such that $\#A = k$. Then

$$V(A) = V(A)^{(k)} = S_A{}^k.$$

Moreover $T(A)_{k-1} = T(A)/T(\mathrm{A})^{(k-1)}$ is homeomorphic to the sphere $S^k$ and we take this homeomorphism as $h(k)/V(A)$ with inverse $\alpha(A)$ say. Patching them up over all $A$ with $\#A = k$, we obtain a homeomorphism

$$h(k) : V^{(k)} \to T_{k-1}^{(k)}$$

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which satisfies (ii) and (iii). Now assume that $h(r)$ has been defined for $r \leqq 2k - 1$, and let $A \subset N$ be such that $\# A = r + 1$. Consider the homotopy equivalence

$$h(r)(A) : V^{(r)}(A) \to T(A)_{k-1}^{(r)}.$$

Note that

$$V(A) = V^{(r+1)}(A) = V^{(r)}(A) \vee S_A{}^{r+1}.$$

Also

$$T(A)_{k-1} = T(\mathrm{A})_{k-1}^{(r+1)} = T(A)_{k-1}^{(r)} \cup e_A{}^{r+1}$$

where it follows from 4.3 that the attaching map of the cell $e_A{}^{r+1}$ is homotopy trivial. Hence $h(r)(A)$ extends to a homotopy equivalence

$$h(r+1)(A) : V(A) \to T(A)_{k-1}$$

with a homotopy inverse $\alpha(A)$, say. It is not difficult to see that these will now patch up to define a homotopy equivalence

$$h(r+1) : V^{(r+1)} \to T_{k-1}^{(r+1)}.$$

Again taking $\rho_A' = \alpha(A) \circ \bar{p}_A \circ h(r+1)$ one can easily see that $\rho_A' \simeq \rho_A$ which in turn implies (iii). This completes the proof of the lemma.

4.5. *Remark*. Clearly there is a family $\{\bar{h}(r)\}_{k \leqq r \leqq 2k-1}$ of homotopy equivalences

$$\bar{h}(r) : T_{k-1}^{(r)} \to V^{(r)}$$

such that for any subset $A$, $\bar{h}(r)(A)$ is the homotopy inverse of $h(r)(A)$.

4.6. THEOREM. *There is a family $\{\mu(D), h(D)\}_D$ where $D$ ranges over subsets of $N$ such that $2k \leqq \# D \leqq 3k - 1$ of maps*

$$\mu(D) : L(D) \to V(D)$$

*and homotopy equivalences*

$$h(D) : C(\mu(D)) \to T(D)_{k-1}^{(3k-1)}$$

*satisfying the following conditions:*

(i) *For any $D \subseteq D_1$,*

$$\mu(D_1)/L(D) \simeq \iota \circ \mu(D)$$

*and*

$$h(D_1)/L(D) \simeq \iota \circ h(D)$$

*where $\iota : V(D) \to V(D_1)$ is the inclusion.*

(ii) $h(D)/V = h(2k - 1)$.

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*Proof*. For any $D$, set $T(D) = X$. Then $X_{2k-1}$ has the homotopy type of a wedge of spheres as seen in 4.3. Indeed $X_{2k-1} \simeq \Sigma L(D)$. Consider the cofibration

$$X^{(2k-1)} \hookrightarrow X \to X_{2k-1} \simeq \Sigma L(D).$$

Since $X^{(2k-1)} = T(D)_{k-1}^{(2k-1)}$ is $(k - 1)$-connected and $X_{2k-1}$ is $(2k - 1)$-connected the Blakers-Massey theorem tells us that the homotopy sequence of the cofibration is exact, for maps of suspensions of dimension $\leqq 3k - 1$ and in particular for maps of $\Sigma L(D)$. Taking

$$\mu'(D) = \partial(\mathrm{Id}_{\Sigma L(D)})$$

we obtain a homotopy equivalence

$$h'(D) : c(\mu'(D)) \to X$$

such that $h'(D)/X^{(2k-1)}$ is identity. Take

$$\mu(D) = \bar{h}(D) \circ \mu'(D) : L(D) \to V(D).$$

Then clearly one obtains a homotopy equivalence

$$h(D) : c(\mu(D)) \to X$$

such that

$$h(D)/V(D) = h(2k - 1).$$

By the naturality of the homotopy exact sequence and the fact that

$$\mu'(D) = \partial(\mathrm{Id}_{\Sigma L(D)})$$

it is easily checked that for any $D \subseteq D_1$,

$$\mu'(D_1)/L(D) = \iota \circ \mu'(D).$$

The corresponding compatibility property for $\{\mu(D), h(D)\}$ now follows.

4.7. In what follows we shall denote $T_{k-1}^{(3k-1)}$ by $Y$. Note that the family $\{h(D)\}$ of homotopy equivalences patch up to define a homotopy equivalence $h : c(\mu) \to Y$ where $\mu : L \to V$ is the map defined by $\{\mu(D)\}_D$ i.e., $\mu|\iota(D) = \mu(D)$. Note that the positive dimensional cells of both $c(\mu)$ and $Y$ are indexed over the subsets $D$ of $N$ such that $k \leqq \#D \leqq 3k - 1$ and $h$ preserves this indexing. This cell structure of $c(\mu)$, (with the orientation induced from $Y$ and $h$) gives a set of generators $\{X_D{}^r\}_D$ of the homology module $H_*(c(\mu))$. The corresponding dual basis elements of $H^*(c(\mu))$ will be denoted by $\bar{X}_D{}^r$. Introduce the notations

4.8 $$\Delta_{A,B}^D = (\bar{X}_A{}^m \cup \bar{X}_B{}^{r-m}) \cap X_D{}^r$$

for every subset $D$ such that $\#D = r$ and for every $A$ and $B$ such that $\#A = m$, $\#B = r - m$, $k \leqq m \leqq 2k - 1$, and $k \leqq r - m \leqq 2k - 1$.

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Clearly

$$(4.9)\qquad \Delta^D_{A,B} = \begin{cases} \pm 1 \text{ if } A \cup B = D \quad \text{and} \quad A \cap B = \emptyset \\ 0 \text{ otherwise} \end{cases}$$

and

$$(4.10)\qquad \Delta^D_{A,B} = (-1)^{(\# A)(\# B)} \Delta^D_{B,A}.$$

Define

$$(4.11)\qquad \tilde{\mu}_D{}^r = \sum_{(A,B)} \Delta^D_{A,B}[\eta_A{}^m, \eta_B{}^{r-m}]$$

where the summation is taken over all ordered paris $(A, B)$ such that $\# A = m$, $\# B = r - m$ and $k \leqq m < r - m \leqq 2k - 1$. And define

$$(4.12)\qquad \tilde{\tilde{\mu}}_D{}^r = \begin{cases} 0 & \text{if } r \text{ is odd} \\ \sum_{\{A,A'\}} \Delta^D_{A,A'} \ [\eta^m_A, \eta^m_{A'}\ ] & \text{if } r = 2m \end{cases}$$

where the summation is taken over all unordered pairs $\{A, A'\}$ with $\# A = \# A' = m$. Let $\bar{\mu}_D{}^r$ be defined by

$$(4.13)\qquad \bar{\mu}_D{}^r = \tilde{\mu}_D{}^r + \tilde{\tilde{\mu}}_D{}^r.$$

For any subset $D$ of $N$ let

$$\iota_D : V(D) \hookrightarrow c(\mu(D))$$

and for any $r$ let

$$\iota_r : V \hookrightarrow c(\mu)^{(r)}$$

denote the inclusion maps. The following theorem now completes the description of cell structure of $Y$.

