J. Austral. Math. Soc. (Series A) **65** (1998), 370–387

L^p-MULTIPLIERS OF MIXED-NORM TYPE ON LOCALLY COMPACT VILENKIN GROUPS

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(Received 15 October 1997; revised 13 August 1998)

Communicated by A. H. Dooley

Abstract

Let G be a locally compact Vilenkin group with dual group Γ. We prove Littlewood-Paley type inequalities corresponding to arbitrary coset decompositions of Γ. These inequalities are then applied to obtain new $L^p(G)$ multiplier theorems. The sharpness of some of these results is also discussed.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 43A70, 43A22.

1. Introduction

Given a sequence $\{g_n\}$ of Fourier multipliers for $L^p(\mathbb{R})$, $1 < p < \infty$, let $g := \sum_{-\infty}^{\infty} g_n \chi_n$, where χ_n denotes the characteristic function of the dyadic interval $[2^n, 2^{n+1}]$ in $\mathbb{R}$. In an earlier paper [OQ] we proved that if the sequence $\{g_n\}$ belongs to a certain mixed-norm space, then g is also an $L^p(\mathbb{R})$ multiplier. A similar result was established for Fourier multipliers for $L^p(G)$-spaces, where G is a locally compact Vilenkin group. In that case we considered the decomposition of Γ, the dual group of G, into sets that are comparable to the dyadic intervals in $\mathbb{R}$.

In this paper we consider essentially the same problem for decompositions of Γ into a union of arbitrary disjoint cosets of subgroups of Γ. The proof of the resulting multiplier theorem, Theorem 5, depends on a one-sided extension of the Littlewood-Paley inequality in the context of Vilenkin groups. This generalizes a similar result of Rubio de Francia for functions in $L^p(\mathbb{R})$, $2 \le p < \infty$. We also prove another one-sided Littlewood-Paley-type inequality for functions in $L^p(G)$, $1 < p < 2$. This inequality is then used to obtain an additional multiplier theorem, Theorem 6. Finally, we discuss the sharpness of some of our results, see Theorems 7, 8 and 9.

https://doi.org/10.1017/S1446788700035941 Published online by Cambridge University Press

2. Definitions and notation

Throughout this paper G will denote a locally compact Vilenkin group, that is to say, G is a locally compact Abelian topological group containing a strictly decreasing sequence of open compact subgroups $(G_n)_{-\infty}^{\infty}$ such that $\cup_{-\infty}^{\infty} G_n = G$ and $\cap_{-\infty}^{\infty} G_n = \{0\}$. In [EG, Section 4.1.4] such groups are called groups with a suitable family of compact open subgroups $(G_{-n})_{-\infty}^{\infty}$. Clearly, such groups are totally disconnected. Examples of locally compact Vilenkin groups are the p-adic numbers and, more generally, the additive group of a local field, see [EG] or [Ta] for further details.

Let Γ denote the dual group of G, and for each $n \in \mathbb{Z}$, let Γ_n denote the annihilator of G_n, that is,

$$\Gamma_n = \{\gamma \in \Gamma : \gamma(x) = 1 \text{ for all } x \in G_n\}.$$

Then we have $\cup_{-\infty}^{\infty} \Gamma_n = \Gamma$, $\cap_{-\infty}^{\infty} \Gamma_n = \{1\}$ and order $(\Gamma_{n+1}/\Gamma_n) =$ order (G_n/G_{n+1}) for all $n \in \mathbb{Z}$.

We choose Haar measures μ on G and λ on Γ so that $\mu(G_0) = \lambda(\Gamma_0) = 1$. Then $\mu(G_n) = (\lambda(\Gamma_n))^{-1}$ for all $n \in \mathbb{Z}$; we set $m_n := \lambda(\Gamma_n)$.

For p with $1 \le p \le \infty$ we shall denote its conjugate by p'; thus $1/p + 1/p' = 1$. For an arbitrary set E we denote its characteristic function by χ_E. The symbols $^\wedge$ and $^\vee$ will be used to denote the Fourier and inverse Fourier transform, respectively. It is easy to see that for each $n \in \mathbb{Z}$ we have

$$(\chi_{\Gamma_n})^{\vee} = (\mu(G_n))^{-1}\chi_{G_n} := \Delta_n.$$

For a definition of the spaces of test functions and distributions on G and Γ, see [Ta]; these spaces will be denoted by $\mathscr{S}(G)$, $\mathscr{S}'(G)$, $\mathscr{S}(\Gamma)$ and $\mathscr{S}'(\Gamma)$. We can also extend the Fourier and inverse Fourier transform to $\mathscr{S}'(G)$ and $\mathscr{S}'(\Gamma)$ in the standard way and the usual properties hold, see [Ta] for details.

Let f be a locally integrable function on G. The function $M_2 f$ is defined on G by

$$M_2 f(x) := \sup_{k\in\mathbb{Z}} \left\{ \frac{1}{\mu(x+G_k)} \int_{x+G_k} |f(y)|^2 d\mu(y) \right\}^{1/2}.$$

Thus $M_2 f = \{M(|f|^2)\}^{1/2}$, where M is the Hardy-Littlewood maximal operator on G.

The sharp function $f^\#$ is defined on G by

$$f^\#(x) := \sup_{n\in\mathbb{Z}} \left\{ \frac{1}{\mu(x+G_n)} \int_{x+G_n} |f(y) - f_{x+G_n}| d\mu(y) \right\},$$

where

$$f_{x+G_n} = \frac{1}{\mu(x+G_n)} \int_{x+G_n} f(y) d\mu(y).$$

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For $1 \le p \le \infty$ let $L^p(G)$ be the space of all p-th integrable functions on G, with obvious modification for $p = \infty$. For a measurable function f on G we set

$$\sigma(f, y) = \mu\{x \in G : |f(x)| > y\}, \qquad y > 0,$$

and

$$f^*(t) = \inf\{y > 0 : \sigma(f, y) \le t\}, \qquad t > 0.$$

For $1 \le p < \infty$ and $1 \le q \le \infty$, the Lorentz space $L^{p,q}(G)$ is the collection of all measurable functions f on G such that $\|f\|^*_{L^{p,q}(G)} < \infty$, where

$$\|f\|^*_{L^{p,q}(G)} = \begin{cases} \left(q/p \int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t}\right)^{1/q} & \text{if } 1 \le p < \infty, \quad 1 \le q < \infty \\ \sup_{t>0} t^{1/p} f^*(t) & \text{if } 1 \le p < \infty, \quad q = \infty. \end{cases}$$

