ON A PROBLEM OF ERDŐS AND SZEKERES

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(received June 27, 1960)

1. Write

$$M(a_1, \ldots, a_n) = \max \prod_{k=1}^{n} |1 - \exp(a_k i\theta)|, \tag{1}$$

$$f(n) = \text{g.l.b. } M(a_1, \ldots, a_n), \tag{2}$$

where the maximum is over all real θ, and the lower bound is over all sets of positive integers $a_1 \leq a_2 \leq \ldots \leq a_n$. The problem of the order of magnitude of $f(n)$ was posed by Erdős and Szekeres [1], side by side with a number of other interesting questions. Writing $g(n) = \log f(n)$, it is obvious that $g(n)$ is sub-additive, in the sense that $g(m+n) \leq g(m) + g(n)$, and also that $g(1) = \log 2$, so that $g(n) \leq n \log 2$. However, they were able to prove the stronger result that

$$g(n) = o(n) \tag{3}$$

for $n \to \infty$, their main tool being the approximation to numbers by rationals; in the other direction, a distinct argument showed that

$$g(n) \geq \tfrac{1}{2} \log(2n). \tag{4}$$

The aim of this note is to improve (3) to

$$g(n) \leq n^{\frac{1}{2}}(\tfrac{1}{2} \log n + 4 \log 2), \tag{5}$$

using a method which is connected more with Fourier series and trigonometric polynomials.

2. The connection with the latter topics appears on taking

Canad. Math. Bull. vol. 4, no. 1, January 1961.

https://doi.org/10.4153/CMB-1961-002-5 Published online by Cambridge University Press

logarithms in (1). We get

$$g(n) = \text{g.l.b. } \max \Sigma_1^n \log|1 - \exp(a_k i\theta)|$$

where the lower bound and maximum are taken over the same sets as in § 1. We re-write this as follows. Define, for non-negative $c_1, \ldots, c_p$, and $1 \leq p \leq n$,

$$(6) \quad N(c_1, \ldots, c_p) = \max_{0 < \theta \leq 2\pi} \Sigma_{k=1}^{p} c_k \log|1 - e^{ki\theta}|.$$

We then write

$$g(p, n) = \text{g.l.b. } N(c_1, \ldots, c_p),$$

where the lower bound is over all sets of p non-negative integers whose sum is n, so that, finally,

$$g(n) = \min_{1 \leq p \leq n} g(p, n).$$

If in (6) we use the Fourier series

$$(7) \qquad \log|1 - e^{i\theta}| = -\Sigma_1^{\infty} m^{-1} \cos m\theta,$$

valid when $\theta \not\equiv 0 \pmod{2\pi}$, re-arrangement gives, formally at any rate,

$$(8) \qquad N(c_1, \ldots, c_p) = -\min_{0 \leq \theta \leq 2\pi} \Sigma_{m=1}^{\infty} m^{-1} j(m\theta),$$

where

$$(9) \qquad j(\phi) = \Sigma_{k=1}^{p} c_k \cos k\phi.$$

This suggests that we should choose the c_k so as to maximize the lower bound of $j(\phi)$, which will be negative if the c_k are not all zero.

3. In making this line of reasoning rigorous, the main obstacle is that the Fourier series (7) is not absolutely convergent. It turns out that the place of the infinite series (7) may be taken by a partial sum of the series, modified by the insertion of certain means.

LEMMA 1. For positive integral M, and real $\theta \not\equiv 0 \pmod{2\pi}$

(10)

$$\log|1 - e^{i\theta}| \leq - \Sigma_{m=1}^{M-1}(1 - m/M)^2 m^{-1} \cos m\theta + M^{-2}(2M - 1)\log 2.$$

For $M = 1$ the sum is empty, and the inequality is then precise if $\theta = \pi$.

It is clearly sufficient to prove this when $0 < \theta \leq \pi$. On this assumption we have

$$\log(1 - e^{i\theta}) = \int_{e^{i\theta}}^{o} \frac{dz}{1 - z}$$

$$= \int_{e^{i\theta}}^{o} \Sigma_0^{N-1} z^k dz + \int_{e^{i\theta}}^{o} \frac{z^N}{1 - z} dz$$

$$= - \Sigma_1^N m^{-1} e^{mi\theta} + R,$$

say. We write

$$R = \left\{\int_{e^{i\theta}}^{-1} + \int_{-1}^{o}\right\} \frac{z^N}{1 - z} dz = R_1 + R_2,$$

say, where R_1 is taken round the unit circle in the positive sense and R_2 along the negative real axis. Putting $z = e^{i\theta}$ we get

$$R_1 = \int_\theta^\pi \frac{e^{Ni\theta} \cdot ie^{i\theta}}{1 - e^{i\theta}} d\theta$$

$$= -\tfrac{1}{2}\int_\theta^\pi e^{(N+\frac{1}{2})i\theta} \operatorname{cosec} \tfrac{1}{2}\theta d\theta.$$

Taking real parts, we therefore get

$$(11)\quad \log|1 - e^{i\theta}| = -\Sigma_{m=1}^{N} m^{-1} \cos m\theta - \int_\theta^\pi \frac{\cos(N + \frac{1}{2})\theta}{2 \sin \frac{1}{2}\theta} d\theta + \int_{-1}^{o} \frac{z^N}{1 - z} dz.$$