4.14. THEOREM. *In Theorem* 4.6 *the family* $\{\mu(D), h(D)\}$ *can be so chosen that if* $\mu_D{}^r = \mu | S_D{}^{r-1}$ *then*

$$[\mu_D{}^r] = \bar{\mu}_D{}^r \quad \textit{in } \pi_{r-1}(V),\ 2k - 1 \leqq r \leqq 3k - 1.$$

We shall prove that, in $\pi_{r-1}(V)$,

$$\begin{cases} (4.14)\ (2k) \quad [\mu_D{}^r] = \bar{\mu}_D{}^r, \quad r = 2k \\ (4.14)\ (r) \quad (\iota_{r-1})_\# [\mu_D{}^r] = (\iota_{r-1})_\# (\bar{\mu}_D{}^r) \quad 2k + 1 \leqq r \leqq 3k - 1. \end{cases}$$

It will then follow that there is a homotopy commutative diagram

$$\begin{tikzcd} L \arrow[r, "\bar{\mu}"] & V \arrow[r, hook] \arrow[d, equal] & c(\bar{\mu}) \arrow[r] \arrow[d, "\lambda"] & \Sigma L \arrow[d, "\wr"] \\ L \arrow[r, "\mu"] & V \arrow[r] & c(\mu) \arrow[r] & \Sigma L \end{tikzcd}$$

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where $\lambda$ is a homotopy equivalence. Hence we can redefine the family $\{\mu(D), h(D)\}$ to satisfy 4.14 also together with 4.6.

4.15. For the proof of 4.14 we shall need to study the structure of $\pi_s(V)$ and various subgroups of it in more detail. Let $\Lambda$ denote the set of all subsets $A$ of $N$ such that $k \leqq \#A \leqq 2k - 1$. Let $E$, $F$ and $G$ be subgroups of $\pi_s(V)$ defined by

$$E = \bigoplus \operatorname{Im} \varphi_A,\ F = \bigoplus_{A<A'} \operatorname{Im} \varphi_{A,A'},\ G = \bigoplus_{\substack{A<A' \\ A''\leqq A'}} \operatorname{Im} \varphi_{AA'A''}$$

with respect to a fixed total ordering $\leqq$ on $\Lambda$. It is easily checked that $F$ and $G$ do not depend upon the total ordering, however. As in Section 3, $\pi_s(V)$ is a direct sum of $E$, $F$ and $G$. Note that $\varphi_{AA'A''}$ is trivial if $s < 3k - 2$ since $V$ is $(k - 1)$-connected. When $s = 3k - 2$, again $\varphi_{AA'A''}$ is trivial if any of the sets $A, A', A''$ has more than $k$ elements, and when all of them have $k$ elements $\operatorname{Im} \varphi_{AA'A''}$ is isomorphic to the infinite cyclic group $\pi_{3k-2}(S^{3k-2})$ and is generated by $[[\eta_A, \eta_{A'}], \eta_{A''}]$. For any subset $C$ of $N$ let $F_C$ and $G_C$ be the subgroups of $F$ and $G$ respectively defined by

$$F_C = \sum_{A \cup A' \subseteq C} \operatorname{Im} \varphi_{AA'},\ G_C = \sum_{A \cup A' \cup A'' \subseteq C} \operatorname{Im} \varphi_{AA'A''}$$

and for any $r \geqq 2k$ let $F_r$ and $G_r$ be the subgroups defined by

$$F_r = \sum_{\#C\leqq r} F_C, \quad G_r = \sum_{\#C\leqq r} G_C.$$

Let $\hat{G}$ be the subgroup of $G$ defined by

$$\hat{G} = \sum_{A \cup A' \cup A''=N} \operatorname{Im} \varphi_{AA'A''}.$$

The elements $\bar{\mu}_D{}^r \in \pi_{r-1}(V)$ define homomorphisms, also denoted by

$$\bar{\mu}_D{}^r : \pi_s(S^{r-1}) \to \pi_s(V)$$

by composition: $x \mapsto \bar{\mu}_D{}^r \circ x$. Let $I_c$ and $I_r$ be the subgroups defined by

$$I_C = \sum_{D\subseteq C} \operatorname{Im} \bar{\mu}_D{}^r, \quad I_r = \sum_{\#C\leqq r} I_C.$$

The elements $\tilde{\bar{\mu}}_D{}^{2k} \in \pi_{2k-1}(V)$ define homomorphisms, say

$$\Theta_D : \pi_k(V) \to \pi_{3k-2}(V),$$

by

$$x \to [\tilde{\bar{\mu}}_D{}^{2k}, x].$$

Let $M$, $M_r$, $M_C$ and $\hat{M}$ be defined by

$$M = \sum_{\#D=2k} \operatorname{Im} \Theta_D,\ M_C = G_C \cap M,\ M_r = G_r \cap M,\ \hat{M} = \hat{G} \cap M.$$

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The lemma below is easily seen using the compatibility of the projection maps $\rho_C : V \to V(C)$.

4.16. LEMMA. *Let* $x \in F_r \oplus M_r$ *be such that for every* $C$ *with* $\# C \leqq r$,

$$(\rho_C)_\#(x) \in I_C \oplus M_C.$$

*Then* $x \in I_r \oplus M_r$.

Now let $J$ be the set of triples $(A, A', A'')$ of $k$-element subsets of $N$ such that

(i) $A \cap A' = \emptyset = A \cap A''$
(ii) $A' \cap A'' =$ a singleton set.

4.17. LEMMA. *If* $n = 3k - 1$, *then* $G$ *is a free abelian group over*

$$\{[[\eta_A, \eta_{A'}], \eta_{A''}]/(A, A', A'') \in J\}$$

*and* $M$ *is a direct summand of* $G$.