Next, the function f^{**} is defined on $\mathbb{R}^+$ by

$$f^{**}(t) = \sup_{t \le \mu(E)} \left\{ \frac{1}{\mu(E)} \int_E |f(x)|^{1/2} d\mu(x) \right\}^2.$$

We denote $\|f^{**}\|^*_{L^{p,q}(\mathbb{R}^+)}$ by $\|f\|_{L^{p,q}(G)}$. It is easy to see that $(f^{**})^* = f^{**}$ and $f^*(t) \le f^{**}(t) \le (f^*)^{**}(t)$ for all $t > 0$. Hence we have

$$\|f\|^*_{L^{p,q}(G)} \le \|f\|_{L^{p,q}(G)} \le \|f^*\|_{L^{p,q}(\mathbb{R}^+)}.$$

By Hardy's inequality we also have

$$\|f^*\|_{L^{p,q}(\mathbb{R}^+)} \le C\|f\|^*_{L^{p,q}(G)}.$$

We note that $L^{p,q}(G) \subseteq L^{p,s}(G)$ if $q \le s$. We equip $L^{p,q}(G)$ with either $\|\cdot\|^*_{L^{p,q}(G)}$ or $\|\cdot\|_{L^{p,q}(G)}$ to define its topology. We observe that $L^{p,p}(G) = L^p(G)$ and we simply denote $\|\cdot\|_{L^{p,q}(G)}$ by $\|\cdot\|_{p,q}$ and $\|\cdot\|_{p,p}$ by $\|\cdot\|_p$ if there is no confusion likely. The same notational simplifications also apply to $\|\cdot\|^*_{L^{p,q}(G)}$.

Let $\phi \in L^\infty(\Gamma)$ and define T_ϕ on $\mathscr{S}(G)$ by $(T_\phi f)^\wedge = \phi\hat{f}$, $f \in \mathscr{S}(G)$. The function ϕ is said to be a multiplier from $L^{p,q}(G)$ into $L^{r,s}(G)$ if there exists a positive constant C so that for all $f \in \mathscr{S}(G)$ we have

$$\|T_\phi f\|_{r,s} \le C\|f\|_{p,q},$$

where $1 \le p, r < \infty$, $1 \le q, s \le \infty$. We say that ϕ is a multiplier of weak type (p, p) if it is a multiplier from $L^p(G)$ to $L^{p,\infty}(G)$. The collection of all multipliers from $L^p(G)$ into $L^p(G)$ is denoted by $\mathscr{M}(L^p(G))$ and the corresponding multiplier norm is denoted by $\|\cdot\|_{\mathscr{M}(L^p)}$.

https://doi.org/10.1017/S1446788700035941 Published online by Cambridge University Press

3. A Littlewood-Paley inequality for arbitrary coset decompositions of Γ; the case $2 \le p < \infty$

Let $\{I_k\}_{k=0}^{\infty}$ be a sequence of mutually disjoint intervals of $\mathbb{R}$. For $f \in L^1(\mathbb{R})$ and $1 < r < \infty$ define the function $\tilde{\Delta}_r f$ on $\mathbb{R}$ by

$$\tilde{\Delta}_r f := \left(\sum_{k=0}^{\infty} |S_{I_k} f|^r \right)^{1/r},$$

where

$$(S_{I_k} f)^{\wedge}(\xi) := \chi_{I_k}(\xi) \hat{f}(\xi).$$

The following result was proved by Rubio de Francia in [R, Theorem 1.2].

THEOREM R. *Let $2 \le p < \infty$. There exists a constant C_p such that*

$$\|\tilde{\Delta}_2 f\|_p \le C_p \|f\|_p, \quad f \in L^p(\mathbb{R}).$$

In [Sj] Sjölin gave a different proof of Theorem R. In this section we use Sjölin's method to obtain an analogue of Theorem R on locally compact Vilenkin groups G.

THEOREM 1. *Let $2 \le p < \infty$ and let $\{\Lambda_k\}_{k=0}^{\infty} := \{\gamma_k + \Gamma_{n_k}\}_{k=0}^{\infty}$ be a decomposition of Γ into mutually disjoint cosets of various subgroups of Γ. For $f \in \mathscr{S}(G)$ define the function Δf on G by*

$$\Delta f := \left(\sum_{k=0}^{\infty} |S_{\Lambda_k} f|^2 \right)^{1/2},$$

where

$$(S_{\Lambda_k} f)^{\wedge}(\gamma) := \chi_{\Lambda_k}(\gamma) \hat{f}(\gamma).$$

Then

$$\|\Delta f\|_p \le C_p \|f\|_p$$

and this inequality can be extended to all $f \in L^p(G)$.

PROOF. It follows immediately from Plancherel's equality that

(1) $$\|\Delta f\|_2 = \|f\|_2.$$

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Thus we may assume that $2 < p < \infty$. For each $k \geq 0$ we define $\psi_k : G \to \mathbb{C}$ by $\psi_k(x) = \gamma_k(x)\Delta_{n_k}(x)$, so that $(\psi_k)^\wedge(\gamma) = \chi_{\Gamma_{n_k}}(\gamma - \gamma_k) = \chi_{\Lambda_k}(\gamma)$. Thus for $f \in \mathscr{S}(G)$ we have

$$\Delta f(x) = \left\{ \sum_{k=0}^{\infty} |\psi_k * f(x)|^2 \right\}^{1/2}.$$

The theorem will follow from the following string of inequalities as in Rubio de Francia [R, p. 5]:

$$\|\Delta f\|_p \leq C\|(\Delta f)^{\#}\|_p \leq C\|M_2 f\|_p \leq C\|f\|_p.$$

It is clear that the last inequality holds as long as $2 < p < \infty$ and we only have to justify the second inequality the proof of which will be given in Lemma 1 below. □

LEMMA 1. *Let $f \in \mathscr{S}(G)$. Then $(\Delta f)^{\#}(x) \leq CM_2 f(x)$ for all $x \in G$.*