We now take (11) with $N = 0, 1, \ldots, M - 1$, multiplying these results respectively by $(2M - 1), (2M - 3), \ldots, 1$, and add. After slight reduction, the result may be written as

$$(12)\quad M^2 \log|1 - e^{i\theta}| = -\Sigma_{m=1}^{M-1}(M - m)^2 m^{-1} \cos m\theta - I_1 + I_2,$$

where

$$I_1 = \int_\theta^\pi (2\sin\tfrac{1}{2}\theta)^{-1} \Sigma_{N=0}^{M-1} (2M - 1 - 2N)\cos(N + \tfrac{1}{2})\theta d\theta ,$$

and

$$I_2 = \int_{-1}^{0} \Sigma_{N=0}^{M-1} (2M - 1 - 2N) \frac{z^N}{1 - z} dz.$$

It is clear that (12) will imply the result of the lemma, namely (10), if we show first that

$$(13) \qquad I_1 \geq 0,$$

and secondly that

$$(14) \qquad I_2 \leq (2M - 1) \log 2.$$

As regards (13), it is sufficient to show that the integrand in I_1 is non-negative. Clearly $\sin\frac{1}{2}\theta > 0$ in the relevant interval. As regards the other factor, simple calculations show that

$$\Sigma_{N=0}^{M-1} (2M - 1 - 2N)\cos(N + \tfrac{1}{2})\theta = \cos\tfrac{1}{2}\theta \sin^2\tfrac{1}{2}M\theta \operatorname{cosec}^2\tfrac{1}{2}\theta,$$

which again is non-negative. This establishes (13). For (14), we observe that, if $-1 < z < 0$,

$$0 < \Sigma_{N=0}^{M-1} (2M - 1 - 2N) z^N < 2M - 1.$$

This follows from the fact that the terms in this sum are alternating in sign, and monotonically decreasing in absolute value. Hence

$$0 < I_2 \leq (2M - 1) \int_{-1}^{0} \frac{dz}{1 - z},$$

which proves (14). This completes the proof of the lemma.

4. We now apply this result to the estimation from above of $N(c_1, \ldots, c_p)$, as given by (6). We have

LEMMA 2. Let $c_0, \ldots, c_p$ be non-negative, and not all zero, and such that

$$(15) \qquad \Sigma_{k=0}^{p} c_k \cos k\phi \geq 0$$

for all real ϕ. Then, for any positive integer M,

$$(16) \qquad N(c_1, \ldots, c_p) \leq c_0 \log M + 2M^{-1} \Sigma_{k=1}^{p} c_k \log 2.$$

The bound (10), inserted in (6), gives, with the notation (9),

$$N(c_1, \ldots, c_p) \leq \max_{0 < \theta \leq 2\pi} \{ -\Sigma_{m=1}^{M-1} (1 - m/M)^2 j(m\theta)\}$$
$$+ \frac{2M - 1}{M^2} \log 2 \, \Sigma_{k=1}^{p} c_k.$$

In deriving (16) we have only to use (15) and the estimate

$$\Sigma_1^{M-1} (1 - m/M)^2 m^{-1} \leq \log M.$$

5. We first use lemma 2 to estimate $g(n)$ when n is a triangular number. We take $c_o = \frac{1}{2}(p + 1)$, $c_k = p + 1 - k$, $k = 1, \ldots, p$, this choice being justified by the identity

$$\Sigma_{k=1}^{p} (p + 1 - k) \cos k\theta + \tfrac{1}{2}(p + 1) = \tfrac{1}{2} \sin^2(p + \tfrac{1}{2})\theta \operatorname{cosec}^2 \tfrac{1}{2}\theta,$$

valid for $0 < \theta < 2\pi$. This gives

$$N(p, p - 1, \ldots, 1) \leq \tfrac{1}{2}(p + 1)\log M + M^{-1}p(p + 1)\log 2.$$

Taking $M = p$, say, we deduce that

$$g(\tfrac{1}{2}p(p + 1)) \leq \tfrac{1}{2}(p + 1)(\log p + 2 \log 2).$$

For the case of general n, let p be the greatest integer such that $\frac{1}{2}p(p + 1) \leq n$, and write

$$n = \tfrac{1}{2}p(p + 1) + q,$$

so that $0 \leq q \leq p$. Then

$$g(n) \leq g(\tfrac{1}{2}p(p + 1)) + g(q)$$
$$\leq \tfrac{1}{2}(p + 1)(\log p + 2 \log 2) + q \log 2.$$

Here we use the bounds $\frac{1}{2}(p + 1) \leq \sqrt{n}$, $q \leq p < \sqrt{(2n)}$, and (5) follows immediately.

6. I conclude with two minor remarks bearing on the precision of our estimate for $g(n)$. Our main result, that for

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$g(\frac{1}{2}p(p+1))$, was in effect that

$$h(\theta) \leq \tfrac{1}{2}(p+1)(\log p + 2\log 2) \tag{17}$$

where $h(\theta) = \Sigma_{k=1}^{p}(p+1-k)\log|1-e^{ki\theta}|$. That the bound on the right of (17) is not very wide of the mark may be seen from the result that

$$\int_0^{\pi}(1-\cos\theta)h(\theta)d\theta = \tfrac{1}{2}\pi p,$$

which follows on use of the Fourier expansion (7). This shows that we must have $h(\theta) > \frac{1}{4}p$ for some θ.

The method of this paper does not seem to yield any improvement on the lower bound (4) for $g(n)$. However it is perhaps worth pointing out that there seems to be a connection with this problem and another very difficult problem, namely that of the minimum of a sum of cosines. Using the Fourier expansion in (1), we need to minimize the upper bound of

$$-\Sigma_{m=1}^{\infty} m^{-1}\Sigma_{k=1}^{n}\cos(a_k m\theta).$$

Treating the terms for which $m = 1$ as perhaps dominant, it is a question of the best possible upper bound for the negative minimum of

$$\Sigma_{k=1}^{n}\cos(a_k\theta),$$

for arbitrary integers $a_1, \ldots, a_n$. A recent work on this topic is that of P. J. Cohen [2].

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https://doi.org/10.4153/CMB-1961-002-5 Published online by Cambridge University Press