*Proof.* $\# A = \# A' = \# A'' = k$ and $A \cup A' \cup A'' = N$ implies, since $n = 3k - 1$, that at least one of the three sets, say $A$, is disjoint from the other two and $A' \cap A'' =$ a singleton set. Now

$$\hat{G} = \bigoplus_{A \cup A' \cup A'' = N} \operatorname{Im} \varphi_{AA'A''}$$

with the restriction that $A < A'$ and $A'' \leqq A'$ would imply

$$\hat{G} = \bigoplus_{(AA', A'') \in J} \operatorname{Im} \varphi_{AA'A''},$$

using the Jacobi identity, if necessary. To see the second part of the lemma let

$$J(D, B) = \{(A, A'A'') \in J/A \cup A' = D, A'' = B\}.$$

Then the family $\{J(D, B)\}$ clearly defines a partition of the set $J$. Moreover, it is easily seen that $\hat{M} = M \cap \hat{G}$ is generated by elements

$$x(D, B) = \sum_{A \cap B = \emptyset} \Delta^D_{A, D-A}[[\eta_A, \eta_{D-A}], \eta_B]$$

where $\# D = 2k$, $\# B = k$ and $D \cup B = N$, where each $x(D, B)$ is an indivisible element in the subgroup

$$\bigoplus_{(A, A', A'') \in J(D, B)} \operatorname{Im} \varphi_{AA'A''}.$$

It follows that $\hat{M}$ is a direct summand of $G$.

4.18. Let $P_N$ denote the symmetric group on $n$ letters. Let $f \in P_N$ and $A$ and $B$ be any two nonempty subsets of $N$. Define the numbers

$n(f)$, $n_A(f)$ and $n(A, B)$ by

$$\begin{cases} n(f) = \text{no. of pairs } (i, j) \text{ such that } i < j \text{ and } f(j) < f(i). \\ n_A(f) = \text{no. of pairs } (i, j) \in A \times A \text{ such that } i < j \text{ and} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad f(j) < f(i). \\ n(A, B) = \text{no. of pairs } (i, j) \in A \times B \text{ such that } j < i. \end{cases}$$

Let

$$\sigma(f) = (-1)^{n(f)}, \sigma_A(f) = (-1)^{n_A(f)} \quad \text{and}$$
$$\Delta(A, B) = (-1)^{n(A,B)}.$$

Note that $\Delta(A, B) = \Delta_{A,B}^{A\cup B}$ as defined in 4.8 when $A \cap B = \emptyset$. The following lemma is easily proved by a simple counting argument:

4.19. LEMMA. *For any $f \in P_N$ and any non empty subset $A \subset N$*

$$\sigma(f) = \sigma_A(f) \cdot \sigma_{N-A}(f) \cdot \Delta(A, N - A) \cdot \Delta(f(A), f(N - A)).$$

4.20. $P_N$ operates on $T$ in an obvious way, inducing an action on the quotient space $Y$. Using the homotopy equivalence $h : c(\mu) \to Y$, we obtain, to each $f \in P_N$, a homotopy equivalence $\bar{f}$ of $c(\mu)$. Clearly $f$ restricts to a homotopy equivalence of $V$ and hence induces an automorphism, which we shall denote by $f_\#$, of $\pi_s(V)$. It is easily seen that for every $A \in \Lambda$

$$f_\#(\eta_A) = \sigma_A(f)\eta_{f(A)}.$$

It follows that the subgroup $\hat{G}$ of $\pi_{3k-2}(V)$ is invariant under this action of $P_N$. Let $A_N$ denote the alternating subgroup of $P_N$.

4.21. LEMMA. *The $A_N$-invariant elements of $G$ are contained in $\hat{M}$.*

*Proof.* Let

$$x = \sum_{(A,A'A'')\in J} n(A, A', A'')[[\eta_A, \eta_{A'}], \eta_{A''}].$$

Let $D$ and $B$ be any two subsets of $N$ with $\#D = 2k$, $\#B = k$ and $D \cap B$ be a singleton set. Then for any two $k$-element subsets $A$ and $A_1$ of $D$, with $A \cap B = A_1 \cap B = \emptyset$, we claim

$$\Delta_{A,D-A}^D n(A_1, D - A_1, B) = \Delta_{A_1,D-A_1}^D n(A, D - A, B).$$

Clearly, there exists $f \in P_N$ such that $f(A) = A_1$, $f(D) = D$ and $f$ is identity on $B$. Since $k \geqq 2$, we can redefine $f$ on $A$ such that $\sigma(f) = 1$ i.e., $f \in A_N$. Since $f/B = \text{id}$, 4.19 yields

$$1 = \sigma(f) = \sigma_D(f) = \sigma_A(f)\sigma_{D-A}(f)\Delta_{A,D-A}^D\Delta_{A_1,D-A_1}^D.$$

On the other hand $x$ is $A_N$-invariant implies that $f_\#(x) = x$ and hence comparing the coefficients of $[[\eta_{A_1}, \eta_{D-A_1}], \eta_B]$ yields

$$\sigma_A(f) \cdot \sigma_{D-A}(f) \cdot n(A, D - A, B) = n(A_1, D - A_1, B).$$

The claim above now follows. Thus it follows that

$$n(A, D - A, B) = \Delta^D_{A,D-A} \cdot n'(D, B)$$

for some integer $n'(D, B)$. Hence

$$x = \sum n'(D, B)x(D, B) \in \hat{M}.$$

4.22. LEMMA. *For $2k \leqq r \leqq 3k - 2$ the statements (4.14)($r'$) for $r' < r$ imply*

$$(4.22)(r) \quad \operatorname{Ker} (\iota_r)_\# = I_r \oplus M$$

*in $\pi_s(V)$ for $2k - 1 \leqq s \leqq 3k - 2$. In particular if $C \subset N$ is a subset with $\# C \leqq r$, then*

$$\operatorname{Ker} (\iota_C)_\# = I_C \oplus M_C.$$

*Proof.* We induct on $r$. For $r = 2k$ we have $[\mu_D{}^{2k}] = \bar{\mu}_D{}^{2k}$ and from 3.2 the conclusion follows. Now assume that for all $r' < r$ the lemma holds and $2k < r \leqq 3k - 2$. From (4.14)($r$) we obtain

$$[\mu_D{}^r] = \bar{\mu}_D{}^r + x_D{}^r,$$

say, for some $x_D{}^r \in \operatorname{Ker} (\iota_{r-1})_\# = I_{r-1}$ in $\pi_{r-1}(V)$ (since in this case $M = 0$). From 3.2 we obtain

$$\begin{aligned}\operatorname{Ker} (\iota_r)_\# &= \sum_{\# C \leqq r} \operatorname{Im} \mu_C{}^r \oplus M = \sum_{\# C < r} \operatorname{Im} \mu_C{}^r \oplus M + \sum_{\# D = r} \operatorname{Im} \mu_D{}^r \\ &= \operatorname{Ker} (\iota_{r-1})_\# + \sum_{D = r} \operatorname{Im} (\bar{\mu}_D{}^r + x_D{}^r) \\ &= \operatorname{Ker} (\iota_{r-1})_\# + \sum_{D = r} \operatorname{Im} \bar{\mu}_D{}^r\end{aligned}$$

(since $x_D{}^r \in \operatorname{Ker} (\iota_{r-1})_\#$)

$$= I_{r-1} \oplus M + \sum_{\# D = r} \operatorname{Im} \bar{\mu}_D{}^r$$

(by the induction hypothesis)

$$= I_r \oplus M.$$

*Proof of* 4.14. Due to the compatibility of the family $\{h(D)\}$, in order to prove (4.14)($r$) for a particular $D \subseteq N$, we can as well assume $D = N$ and hence $r = n \leqq 3k - 1$.