PROOF. Take any $x_0 \in G$ and let $I_0 := x_0 + G_{k_0}$ be a coset containing x_0. Decompose f into

$$f = f\chi_{I_0} + f\chi_{G\setminus I_0} := g + h.$$

Let

$$a := \left(\sum_{k \in S_0} |\psi_k * h(x_0)|^2 \right)^{1/2}.$$

where $S_0 = \{k : n_k \leq k_0\}$; that is to say, we sum over those values of k for which the corresponding function ψ_k has the property:

$$G_{k_0} \subset G_{n_k} = \text{supp}\,(\psi_k).$$

For every $x \in G$ we have

$$(\dagger) \qquad |\Delta f(x) - a| \leq |\Delta f(x) - \Delta h(x)| + |\Delta h(x) - a|.$$

We analyze each of the two terms in $(\dagger)$. By the ℓ^2-triangle inequality we have

$$\begin{aligned} \Delta f(x) &= \left(\sum_k |\psi_k * g + \psi_k * h|^2(x) \right)^{1/2} \\ &\leq \left(\sum_k |\psi_k * g(x)|^2 \right)^{1/2} + \left(\sum_k |\psi_k * h(x)|^2 \right)^{1/2} \\ &= \Delta g(x) + \Delta h(x), \end{aligned}$$

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that is,

$$\Delta f(x) - \Delta h(x) \leq \Delta g(x).$$

Similarly,

$$\Delta h(x) = \Delta(f - g)(x) \leq \Delta f(x) + \Delta g(x)$$

so that

$$\Delta h(x) - \Delta f(x) \leq \Delta g(x).$$

Therefore,

$$|\Delta f(x) - \Delta h(x)| \leq \Delta g(x).$$

For the second term in (†) we have

$$\begin{aligned}
|\Delta h(x) - a| &= \left| \left(\sum_k |\psi_k * h(x)|^2 \right)^{1/2} - \left(\sum_{k \in S_0} |\psi_k * h(x_0)|^2 \right)^{1/2} \right| \\
&= \left| \left(\sum_k |\psi_k * h(x)\overline{\gamma_k(x)}|^2 \right)^{1/2} - \left(\sum_{k \in S_0} |\psi_k * h(x_0)\overline{\gamma_k(x_0)}|^2 \right)^{1/2} \right| \\
&\leq \left(\sum_{k \notin S_0} |\psi_k * h(x)|^2 \right)^{1/2} + \left(\sum_{k \in S_0} |F_k(x)|^2 \right)^{1/2},
\end{aligned}$$

where

$$\begin{aligned}
F_k(x) &:= \psi_k * h(x)\overline{\gamma_k(x)} - \psi_k * h(x_0)\overline{\gamma_k(x_0)} \\
&= \int_G \psi_k(x - y)h(y)\overline{\gamma_k(x)}dy - \int_G \psi_k(x_0 - y)h(y)\overline{\gamma_k(x_0)}dy \\
&= \int_G [\Delta_{n_k}(x - y) - \Delta_{n_k}(x_0 - y)]\overline{\gamma_k(y)}h(y)dy.
\end{aligned}$$

Thus we see that

$$\begin{aligned}
|\Delta f(x) - a| &\leq \Delta g(x) + \left(\sum_{k \notin S_0} |\psi_k * h(x)|^2 \right)^{1/2} + \left(\sum_{k \in S_0} |F_k(x)|^2 \right)^{1/2} \\
&:= A_1(x) + A_2(x) + A_3(x).
\end{aligned}$$

We now consider in turn $\dfrac{1}{\mu(I_0)}\displaystyle\int_{I_0} A_i(x)dx, \quad i = 1, 2, 3.$

We have

$$\begin{aligned}
\frac{1}{\mu(I_0)}\int_{I_0} A_1(x)dx &= m_{k_0}\int_{I_0} \Delta g(x)dx \\
&\le m_{k_0}\left(\int_{I_0} |\Delta g(x)|^2 dx\right)^{1/2}\left(\int_{I_0} dx\right)^{1/2} \\
&\le (m_{k_0})^{1/2}\left(\int_G \sum_k |\psi_k * g(x)|^2 dx\right)^{1/2} \\
&\le (m_{k_0})^{1/2}\left(\sum_k \int_\Gamma |(\psi_k * g)^\wedge(\gamma)|^2 d\gamma\right)^{1/2} \\
&\le (m_{k_0})^{1/2}\left(\int_\Gamma |\hat{g}(\gamma)|^2 d\gamma\right)^{1/2} \\
&= (m_{k_0})^{1/2}\left(\int_{I_0} |f(x)|^2 dx\right)^{1/2} \qquad \text{since} \quad g(x) = f(x)\chi_{I_0}(x) \\
&= \left(\frac{1}{\mu(I_0)}\int_{I_0} |f(x)|^2 dx\right)^{1/2} \\
&\le CM_2 f(x_0).
\end{aligned}$$

To find an estimate for

$$\frac{1}{\mu(I_0)}\int_{I_0} A_2(x)dx = \frac{1}{\mu(I_0)}\int_{I_0}\left(\sum_{k\notin S_0} |\psi_k * h(x)|^2\right)^{1/2} dx$$

we observe that for $x \in I_0$ and $k \notin S_0$ we have

$$\begin{aligned}
|\psi_k * h(x)| &= |\psi_k * h(x)\overline{\gamma_k(x)}| \\
&= \left|\int_G \psi_k(x-y)h(y)\overline{\gamma_k(x)}\,dy\right| \\
&= \left|\int_G \Delta_{n_k}(x-y)\overline{\gamma_k(y)}h(y)\,dy\right| \\
&= \left|\int_{G\setminus I_0} \Delta_{n_k}(x-y)\overline{\gamma_k(y)}f(y)\,dy\right|.
\end{aligned}$$