*Step* I. Here we claim that for any subset $A$ of $N$ with $k \leqq \# A \leqq 2k - 1$,

(a) $\operatorname{Im} \varphi_A$-component of $[\mu_N]$ is zero and
(b) $\operatorname{Im} \varphi_{A,N-A}$-component of $[\mu_N] = \Delta^N_{A,N-A}[\eta_A, \eta_{N-A}]$.

So let $\beta_A$ denote the composite map

$$Y \xrightarrow{\bar{p}_A} T(A)_{k-1} \longrightarrow S_A$$

where the second arrow indicates the quotient map collapsing all but the top dimensional cell in $T(A)_{k-1}$ to a point. Let $q_A : V \to S_A$ be the quotient map collapsing all but the sphere $S_A$ to a point in $V$. Then from 4.4 (iii) it follows that

$$\beta_A \circ h \circ \iota : V \to S_A$$

is homotopic to $q_A$ where $\iota : V \to c(\mu)$ is the inclusion. Now let $f$ be the composite map

$$c(\mu) \xrightarrow{\quad h \quad} Y \xrightarrow{\quad \Delta \quad} Y \times Y \xrightarrow{\beta_A \times \beta_{N-A}} S_A \times S_{N-A}$$

where $\Delta$ is the diagonal map. (Note that $k \leqq \#(N - A) \leqq 2k - 1$, since $2k \leqq n \leqq 3k - 1$.) Let

$$j : S_A \vee S_{N-A} \to S_A \times S_{N-A}$$

be the natural inclusion. It follows from the above observation that

$$f \circ \iota = ((\beta_A \circ h \circ \iota) \times (\beta_{N-A} \circ h \circ \iota)) \circ \Delta$$
$$\simeq (q_A \times q_{N-A}) \circ \Delta = j \circ (q_A \vee q_{N-A})$$

where

$$q_A \vee q_{N-A} : V \to S_A \vee S_{N-A}$$

is again the obvious quotient map. It is easily seen that $f$ induces isomorphism of the top dimensional homology groups and hence using 3.3 and 3.4 we conclude that

$$(\rho_A \vee \rho_{N-A})_\#[\mu_N] = f_n' \cdot (\iota_{n-1})_\#[\mu_N]$$

(where $f'$ is the restriction of $f$ to $c(\mu)^{(n-1)} = \Delta^N_{A,N-A}[\eta_A, \eta_{N-A}]$).

Now using 3.1 the proof of Step I is completed.

*Step* II. For any subset $C$ of $N$ with $\#C < n$,

$$(\rho_C)_\#[\mu_N] \in \operatorname{Ker} (\iota_C)_\#$$

where $\iota_C$ is the identity map if $\#C < 2k$.

As in Step I, using 4.4(iii) we first observe that the map $\alpha(C) \circ \bar{p}_c \circ h$ is homotopic to $\iota_C \circ \rho_C$ and hence

$$(\iota_C)_\# \circ (\rho_C)_\#[\mu_N] = 0.$$

We shall now complete the proof of 4.14. Consider the case $n = 2k$. Step II, together with 3.1 implies that Im $\varphi_{AA'}$-components of $[\mu_N]$, for $A \cap A' \neq \emptyset$, are zero. Then Step I proves 4.14($2k$). Now assume that we have proved (4.14)($r$) for all $r < n$ and $2k < n \leqq 3k - 1$. From Step I we have $[\mu_N] = \bar{\mu}_N + x$ for some $x = x_1 + x_2$ say, with $x_1 \in F_{n-1} + G_{n-1}$ and $x_2 \in \hat{G}$ (of course $G_{n-1}$ and $\hat{G}$ are zero unless $n = 3k - 1$).

From Step II we obtain that for every proper subset $C$ of $N$

$$(\rho_C)_\#[\mu_N] = (\rho_C)_\#(x) = (\rho_C)_\#(x_1) \in \operatorname{Ker}(\iota_C)_\#.$$

By induction hypothesis and Lemma 4.22 it follows that

$$(\rho_C)_\#(x_1) \in I_C \oplus M_C.$$

Hence by 4.16 we have

$$x_1 \in I_{n-1} \oplus M_{n-1}.$$

Hence

$$x_1 \in \operatorname{Ker}(\iota_{n-1})_\#.$$

Since, obviously $\hat{M} \subset \operatorname{Ker}(\iota_{n-1})_\#$, we shall show that $x_2 \in \hat{M}$ which then completes the proof of 4.14.

Let $f \in A_N$ be any. If $\bar{f}$ is the homotopy equivalence defined by $f$ on $c(\mu)$ as in 4.20, then since $\bar{f}$ induces the identity automorphism on the top dimensional homology group, it follows that

$$\bar{f}_\#(\iota_{n-1})_\#[\mu_N] = (\iota_{n-1})_\#[\mu_N].$$

Hence

$$(\iota_{n-1})_\# \circ f_\#[\mu_N] = (\iota_{n-1})_\#[\mu_N].$$

Also using 4.19 one can verify easily that $f_\#(\bar{\mu}_N) = \bar{\mu}_N$. Moreover, obviously $F_{n-1}$ and $G_{n-1}$ are invariant under $f_\#$. Hence we have both $x_1$ and $f_\#(x_1) \in \operatorname{Ker}(\iota_{n-1})_\#$. Thus it follows that

$$f_\#(x_2) - x_2 \in \operatorname{Ker}(\iota_{n-1})_\# = I_{n-1} \oplus M.$$

But both $x_2$ and $f_\#(x_2) \in \hat{G}$ and hence $f_\#(x_2) - x_2 \in \hat{M}$. Since this is true for every $f \in A_N$, summing over $f \in A_N$ and using 4.21, we obtain $(n!/2)x_2 \in \hat{M}$. By 4.17 it follows that $x_2 \in \hat{M}$, as claimed. This completes the proof of 4.14.