For $x \in I_0 = x_0 + G_{k_0}$ and $y \notin I_0$ we have $x - y \notin G_{k_0}$. Also, $k \notin S_0$ implies that $G_{n_k} \subset G_{k_0}$. Thus $x - y \notin G_{n_k}$ and, hence, $\Delta_{n_k}(x-y) = 0$. That is,

$$\frac{1}{\mu(I_0)}\int_{I_0} A_2(x)dx = 0.$$

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To find an estimate for

$$\frac{1}{\mu(I_0)}\int_{I_0} A_3(x)dx = \frac{1}{\mu(I_0)}\int_{I_0}\left(\sum_{k\in S_0}|F_k(x)|^2\right)^{1/2} dx$$

we observe that

(i) if $x \in I_0$ and $k \in S_0$ and $y \in x_0 + G_{n_k}$ then we have $x - y \in G_{n_k}$ and $x_0 - y \in G_{n_k}$, so that

$$\Delta_{n_k}(x-y) - \Delta_{n_k}(x_0 - y) = m_{n_k} - m_{n_k} = 0.$$

(ii) if $x \in I_0$ and $k \in S_0$ and $y \notin x_0 + G_{n_k}$ then $x - y \notin G_{n_k}$ and $x_0 - y \notin G_{n_k}$, so that

$$\Delta_{n_k}(x-y) - \Delta_{n_k}(x_0 - y) = 0.$$

We see that for $x \in I_0$ and $k \in S_0$ we have $F_k(x) = 0$, so that

$$\frac{1}{\mu(I_0)}\int_{I_0} A_3(x)dx = 0.$$

Thus we may conclude that

$$\frac{1}{\mu(I_0)}\int_{I_0} |\Delta f(x) - a|dx \leq CM_2 f(x_0),$$

so that

$$(\Delta f)^{\#}(x_0) \leq CM_2 f(x_0).$$

This completes the proof of the Lemma. □

4. A Littlewood-Paley-type inequality for arbitrary coset decompositions of Γ; the case $1 < p < 2$

For the case $1 < p < 2$, Rubio de Francia conjectured that for each $f \in L^p(\mathbb{R})$ we have

$$\|\tilde{\Delta}_{p'} f\|_p \leq C_p \|f\|_p.$$

In this section we shall prove an inequality that is related to but weaker than the inequality in Rubio de Francia's conjecture.

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THEOREM 2. *Let* $1 < p < 2$ *and let* $\{\Lambda_k\}_{k=0}^{\infty} := \{\gamma_k + \Gamma_{n_k}\}_{k=0}^{\infty}$ *be a decomposition of* Γ *into mutually disjoint cosets of various subgroups of* Γ. *If* T *is the operator defined for a simple function* f *on* G *by* $Tf = \{\sum_0^{\infty} |S_{\Lambda_k} f|^{p'}\}^{1/p'}$, *then* $\|Tf\|_{p,p'} \le C\|f\|_p$. *Hence* T *can be extended to a bounded operator from* $L^p(G)$ *into* $L^{p,p'}(G)$.

PROOF. For each $k \ge 0$, $x \in G$ and $f \in \mathscr{S}(G)$ we have

$$
\begin{aligned}
|S_{\Lambda_k} f(x)| &= |\psi_k * f(x)| \\
&= \left| \int_G \gamma_k(x-y)\Delta_{n_k}(x-y) f(y)\, dy \right| \\
&\le \Delta_{n_k} * |f|(x) \qquad (*) \\
&\le Mf(x).
\end{aligned}
$$

Thus,

$$(**) \qquad \sup_k |S_{\Lambda_k} f(x)| \le Mf(x)$$

so that the mapping

$$(2) \qquad f \to \sup_k |S_{\Lambda_k} f| \quad \text{is of weak type } (1,1).$$

We now choose θ such that $1/p = 1 - \theta/2$, that is, $\theta = 2(1 - 1/p)$; then $0 < \theta < 1$. Let $\Omega := \{z \in \mathbb{C} : 0 \le \operatorname{Re} z \le 1\}$ and let $f \in \mathscr{S}(G)$ such that $\|f\|_p = 1$. For $z \in \Omega$ define the function f_z on G by

$$f_z(x) = \begin{cases} \frac{f(x)}{|f(x)|} |f(x)|^{p(1-z/2)} & \text{if } f(x) \ne 0 \\ 0 & \text{if } f(x) = 0. \end{cases}$$

Then $f_z \in \mathscr{S}(G)$ for each $z \in \Omega$. Moreover we have $f_\theta = f$, $\|f_{it}\|_1 = 1$ and $\|f_{1+it}\|_2 = 1$. For $N \in \mathbb{N}$, $z \in \Omega$ and $x \in G$ define the sequence $\{(T_N f_z)_k(x)\}_{k=0}^{\infty}$ by

$$(T_N f_z)_k(x) = \begin{cases} S_{\Lambda_k} f_z(x) & \text{if } 0 \le k \le N \\ 0 & \text{if } k > N. \end{cases}$$

Let $[\ell^\infty, \ell^2]_\theta$ be the complex interpolation space. Then $[\ell^\infty, \ell^2]_\theta = \ell^{p'}$ (see [Tr, 1.18.1, (12)]). For each $x \in G$ define $U_N f_\theta(x)$ by

$$
\begin{aligned}
U_N f_\theta(x) &:= \left\| \{(T_N f_\theta)_k(x)\}_{k=0}^{\infty} \right\|_{[\ell^\infty, \ell^2]_\theta} \\
&= \sum_{k=0}^{N} \left(\left| S_{\Lambda_k} f_\theta(x) \right|^{p'} \right)^{1/p'}.
\end{aligned}
$$

It follows from [Tr, 1.10.3, (9)] that

$$\log U_N f_\theta(x) \le \int_{-\infty}^{\infty} P_0(\theta, t) \log \|\{(T_N f_{it})_k(x)\}\|_{\ell^\infty} dt$$
$$+ \int_{-\infty}^{\infty} P_1(\theta, t) \log \|\{(T_N f_{1+it})_k(x)\}\|_{\ell^2} dt, \tag{3}$$

where $P_0(\theta, t) \ge 0$, $P_1(\theta, t) \ge 0$, $\int_{-\infty}^{\infty} P_0(\theta, t)dt = 1 - \theta$ and $\int_{-\infty}^{\infty} P_1(\theta, t)dt = \theta$.

Thus, taking exponentials in (3) we have

$$U_N f_\theta(x) \le \left[\left\{\exp\left(\frac{1}{(1-\theta)} \int_{-\infty}^{\infty} P_0(\theta, t) \log \|\{(T_N f_{it})_k(x)\}\|_{\ell^\infty}^{1/2} dt\right)\right\}^2\right]^{(1-\theta)}$$
$$\times \left[\left\{\exp\left(\frac{1}{\theta} \int_{-\infty}^{\infty} P_1(\theta, t) \log \|\{(T_N f_{1+it})_k(x)\}\|_{\ell^2}^{1/2} dt\right)\right\}^2\right]^{\theta}.$$

It follows from Jensen's inequality that

$$U_N f_\theta(x) \le \{H_{N,0}(x)\}^{(1-\theta)}\{H_{N,1}(x)\}^{\theta},$$

where

$$H_{N,0}(x) = \left(\frac{1}{1-\theta} \int_{-\infty}^{\infty} P_0(\theta, t) \|\{(T_N f_{it})_k(x)\}\|_{\ell^\infty}^{1/2} dt\right)^2$$

and

$$H_{N,1}(x) = \left(\frac{1}{\theta} \int_{-\infty}^{\infty} P_1(\theta, t) \|\{(T_N f_{1+it})_k(x)\}\|_{\ell^2}^{1/2} dt\right)^2.$$

For each measurable subset E of G we have

$$\left(\frac{1}{\mu(E)} \int_E (U_N f_\theta(x))^{1/2} dx\right)^2$$
$$\le \left(\frac{1}{\mu(E)} \int_E \{H_{N,0}(x)\}^{(1-\theta)/2}\{H_{N,1}(x)\}^{\theta/2} dx\right)^2$$
$$\le \left(\frac{1}{\mu(E)} \int_E \{H_{N,0}(x)\}^{1/2} dx\right)^{2(1-\theta)} \left(\frac{1}{\mu(E)} \int_E \{H_{N,1}(x)\}^{1/2} dx\right)^{2\theta}.$$

It follows that for $y > 0$

$$(U_N f_\theta)^{**}(y) = \sup_{y \le \mu(E)} \left(\frac{1}{\mu(E)} \int_E (U_N f_\theta(x))^{1/2} dx\right)^2$$
$$\le \{H_{N,0}^{**}(y)\}^{(1-\theta)}\{H_{N,1}^{**}(y)\}^{\theta}.$$

Since $(U_N f_\theta)^{**} = \{(U_N f_\theta)^{**}\}^*$ we have, for $1 < p < \infty$,

$$\begin{aligned}\|U_N f_\theta\|_{p,p'} &= \left(\frac{p'}{p}\int_0^\infty (t^{1/p}(U_N f_\theta)^{**}(t))^{p'}\frac{dt}{t}\right)^{1/p'}\\ &\le \left(\frac{p'}{p}\int_0^\infty \left(t^{1/p}\{H_{N,0}^{**}(t)\}^{(1-\theta)}\{H_{N,1}^{**}(t)\}^{\theta}\right)^{p'}\frac{dt}{t}\right)^{1/p'}\\ &= \left(\frac{p'}{p}\int_0^\infty \{tH_{N,0}^{**}(t)\}^{(1-\theta)p'}\{t^{1/2}H_{N,1}^{**}(t)\}^{\theta p'}\frac{dt}{t}\right)^{1/p'}\\ &= \left(\frac{p'}{p}\int_0^\infty \{tH_{N,0}^{**}(t)\}^{(1-\theta)p'}\{t^{1/2}H_{N,1}^{**}(t)\}^{2}\frac{dt}{t}\right)^{1/p'},\end{aligned}$$

since $\theta p' = 2(1-1/p)p' = 2$. By Hölder's inequality we have

$$\begin{aligned}\|U_N f_\theta\|_{p,p'} &\le \sup\{tH_{N,0}^{**}(t)\}^{(1-\theta)}\left(\frac{p'}{p}\int_0^\infty \{t^{1/2}H_{N,1}^{**}(t)\}^2\frac{dt}{t}\right)^{\theta/2}\\ &\le B\|H_{N,0}\|_{1,\infty}^{(1-\theta)}\|H_{N,1}\|_2^{\theta},\end{aligned}$$

where we use $1/p' = \theta/2$ in the first inequality.

We shall estimate $\|H_{N,0}\|_{1,\infty}$. For $y > 0$ we have

$$\begin{aligned}H_{N,0}^{**}(y) &= \sup_{y\le\mu(E)}\left(\frac{1}{\mu(E)}\int_E |H_{N,0}(x)|^{1/2}d\mu(x)\right)^2\\ &= \sup_{y\le\mu(E)}\left[\frac{1}{\mu(E)}\int_E\left(\frac{1}{(1-\theta)}\int_{-\infty}^{\infty}P_0(\theta,t)\|\{(T_N f_{it})_k(x)\}\|_{\ell^\infty}^{1/2}dt\right)d\mu(x)\right]^2\\ &= \left[\frac{1}{(1-\theta)}\int_{-\infty}^{\infty}P_0(\theta,t)\left(\sup_{y\le\mu(E)}\frac{1}{\mu(E)}\int_E\|\{(T_N f_{it})_k(x)\}\|_{\ell^\infty}^{1/2}d\mu(x)\right)dt\right]^2,\end{aligned}$$

where the last equality follows from Fubini's theorem. Now

$$\|\{(T_N f_{it})_k(x)\}\|_{\ell^\infty} = \sup_{0\le k\le N}|S_{\Lambda_k} f_{it}(x)| \le \sup_{0\le k}|S_{\Lambda_k} f_{it}(x)| := F_{it}(x).$$

Therefore

$$\begin{aligned}H_{N,0}^{**}(y) &\le \left[\frac{1}{(1-\theta)}\int_{-\infty}^{\infty}P_0(\theta,t)\left(\sup_{y\le\mu(E)}\frac{1}{\mu(E)}\int_E (F_{it}(x))^{1/2}d\mu(x)\right)dt\right]^2\\ &\le \left[\frac{1}{(1-\theta)}\int_{-\infty}^{\infty}P_0(\theta,t)\,((F_{it})^{**}(y))^{1/2}\,dt\right]^2\\ &\le (F_{it})^{**}(y).\end{aligned}$$

Consequently we have

$$\|H_{N,0}\|_{1,\infty} \le \|F_{it}\|_{1,\infty} = \Big\| \|\{S_{\Lambda_k} f_{it}\}\|_{\ell^\infty} \Big\|_{1,\infty} \le C\|f_{it}\|_1 = C,$$

where the last inequality follows from (2). Similarly we have

$$\begin{aligned} \|H_{N,1}\|_2 &\le C\left(\frac{1}{\theta}\int_{-\infty}^{\infty} P_1(\theta,t)\|f_{1+it}\|_2^{1/2}dt\right)^2 \\ &\le C\|\Delta f_{1+it}\|_2 \\ &= C\|f_{1+it}\|_2 \qquad \text{(using (1))} \\ &= C. \end{aligned}$$

It follows that $\|U_N f_\theta\|_{p,p'} \le C$ if $\|f\|_p = 1$. Since $\|Tf\|_{p,p'} = \lim_{N\to\infty} \|U_N f_\theta\|_{p,p'}$, we have $\|Tf\|_{p,p'} \le C\|f\|_p$ for $f \in \mathscr{S}(G)$. Now $\mathscr{S}(G)$ is dense in $L^p(G)$ and so T can be extended to all functions in $L^p(G)$ and our proof is complete. □

We observe that inequality (**) in the proof of Theorem 2 above shows that for each $r > 1$ we have

$$\left\| \sup_k |S_{\Lambda_k} f| \right\|_r \le \|Mf\|_r \le C\|f\|_r. \tag{4}$$

Interpolation between (1) and (4) yields the following theorem.