## 5. A realization theorem.

5.1. Let $R$ be a commutative ring with a unit $1 \in R$. By a Poincaré duality algebra $P^*$ of dimension $n$ (i.e., $P^*$ is a $PD^n$-algebra) over $R$, we mean a graded, associative, anticommutative algebra

$$P^* = \sum_{0 \leqq i \leqq n} P^{(i)}$$

such that

(i) $P^{(n)} = R$ and

(ii) the pairing $P^{(i)} \otimes P^{(n-i)} \to P^{(n)} = R$ given by $x \otimes y \mapsto x \cup y$ is dual, i.e., defines an isomorphism

$$P^{(i)} \simeq \operatorname{Hom}(P^{(n-i)}; R)$$

for every $i$. Here $\cup$ denotes the multiplication in $P^*$. An algebra map $\alpha^* : P^* \to Q^*$ between two graded algebras $P^*$ and $Q^*$ is said to be split injective if $\alpha^{(i)} : P^{(i)} \to Q^{(i)}$ is split injective for every $i$.

5.2. *Examples.* (i) Let $R$ be a field. Then for any $n$-dimensional Poincaré complex $X$, $H^*(X; R)$ is a $PD^n$-algebra over $R$, where the action of $\pi_1(X)$ on $R$ is trivial.

(ii) Let $R_{k,n}[X] = R[X]/(X^{n+1} - 1)$ be the truncated polynomial algebra over $R$ generated by an indeterminate $X$ of degree $k$. Let $k$ be even or $2 = 0$ in $R$. Then $R_{k,n}[X]$ is a $PD^{kn}$-algebra over $R$ with the obvious gradation. If $R$ is a field then

$$R_{k,n}[X] \simeq H^*(P^n(\mathbf{F}); R)$$

where $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$ according as $k = 1, 2$ or $4$.

(iii) For the $n$-dimensional torus $T$, $H^*(T)$ and $H^*(T; \mathbf{Z}_2)$ are $PD^n$-algebras over $\mathbf{Z}$ and $\mathbf{Z}_2$ respectively. If $X$ is a Poincaré complex of dimension $n$ and $f : T \to X$ is a degree one map, then $H^*(X)$ is also a $PD^n$-algebra over $\mathbf{Z}$ and

$$f^* : H^*(X) \to H^*(T)$$

is split injective. A similar statement holds with $\mathbf{Z}_2$ coefficients also.

This last example motivates the question of realization we have formulated in the introduction. The main theorem of this section will answer this question partially. Before that we have:

5.3. Proposition. *Let $c(\mu)$ be as in 4.14 and let $\Psi$ be a quadratic on $H^*(c(\mu); \mathbf{Z}_2)$ such that $\Psi(\bar{X}_A) = 0$ for every $A$. Suppose*

$$\alpha^* : Q^* \to H^*(c(\mu))$$

*is an algebra homomorphism, which is split injective, such that*

$$\alpha^* \otimes \bar{1}: Q^* \otimes \mathbf{Z}_2 \to H^*(c(\mu)) \otimes \mathbf{Z}_2 = H^*(c(\mu); \mathbf{Z}_2)$$

*is $\Psi$-invariant. Then there exists a complex $X$ and a map $f : c(\mu) \to X$ such that $f^* = \alpha^*$ where $H^*(X)$ is identified with $Q^*$ by an algebra isomorphism.*

*Proof.* Choose a basis $\{\bar{a}_i^r\}$ for each $Q^{(r)}$ over $\mathbf{Z}$. Note that $Q^{(r)}$ is a free abelian group of finite rank, and also $Q^{(r)} = 0$ for $1 \leqq r \leqq k - 1$. Define the graded module $Q_* = \text{Hom}\,(Q_*; \mathbf{Z})$ and $\alpha_* : H_*(c(\mu)) \to Q_*$ by the formula

$$\alpha_*(X)(\bar{a}) = \alpha^*(\bar{a})(X) = \alpha^*(\bar{a}) \cap X$$

for every $x \in H_r(C(\mu))$, $\bar{a} \in Q^{(r)}$, and for every $r$. The basis $\{\bar{a}_i^r\}$ of $Q^{(r)}$ gives a dual basis $\{a_i^r\}$ of $Q_{(r)}$. We shall use the notation $\bar{a} \cap b = b(\bar{a})$ for every $b \in Q_{(r)}$ and $\bar{a} \in Q^{(r)}$. In this notation $\bar{a}_j^r \cap a_i^r = \delta_{ij}$. By tensoring with $\mathbf{Z}_2$ these give rise to bases of $Q^* \otimes \mathbf{Z}_2$ and $Q_* \otimes \mathbf{Z}_2$ etc. We shall drop the notation $\otimes \mathbf{Z}_2$ from now on and denote by the same symbol the corresponding elements with $\mathbf{Z}$ or $\mathbf{Z}_2$ coefficients, the precise meaning being clear from the context. The hypothesis that $\alpha^* \otimes \bar{1}$ (in our new notation, simply $\alpha^*$) is $\Psi$-invariant defines a quadratic $\Psi'$ on $Q^*$ such that

5.4 $\quad \alpha^* \circ \Psi' = \Psi \circ \alpha^*.$

Introduce the notations

5.5 $\quad \alpha_{A\,t} = \alpha^*(\bar{a}_t^{\,r}) \cap X_A^{\,r} \quad (\text{for } \#A = r)$

and

5.6 $\quad \Delta_{t,s}^{\,j} = (\bar{a}_t^{\,m_t} \cup \bar{a}_s^{\,m_s}) \cap a_j^{\,r}$

for every $t$, $s$ and $j$ such that $m_t + m_s = r_j$. Then clearly

$$(5.7) \quad \alpha^*(\bar{a}_j^{\,r}) = \sum_A \alpha_{A\,j} \bar{X}_A^{\,r}, \quad \alpha_*(X_A^{\,r}) = \sum_j \alpha_{A\,j}\, a_j^{\,r}$$

and

5.8 $\quad \Delta_{t,s}^{\,j} = (-1)^{m_t \cdot m_s} \Delta_{s,t}^{\,j}.$

Moreover since $\alpha^*$ is split injective and $\bar{X} \cup \bar{X}$ is even for every $\bar{X} \in H^*(c(\mu))$, it follows that

5.9 $\quad \Delta_{t,t}^{\,j}$ is even for every $j$ and every $t$.