THEOREM 3. *Let $1 < p < 2$ and let $\{\Lambda_k\}_{k=0}^\infty$ be as in Theorem 2. If $s > p'$, then*

$$\left\| \sum_{k=0}^{\infty} \left(|S_{\Lambda_k} f|^s\right)^{1/s} \right\|_p \le C\|f\|_p.$$

Another result we can derive from inequality (*) in the proof of Theorem 2 is the following theorem.

THEOREM 4. *Assume $\{\Lambda_k\}_{k=0}^\infty = \{\gamma_k + \Gamma_{n_0}\}_{k=0}^\infty$ for some fixed n_0 (i.e. we have a partition of Γ into the cosets of a fixed subgroup Γ_{n_0} of Γ). Then*

$$\left\| \sum_{k=0}^{\infty} \left(|S_{\Lambda_k} f|^{p'}\right)^{1/p'} \right\|_p \le C\|f\|_p \tag{5}$$

PROOF. According to (*) in the proof of Theorem 2, we have for every $k \ge 0$,

$$|S_{\Lambda_k} f(x)| \le \Delta_{n_0} * |f|(x),$$

so that

$$\left\| \sup_k |S_{\Lambda_k} f| \right\|_1 \le \left\| \Delta_{n_0} * |f| \right\|_1 \le \|\Delta_{n_0}\|_1 \|f\|_1 = \|f\|_1. \tag{6}$$

Interpolation between (1) and (6) yields (5). □

REMARK. Note that a slight generalization of this result can be obtained by considering partitions $\{\Lambda_k\}_{k=0}^{\infty} = \{\gamma_k + \Gamma_{n_k}\}_{k=0}^{\infty}$ satisfying the condition $\sup_k \lambda(\gamma_k + \Gamma_{n_k}) = \sup_k m_{n_k} = m_{n_\alpha}$ for some $n_\alpha \in \mathbb{Z}$. In this case we have for each $k \geq 0$,

$$|S_{\Lambda_k} f(x)| \leq \Delta_{n_k} * |f|(x)$$

so that

$$\sup_k |S_{\Lambda_k} f(x)| \leq \sum_{t=0}^{\alpha} \Delta_{n_t} * |f|(x).$$

Therefore,

$$\left\|\sup_k |S_{\Lambda_k} f| \right\|_1 \leq \sum_{t=0}^{\alpha} \left\| \Delta_{n_t} * |f| \right\|_1 \leq C\|f\|_1,$$

yielding again (5).

5. Multipliers on $L^p(G)$

In [OQ] we considered the decomposition of Γ into disjoint sets $\Gamma_{k+1}\backslash\Gamma_k$ and in [OQ, Theorem 2.1] the following multiplier theorem was proved.

THEOREM OQ. *Let* $1 < p < \infty$ *and let* $\{\phi_k\}_{k=-\infty}^{\infty} \in \ell^s(\mathcal{M}(L^p(G)))$ *for some* $0 < s \leq |2p/(2-p)|$. *If* $\phi := \sum_{-\infty}^{\infty} \phi_k \chi_{\Gamma_{k+1}\backslash\Gamma_k} \in L^\infty(\Gamma)$ *then* $\phi \in \mathcal{M}(L^p(G))$.

As an application of Theorem 1 we prove a comparable result for decompositions of Γ as considered in the present paper, see Theorem 5. Our proof was motivated by [CFF, Theorem 2] and is similar to that of [OQ, Theorem 2.1]. We shall discuss the sharpness of Theorem 5 in Theorem 8.

THEOREM 5. *Let* $\{\Lambda_k\}_{k=0}^{\infty} = \{\gamma_k + \Gamma_{n_k}\}_{k=0}^{\infty}$ *be as in Theorem* 1 *and let* $1 < p < \infty$. *Let* $\{\phi_k\}_{k=0}^{\infty} \in \ell^s(\mathcal{M}(L^p(G)))$ *for* $s = |p/(2-p)|$ *and assume* $\phi := \sum_{k=0}^{\infty} \phi_k \chi_{\Lambda_k} \in L^\infty(\Gamma)$. *Then* $\phi \in \mathcal{M}(L^p(G))$.

PROOF. We may assume that $2 < p < \infty$ and that $s = p/(p-2)$. Take any $f \in \mathcal{S}(G)$. A direct computation for the cases $p = 2$ and $p = \infty$, followed by an interpolation argument shows that the following inequality holds:

$$\|(\phi \hat{f})^\vee\|_p^{p'} = \left\| \sum_k \psi_k * (\phi_k \chi_{\Lambda_k} \hat{f})^\vee \right\|_p^{p'} \leq C \sum_k \|(\phi_k \chi_{\Lambda_k} \hat{f})^\vee\|_p^{p'}.$$

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Therefore,

$$\begin{aligned}
\|(\phi\hat{f})^{\vee}\|_p^{p'} &\le C\sum_k \|\phi_k\|^{p'}_{\mathscr{M}(L^p)}\|S_{\Lambda_k}f\|_p^{p'} \\
&\le C\left(\sum_k \|\phi_k\|^{p'/(2-p')}_{\mathscr{M}(L^p)}\right)^{2-p'}\left(\sum_k \|S_{\Lambda_k}f\|_p^p\right)^{p'/p} \\
&= C\left(\sum_k \|\phi_k\|^{s}_{\mathscr{M}(L^p)}\right)^{p'/s}\left(\int_G\sum_k |S_{\Lambda_k}f(x)|^p\,dx\right)^{p'/p} \\
&\le C\left(\sum_k \|\phi_k\|^{s}_{\mathscr{M}(L^p)}\right)^{p'/s}\left(\int_G\left\{\sum_k |S_{\Lambda_k}f(x)|^2\right\}^{p/2}dx\right)^{p'/p} \\
&\le C\|f\|_p^{p'}.
\end{aligned}$$

where the penultimate inequality holds because $2 < p$, while the final inequality follows from Theorem 1. □

As an additional application of Theorems 1 and 2 we have

THEOREM 6. *Let* $\{\Lambda_k\}_{k=0}^{\infty}$ *be a decomposition of* Γ *as in Theorem* 1.

(i) *If* $\{a_k\}_{k=0}^{\infty} \in \ell^2$, *then* $\sum_{k=0}^{\infty} a_k\chi_{\Lambda_k}$ *is a multiplier on* $L^p(G)$ *for* $1 < p < \infty$.

(ii) *If* $\{a_k\}_{k=0}^{\infty} \in \ell^s$ *for some* $s > 2$, *then* $\sum_{k=0}^{\infty} a_k\chi_{\Lambda_k}$ *is a multiplier from* $L^p(G)$ *into* $L^{p,p'}(G)$ *for* $2s/(2+s) \le p \le 2$.

PROOF. (i) It follows from Theorem 1 that for $2 \le p < \infty$ we have

$$\left\|\sum_{k=0}^{\infty}(a_k\chi_{\Lambda_k}\hat{f})^{\vee}\right\|_p \le \left\{\sum_{k=0}^{\infty}|a_k|^2\right\}^{1/2}\left\|\left\{\sum_{k=0}^{\infty}|S_{\Lambda_k}f|^2\right\}^{1/2}\right\|_p \le C\|f\|_p.$$

Hence $\sum_{k=0}^{\infty} a_k\chi_{\Lambda_k}$ is a multiplier on $L^p(G)$ for $2 \le p < \infty$. The case $1 < p < 2$ follows from duality.

(ii) Applying real interpolation (see [Tr, 1.18.6, Theorem 2]) to the inequalities obtained from the cases $p = 2$ and $p = r^*$ for some $r^* > r$ of Theorem 1, we obtain

$$\left\|\left\{\sum_{k=0}^{\infty}|S_{\Lambda_k}f|^2\right\}^{1/2}\right\|_{r,q} \le C\|f\|_{r,q} \tag{7}$$

for $2 < r < \infty$ and $1 \le q < \infty$. Also, an argument as in [St, Chapter IV, 5.3.1] shows that for all $f, g \in \mathscr{S}(G)$

$$\int_G f(x)\overline{g(x)}dx = \sum_{k=0}^{\infty}\int_G S_{\Lambda_k}f(x)\overline{S_{\Lambda_k}g(x)}dx. \tag{8}$$

Next, a standard argument using (7), (8) and the converse of Hölder's inequality for Lorentz spaces shows that for $1 < p < 2$

$$\|f\|_{p,p'} \leq C \left\| \left(\sum_{k=0}^{\infty} |S_{\Lambda_k} f|^2 \right)^{1/2} \right\|_{p,p'}. \tag{9}$$

Finally, set $t = 2s/(2+s)$; using inequality (9), Hölder's inequality and Theorem 2 (see the proof in [CFF, p.341]) shows that

$$\left\| \sum_{k=0}^{\infty} (a_k \chi_{\Lambda_k} \hat{f})^{\vee} \right\|_{t,t'} \leq C \left\| \left(\sum_{k=0}^{\infty} |a_k S_{\Lambda_k} f|^2 \right)^{1/2} \right\|_{t,t'} \leq C \|f\|_t .$$

Hence $\sum_{k=0}^{\infty} a_k \chi_{\Lambda_k}$ is a multiplier from $L^t(G)$ into $L^{t,t'}(G)$. The result now follows because $\sum_{k=0}^{\infty} a_k \chi_{\Lambda_k}$ is also a multiplier on $L^2(G)$. □

6. Sharpness of certain results

The following theorem shows that Theorem 2 is sharp in a certain sense.

THEOREM 7. *Let $1 < p < 2$ and let $s < p'$. There exists a decomposition $\{\Lambda_k\}_{k=0}^{\infty}$ of Γ into mutually disjoint cosets of various subgroups of Γ such that the mapping $f \to \{\sum_{k=0}^{\infty} |S_{\Lambda_k} f|^s\}^{1/s}$ is not bounded from $L^p(G)$ to $L^{p,p'}(G)$.*

PROOF. Take $\{\Lambda_k\}_{k=0}^{\infty} = \{\gamma_k + \Gamma_0\}_{k=0}^{\infty}$, that is, partition Γ into the cosets of Γ_0 and choose the γ_k in such a way that for each $l \geq 0$, we have

$$\bigcup_{0 \leq k < m_l} \gamma_k + \Gamma_0 = \Gamma_l .$$

Next, for $l \geq 0$, let $f_l(x) = \Delta_l(x)$, so that $\|f_l\|_p = (m_l)^{1/p'}$ and $(f_l)^{\wedge}(\gamma) = \chi_{\Gamma_l}(\gamma)$. Then

$$S_{\Lambda_k} f_l(x) = \begin{cases} \chi_{G_0}(x)\gamma_k(x) & \text{if } k < m_l \\ 0 & \text{if } k \geq m_l . \end{cases}$$

Therefore,

$$\left\| \left(\sum_{k=0}^{m_l - 1} |S_{\Lambda_k} f_l|^s \right)^{1/s} \right\|_{p,p'}^{*} = (m_l)^{1/s} .$$

If there were a constant C such that

$$\left\| \left(\sum_{k=0}^{m_l-1} |S_{\Lambda_k} f_l(x)|^s \right)^{1/s} \right\|_{p,p'}^* \le C\|f_l\|_p,$$

then we would have $(m_l)^{1/s} \le C(m_l)^{1-1/p}$ for all $l \ge 0$. But this is impossible because $s < p'$. □

Theorem 7 has the following obvious corollary which shows that Theorem 1 is not necessarily true if $1 < p < 2$.