Let $W_r = \vee_t S_t^{\,r}$ where $t$ ranges over the indexing set which is the same as that of the basis $\{\bar{a}_t^{\,r}\}$ of $Q^{(r)}$ and let

$$W = \bigvee_{k \leqq r \leqq 2k-1} W_r.$$

Let $M_r = \vee_j S_j^{\,r-1}$ where $j$ ranges over the indexing set which is the same as that of the basis $\{\bar{a}_j^{\,r}\}$ of $Q^{(r)}$ for $2k \leqq r \leqq 3k - 1$. Let

$$M = \bigvee_{2k \leqq r \leqq 3k-1} M_r.$$

Let $\tilde{\nu}_j^{\,r}$ and $\tilde{\tilde{\nu}}_j^{\,r}$ be the elements of $\pi_{r-1}(W)$ given by

$$(5.10) \quad \tilde{\nu}_j^{\,r} = \sum_{(t,s)} \Delta_{t,s}^{\,j}[\eta_t^{\,m_t}, \eta_s^{\,m_s}]$$

where the summation is taken over all ordered pairs $(t, s)$ such that $k \leqq m_t < m_s \leqq 2k - 1$ and $m_t + m_s = r$; and

$$(5.11) \quad \tilde{\tilde{\nu}}_j^{\,r} = \begin{cases} \dfrac{1}{2} \sum\limits_{(t,t')} \Delta_{t,t'}^{\,j}[\eta_t^{\,m}, \eta_{t'}^{\,m}] & \text{if } r = 2m \text{ for some } m \text{ even.} \\ \sum\limits_{t<t'} \Delta_{t,t'}^{\,j}[\eta_t^{\,m}, \eta_{t'}^{\,m}] + \sum\limits_t \Psi(\bar{a}_t^{\,m}) \cap a_j^{\,r}[\eta_t^{\,m}, \eta_t^{\,m}] & \text{if } r = 2m \text{ with } m \text{ odd.} \\ 0 \quad \text{otherwise.} \end{cases}$$

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Note that because of 5.9 and 5.8 the first part of the definition of $\tilde{\tilde{\nu}}_j{}^r$ makes sense. Now let $\nu : M \rightarrow W$ be a map such that for every $j$

$$[\nu_j{}^r] = \tilde{\nu}_j{}^r + \tilde{\tilde{\nu}}_j{}^r$$

and let $X = c(\nu)$. Then clearly $H_*(c(\nu))$ can be identified with $Q_*$ by identifing the bases of $H_*(c(\nu))$ given by the obvious cell structure of $c(\nu)$ with the chosen basis $\{a_t{}^m\}$ of $Q_*$. Now define two maps $f_1 : V \rightarrow W$ and $f_2 : L \rightarrow M$ such that

$$(5.12)\quad \begin{aligned} (f_1)_\#(\eta_A{}^m) &= \sum_t \alpha_{At}\eta_t{}^m \quad k \leqq m \leqq 2k-1 \\ (f_2)_\#(\eta_D{}^{r-1}) &= \sum_j \alpha_{Dj}\eta_j{}^{r-1} \quad 2k \leqq r \leqq 3k-1. \end{aligned}$$

We claim that

$$(5.13)\quad \nu \circ f_2 \simeq f_1 \circ \mu.$$

Granting this for a moment, it follows that there is a map of cofibrations

$$\begin{tikzcd} L \arrow[r] \arrow[d, "f_2"'] & V \arrow[r, hook] \arrow[d, "f_1"'] & c(\mu) \arrow[r] \arrow[d, "f"] & c(\mu)/V \arrow[r, "\simeq"] \arrow[d] & \Sigma L \arrow[d, "\Sigma f_2"] \\ M \arrow[r] & W \arrow[r, hook] & X \arrow[r] & c(\nu)/W \arrow[r, "\simeq"] & \Sigma M \end{tikzcd}$$

and hence one can easily deduce that $f_* = \alpha_*$ and hence $f^* = \alpha^*$. It now follows that, the identification of $H^*(c(\mu))$ with $Q^*$ is actually an algebra isomorphism also, as desired.

The proof of 5.13 is very much computational and we shall give it at the end of this paper. First we shall state and prove the main theorem of this section using Proposition 5.3 (and hence assuming 5.13).

5.14. THEOREM. *Let $P^*$ be a $PD^n$-algebra over $\mathbf{Z}$ and*

$$\alpha^* : P^* \rightarrow \Lambda^*(n; \mathbf{Z})$$

*be an algebra map which is split injective. Assume that*

(i) *$\alpha^*$ is $\Psi_{\mathscr{E}}$-invariant for some natural basis $\mathscr{E}$ of $\Lambda^*(n, \mathbf{Z}_2)$ and*

(ii) *for some $k \geqq 2$, $P^{(i)} = 0$ for $0 < i \leqq k - 1$ and $n \leqq 3k$.*

*Then there is a Poincare complex $X$ and a map $f : T \rightarrow X$ such that $f^* = \alpha^*$ where $H^*(X)$ is identified with $P^*$ by some algebra isomorphism $H^*(X) \simeq P^*$, and $\Lambda^*(n; \mathbf{Z})$ is identified with $H^*(T)$ using the basis $\mathscr{E}$ and a product structure on $T$.*

5.15. *Remark.* (1) If we drop the "Poincaré duality" from the hypothesis as well as from the conclusion, the theorem would still be true.

(2) The necessity of condition i) above is seen in 2.9. Later in this section we shall illustrate that this is actually non-vacuous.

(3) The theorem is true for $k = 1$ also. In this case for $P^*$ to be a $PD^n$-algebra satisfying the conditions of 5.14, there are only a few candidates and a complete classification can be obtained easily (see eg. [**8**]).

*Proof of* 5.14. Let $\underline{P}^*$ be the algebra obtained by truncating $P^*$ at the $n$th stage i.e.,

$$\underline{P}^{(i)} = \begin{cases} P^{(i)} & \text{for } i < n \\ 0 & \text{for } i = n \end{cases}$$

and let $\underline{\alpha}^* : \underline{P}^* \rightarrow H^*(T^{(n-1)}; \mathbf{Z})$ be the homomorphism defined by the composite,

$$P^* \xrightarrow{\alpha^*} H^*(T; \mathbf{Z}) \xrightarrow{\iota^*} H^*(T^{(n-1)}; \mathbf{Z}).$$

Let $Y = T_{k-1}^{(n-1)}$ and $q : T^{(n-1)} \rightarrow Y$ be the quotient map. Since $\underline{P}^{(i)} = 0$ for $1 \leqq i \leqq k - 1$ and $q^*$ is an isomorphism in dimension $\geqq k$, it follows that there is a unique homomorphism $\beta^* : \underline{P}^* \rightarrow H^*(Y)$ such that $q^* \circ \beta^* = \underline{\alpha}^*$. Clearly $\beta^*$ is split injective. Let $\Psi_1$ be the quadratic on $H^*(Y; \mathbf{Z}_2)$ given by 2.15 such that

$$q^* \circ \Psi_1 = \Psi_\xi \circ q^*.$$

The homotopy equivalence $h : c(\mu) \rightarrow Y$ now defines a quadratic $\Psi$ on $H^*(c(\mu); \mathbf{Z}_2)$, which clearly vanishes on the basic elements (i.e., $\Psi(\bar{X}_A) = 0$ etc.). If $\lambda^* = h^* \circ \beta^*$, then clearly $\lambda^*$ is $\Psi$-invariant. Hence we can apply Proposition 5.3 to obtain a map $f' : c(\mu) \rightarrow X'$ such that $f'^* = \lambda^*$ under some identification $H^*(\underline{X}') \rightleftarrows P^*$. Take $f_1 = f' \circ \bar{h} \circ q$ (where $\bar{h}$ is the homotopy inverse of $h$). Then $f_1{}^* = \underline{\alpha}^*$. If $\varphi : S^{n-1} \rightarrow T^{(n-1)}$ denotes the attaching map of the $n$-cell of $T$, then take

$$X = X' \bigcup_{f_1 \circ \varphi} e^n.$$

Then clearly $f_1$ extends to a map $f : T \rightarrow X$ such that $f^* = \alpha^*$. That $X$ is a Poincaré complex now follows from the fact that $X$ is simply connected and $P^*$ is a $PD^n$-algebra over $\mathbf{Z}$.