COROLLARY. *Let $1 < p < 2$. Then there exists a decomposition $\{\Lambda_k\}_{k=0}^{\infty}$ of Γ into mutually disjoint cosets of various subgroups of Γ such that the mapping $f \to \Delta f$ is not bounded on $L^p(G)$, where Δf is as defined in Theorem 1.*

Next we prove the sharpness of Theorem 5. The example constructed in the proof of Theorem 8 below is analogous to [CFF, Example 2].

THEOREM 8. *Let $1 < p < \infty$ and assume that $q > s = |p/(2-p)|$. Then there exists a decomposition $\{\Lambda_k\}_{k=0}^{\infty}$ of Γ as in Theorem 1 and functions $\{\phi_k\} \in \mathscr{M}(L^p(G))$ such that*

(a) *$\operatorname{supp}\phi_k = \Lambda_k$ for all non-negative integers k,*
(b) *$\{\phi_k\} \in l^q(\mathscr{M}(L^p(G)))$,*
(c) *if $\phi := \sum_0^{\infty} \phi_k$ then $\phi \in L^{\infty}(\Gamma)$ and $\phi \notin \mathscr{M}(L^p(G))$.*

PROOF. We assume that $1 < p < 2$ so that $s = p/(2-p)$. Take $\{\Lambda_k\}_{k=0}^{\infty} = \{\gamma_k + \Gamma_0\}_{k=0}^{\infty}$ and choose the γ_k so that for each $l \ge 0$, we have

$$\bigcup_{0 \le k < m_l} \gamma_k + \Gamma_0 = \Gamma_l.$$

Choose α so that $1/q < \alpha < 1/s$. For each $k \ge 0$ choose an $x_k \in G_{-k} \setminus G_{-k+1}$ and define the functions $\phi_k : \Gamma \to \mathbb{C}$ by

$$\phi_k(\gamma) = (k+1)^{-\alpha}\overline{\gamma(x_k)}\chi_{\Lambda_k}(\gamma).$$

Then we have $(\phi_k)^{\vee}(x) = (k+1)^{-\alpha}\gamma_k(x)\chi_{G_0}(x - x_k)$, so that $\|\phi_k\|_{\mathscr{M}(L^p)} \le \|\phi_k\|_1 = (k+1)^{-\alpha}$. Hence the sequence $\{\phi_k\}$ satisfies conditions (a) and (b).

Moreover, if we define $\phi := \sum_0^{\infty} \phi_k$, then it can be shown as in [OQ, Theorem 2.2] that $\phi \notin \mathscr{M}(L^p(G))$. This completes the proof of Theorem 8. □

Our last result shows that Theorem 6 is also best possible in a certain sense.

https://doi.org/10.1017/S1446788700035941 Published online by Cambridge University Press

THEOREM 9. *Let G be the dyadic group. Let $2 < s < \infty$ and let $p = 2s/(2+s)$. Then there exists a sequence $\{a_k\}_{k=1}^{\infty} \in \ell^s$ and a decomposition of Γ as in Theorem 1 so that $\sum_{k\in\mathbb{N}} a_k \chi_{\Lambda_k}$ is not a multiplier from $L^r(G)$ into $L^{r,r'}(G)$ for any r such that $1 < r < p$.*

PROOF. Following [GI, Example 5.2], we construct Rudin-Shapiro-like polynomials on G as follows:

For $0 \le n$, fix γ_0^n in $\Gamma_{2n+2}\backslash\Gamma_{2n+1}$ and let

$$\rho_0^n = \sigma_0^n = \chi_{G_n}\gamma_0^n.$$

Next, for $k = 1, \dots, n+1$, set

$$\rho_k^n = \rho_{k-1}^n + \gamma_k^n \sigma_{k-1}^n,$$

and

$$\sigma_k^n = \rho_{k-1}^n - \gamma_k^n \sigma_{k-1}^n,$$

where γ_k^n are chosen from Γ_{2n+1} such that $(\rho_k^n)^\wedge$ and $(\sigma_k^n)^\wedge$ are both constant and non-zero on precisely 2^k cosets of Γ_n in $\Gamma_{2n+2}\backslash\Gamma_{2n+1}$. Now define Θ on Γ by

$$\Theta(\gamma) = \begin{cases} \operatorname{sgn}(\rho_{n+1}^n)^\wedge(\gamma) & \text{if } \gamma \in \Gamma_{2n+2}\backslash\Gamma_{2n+1}, \quad n \ge 0 \\ 0 & \text{otherwise.} \end{cases}$$

Choose q such that $r < q < p$ and choose α so that $q < 2/(2-\alpha) < p$; then $0 < \alpha < 1$. Define Φ on Γ by

$$\Phi(\gamma) = \sum_{n\in\mathbb{N}} 2^{(\alpha-1)n/2} \chi_{\Gamma_{2n}\backslash\Gamma_{2n-1}}(\gamma)\Theta(\gamma).$$

Note that for $n \ge 1$, $\Phi(\gamma)$ is constant $(= \pm 2^{(\alpha-1)n/2})$ on the 2^n cosets of Γ_{n-1} in $\Gamma_{2n}\backslash\Gamma_{2n-1}$ and is zero elsewhere. Denote the 2^n cosets of Γ_{n-1} in $\Gamma_{2n}\backslash\Gamma_{2n-1}$ by $\Lambda_{(2n,k)}$ for $k = 1, \dots, 2^n$. Now define the sequence $\{a_{(2n,k)}\}$ such that

$$|a_{(2n,k)}| = 2^{(\alpha-1)n/2} \qquad \text{for} \quad n \in \mathbb{N}, \ k = 1, \dots, 2^n$$

and satisfying

$$\sum_{n\in\mathbb{N}} \sum_{k=1}^{2^n} a_{(2n,k)} \chi_{\Lambda_{(2n,k)}}(\gamma) = \sum_{n\in\mathbb{N}} 2^{(\alpha-1)n/2} \chi_{\Gamma_{2n}\backslash\Gamma_{2n-1}}(\gamma)\Theta(\gamma).$$

It is easy to see that

$$\sum_{n\in\mathbb{N}}\sum_{k=1}^{2^n}|a_{(2n,k)}|^s < \infty.$$

Now suppose Φ were a multiplier from $L^r(G)$ to $L^{r,r'}(G)$, then Φ would be a multiplier on $L^q(G)$ because Φ is a multiplier on $L^2(G)$ and $r < q < p < 2$. But by [GI, Example 5.2] Φ is not a multiplier on $L^q(G)$. Hence we have a contradiction. □

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