We shall now give an example to illustrate that there do exist $PD^n$-algebras $P^*$ over $\mathbf{Z}$ and split monomorphisms

$$\alpha^* : P^* \rightarrow \Lambda^*(n; \mathbf{Z})$$

which are not invariant under $\Psi_{\mathscr{E}}$ for any basis $\mathscr{E}$.

5.16. *Example*. Let $n = 3k$ and $k \geqq 2$. Let $P^*$ be the subalgebra of $\Lambda^*(n; \mathbf{Z})$ generated by $\bar{e}_A + \bar{e}_B$ and $\bar{e}_{B \cup C}$ for any three mutually disjoint subsets $A$, $B$, $C$ of $N$ with $k$ elements each. Let $\alpha^*$ be the inclusion homomorphism. The assertion that $P^*$ is a $PD^n$-algebra and $\alpha^*$ is split injective is trivial. That $P^*$ is not $\Psi_{\mathscr{E}}$-invariant is also trivial. To show that $P^*$ is not $\Psi_{\mathscr{E}'}$-invariant for any $\mathscr{E}'$, we argue as follows:

For any $n \times n$ matrix $M$ over $\mathbf{Z}_2$ the Laplacian identities can be given as follows (see e.g. [**3**], Theorem 8.2). For any two subsets $D$, $D'$ of $N$ with $\#D = \#D' = m$ let $M_{D,D'}$ denote the $m \times m$ minor of $M$ with rows defined by $D$ and columns defined $D'$. Then for any two fixed subsets $F$ and $F'$ of $N$ with $\#F = m$ and $\#F' = n - m$, we have

$$(5.17)\quad \sum_{\substack{G \subset N \\ \#G=m}} |M_{G,F}| \cdot |M_{G^c,F'}| = \begin{cases} |M| & \text{if } F \cap F' = \emptyset \\ 0 & \text{if } F \cap F' \neq \emptyset \end{cases}$$

where $G^c$ denotes $N - G$.

Now let $M$ be the matrix of change of basis i.e.,

$$(e_1', \ldots, e_n')M = (e_1, \ldots, e_n).$$

Then it follows that in the exterior algebra $\Lambda^*(n, \mathbf{Z}_2)$ we have

$$(5.18)\quad \bar{e}_D = \sum_{\substack{D' \\ \#D'=\#D}} |M_{D',D}|\bar{e}_{D'}' \quad \text{for every } D \subseteq N.$$

Hence from 2.11 one obtains

$$(5.19)\quad \Psi_{\mathscr{E}}(\bar{e}_A + \bar{e}_B) = \sum_{\substack{D,D' \\ \#D=\#D'=k}} |M_{D,A}| \cdot |M_{D',B}|\bar{e}_D' \wedge \bar{e}_{D'}'$$

$$= \sum_{\substack{E \\ \#E=2k}} \left( \sum_{\substack{D,D' \\ D \cup D'=E \\ \#D=\#D'=k}} |M_{D,A}| \cdot |M_{D',B}| \right) \bar{e}_E' = \sum_{\substack{E \\ \#E=2k}} |M_{E,A \cup B}|\bar{e}_E'$$

(by applying 5.17 to the matrix $M_{E,A \cup B}$ in place of $M$).

If $P^*$ were $\Psi_{\mathscr{E}'}$-invariant, we should have

(i) $\Psi_{\mathscr{E}'}(\bar{e}_A + \bar{e}_B) = 0$ or

(ii) $\Psi_{\mathscr{E}'}(\bar{e}_A + \bar{e}_B) = \bar{e}_{B \cup C}$.

Now (i) implies $|M_{E,A \cup B}| = 0$ for every $E$ with $\#E = 2k$, and hence

$$0 = \sum_E |M_{E,A \cup B}| \cdot |M_{E^c,C}| = |M| \quad \text{(again by 5.17)}$$

which is a contradiction to the fact that $|M| = 1$. On the other hand (ii) implies (using 5.18 for $D = B \cup C$)

$$|M_{E,A \cup B}| = |M_{E,B \cup C}| \quad \text{for every } E \text{ with } \#E = 2k$$

and hence

$$|M| = \sum_E |M_{E,A \cup B}|\, |M_{E^c,C}| = \sum_E |M_{E,B \cup C}|M_{E^c,C}| = 0$$

which is again a contradiction.

https://doi.org/10.4153/CJM-1982-054-6 Published online by Cambridge University Press

**6. Completion of proof of 5.3.** It remains to prove the claim 5.13: $\nu \circ f_2 \simeq f_1 \circ \mu$. Clearly it suffices to show that

(6.1) $\quad (f_1)_\#(\tilde{\mu}_D{}^r) = \sum_j \alpha_{Dj}\tilde{\nu}_j{}^r$ and

(6.2) $\quad (f_1)_\#(\tilde{\tilde{\mu}}_D{}^r) = \sum_j \alpha_{Dj}\tilde{\tilde{\nu}}_j{}^r$

for every $D$.

6.3. LEMMA. *For any $D$ and any two fixed $t$ and $s$ with $m_t + m_s = \#D = r$*

$$\sum_{(A,B)} \Delta^D_{A,B}\alpha_{A\,t}\alpha_{B\,s} = \sum_j \Delta^j_{t,s}\alpha_{Dj}.$$

*Proof.*

$$\begin{aligned}
\text{L.H.S.} &= \sum_{(A,B)} (\bar{X}_A{}^m \cup \bar{X}_B{}^{r-m}) \cap X_D{}^r \cdot \alpha_{A\,t}\alpha_{B\,s} \\
&= \left(\left(\sum_A \alpha_{A\,t}\bar{X}_A{}^m\right) \cup \left(\sum_B \alpha_{B\,s}\bar{X}_B{}^{r-m}\right)\right) \cap X_D{}^r \\
&= (\alpha^*(\bar{a}_t{}^m) \cup \alpha^*(\bar{a}_s{}^{r-m})) \cap X_D{}^r = \alpha^*(\bar{a}_t{}^m \cup \bar{a}_s{}^{r-m}) \cap X_D{}^r \\
&= \alpha^*\left(\sum_j \Delta^j_{t,s}\bar{a}_j{}^r\right) \cap X_D{}^r = \sum_j \Delta^j_{t,s}\alpha^*(\bar{a}_j{}^r) \cap X_D^r \\
&= \text{R.H.S.}
\end{aligned}$$

Hence

$$\begin{aligned}
(f_1)_\#(\tilde{\mu}_D{}^r) &= \sum_{(A,B)} \Delta^D_{A,B}[(f_1)_\#(\eta_A{}^m), (f_1)_\#(\eta_B{}^{r-m})] \\
&= \sum_{(A,B)} \Delta^D_{A,B}\left[\sum_t \alpha_{A,t}\eta_t{}^m, \sum_s \alpha_{B\,s}\eta_s{}^{r-m}\right] \\
&= \sum_{(A,B)} \Delta^D_{A,B}\left(\sum_{(t,s)} \alpha_{A\,t} \cdot \alpha_{B\,s}[\eta_t{}^m, \eta_s{}^{r-m}]\right) \\
&= \sum_{(t,s)} \left(\sum_{(A,B)} \Delta^D_{A,B} \cdot \alpha_{A,t}\alpha_{B\,s}\right) \cdot [\eta_t{}^m, \eta_s{}^{r-m}] \\
&= \sum_{(t,s)} \left(\sum_j \Delta^j_{t,s}\alpha_{Dj}\right)[\eta_t{}^m, \eta_s{}^{r-m}] \quad \text{(by 6.3)} \\
&= \sum_j \alpha_{Dj} \sum_{(t,s)} \Delta^j_{t,s}[\eta_t{}^m, \eta_s{}^{r-m}] = \sum_j \alpha_{Dj}\eta_j{}^r
\end{aligned}$$

which proves 6.1.

In order to prove 6.2, consider the case $r = 2m$ when $m$ is even. Now $\Delta^D_{A,A} = 0$ and $\Delta^D_{A,A'} = \Delta^D_{A',A}$ and hence

$$\tilde{\tilde{\mu}}_D{}^r = \frac{1}{2}\sum_{(A,A')} \Delta^D_{A,A'}[\eta_A{}^m, \eta_{A'}{}^m].$$

https://doi.org/10.4153/CJM-1982-054-6 Published online by Cambridge University Press

Thus exactly as in the proof of 6.1 above we can see that

$$(f_1)_{\#}(\tilde{\bar{\mu}}_D{}^r) = \sum_j \alpha_{Dj}\tilde{\bar{\nu}}_j{}^r.$$

Now consider the case when $r = 2m$ with $m$ odd. Let

$$\Omega^{t,t'}_{A,A'} = \alpha_{A,t}\alpha_{A',t'} - \alpha_{A',t}\cdot\alpha_{A,t'} \quad \text{and}$$

$$\omega^t_{A,A'} = \alpha_{At}\cdot\alpha_{A't}.$$

Now

$$\Delta^D_{A,A'} = -\Delta^D_{A',A}$$

and hence

$$\sum_{\{A,A'\}} \Delta^D_{A,A'}\Omega^{t,t'}_{A,A'} = \sum_{(A,A')} \Delta^D_{A,A'}\cdot\alpha_{At}\alpha_{A't'} = \sum_j \Delta^j_{t,t'}\alpha_{Dj}$$

(by 6.3). Hence

$$(6.4)\quad \sum_{\{A,A'\}} \Delta^D_{A,A'}\left(\sum_{t<t'} \Omega^{t,t'}_{A,A'}[\eta_t{}^m, \eta_{t'}{}^m]\right)$$
$$= \sum_{t<t'}\left(\sum_{\{A,A'\}} \Delta^D_{A,A'}\Omega^{t,t'}_{A,A'}\right)[\eta_t{}^m, \eta_{t'}{}^m]$$
$$= \sum_{t<t'}\left(\sum_j \Delta^j_{t,t'}\alpha_{Dj}\right)[\eta_t{}^m, \eta_{t'}{}^m] = \sum_j \alpha_{Dj}\left(\sum_{t<t'} \Delta^j_{t,t'}[\eta_t{}^m, \eta_{t'}{}^m]\right).$$

On the other hand,

$$\sum_{\{A,A'\}} \Psi^t_{A,A'}\Delta^D_{A,A'} = \sum_{\{A,A'\}} \alpha_{At}\alpha_{A't}(\bar{X}_A{}^m \cup \bar{X}_{A'}{}^m) \cap X_D{}^r$$
$$= \left(\Psi\left(\sum_A \alpha_{At}\bar{X}_A{}^m\right)\right) \cap X_D{}^r$$

(by 2.10 and 2.11)

$$= (\Psi(\alpha^*(\bar{a}_t{}^m)) \cap X_D{}^r = (\alpha^* \circ \Psi'(\bar{a}_t{}^m)) \cap X_D{}^r$$
$$= \alpha^*\left(\sum_j (\Psi'(\bar{a}_t{}^m) \cap a_j{}^r)\bar{a}_j{}^r\right) \cap X_D{}^r$$
$$= \sum_j (\Psi'(\bar{a}_t{}^m) \cap a_j{}^r)\cdot(\alpha^*(\bar{a}_j{}^r) \cap X_D{}^r) = \sum_j \alpha_{Dj}\Psi'(\bar{a}_t{}^m) \cap a_j{}^r.$$

Hence

$$(6.5)\quad \sum_{\{A,A'\}} \Delta^D_{A,A'}\cdot\sum_t \Psi^t_{A,A'}[\eta_t{}^m,\eta_t{}^m] = \sum_j \alpha_{Dj}\sum \Psi'(\bar{a}_t{}^m) \cap a_j{}^r[\eta_t{}^m, \eta_t{}^m].$$

Now

$$(f_1)_\#(\tilde{\tilde{\mu}}_D{}^r) = \sum_{\{A,A'\}} \Delta^D_{A,A'}[(f_1)_\#(\eta_A{}^m), (f_1)_\#(\eta_{A'}{}^m)]$$

$$= \sum_{\{A,A'\}} \Delta^D_{A,A'}\left( \sum_{t<t'} \Omega^{t,t'}_{A,A'}[\eta_t{}^m, \eta_{t'}{}^m] + \sum_t \Psi^t_{A,A'}[\eta_t{}^m, \eta_t{}^m]\right)$$

$$= \sum_j \alpha_{Dj}\left( \sum_{t<t'} \Delta^j_{t,t'}[\eta_t{}^m, \eta_{t'}{}^m] + \sum_t \Psi'(\bar{a}_t{}^m) \cap a_j{}^r[\eta_t{}^m, \eta_t{}^m]\right)$$

(by 6.4 and 6.5)

$$= \sum_j \alpha_{Dj}\tilde{\tilde{\nu}}_j{}^r.$$

This completes the proof of claim 5.13 and thereby the proofs of 5.3 and 5.14 also.

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