Can. J. Math.Vol. **45** (4), 1993 pp. 727–739

SYNERGY IN THE THEORIES OF GRÖBNER BASES AND PATH ALGEBRAS

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ABSTRACT A general theory for Grobner basis in path algebras is introduced which extends the known theory for commutative polynomial rings and free associative algebras

0. **Introduction.** A primary goal of the theory of Gröbner bases is to systematically construct a set of generators for an ideal in a ring, which is a particularly amenable to computation. The theory began in earnest with Buchberger's algorithm for producing Grobner bases of ideals in the polynomial ring ([3], [9], [11]). Mora pointed out ([6]) that the algorithm could be adapted to ideals in the free associative algebra where, it turns out, the theory is an implementation of Bergman's Diamond Lemma ([2]). These same ideas have been exploited by Ufnarovskii to analyze the growth of finitely generated algebras ([14]). The authors' interest originally arose from the design of a computer program to calculate invariants for finite dimensional algebras. Here we required a Buchberger-type algorithm for path algebras. Towards this end, we show how the theory can be extended to path algebras. We hope that an individual interested in computational algebra will be attracted to new applications for Gröbner bases and that a ring theorist working on the representation theory of algebras will discover a powerful new tool.

Since path algebras are quotients of free algebras and since Mora [6] presents a method for finding a Gröbner basis for free associative algebras over a field when such a basis is finite, we briefly justify the need for the extension of the theory to path algebras. First, it should be noted that free algebras are special cases of path algebras. Thus one might view our work as an extension of the free algebra theory. Second, when viewing path algebras as quotient rings of free associative algebras, the additional monomial relations substantially increase the computational complexity of finding a Gröbner basis. (Technically, one must also add variables for the vertices of the graph, monomial relations for the orthogonality of the vertex variables, and the relation that the sum of the vertex variables equals 1.) A third reason to introduce a theory for path algebras is that, in the case of finite dimensional algebras over algebraically closed fields, there is a graph called the quiver of the algebra which is an isomorphism invariant of the algebra. Moreover, the given algebra is a homomorphic image of the path algebra of its quiver. This point of

The first and last authors were partially supported supported by grants from the National Science Foundation

Received by the editors October 21, 1991

AMS subject classification 13P10, 16D15, 68Q40

view enhances the speed of our real programs It is also inevitable in our algorithm which computes minimal projective resolutions for modules over finite dimensional images of path algebras (see [5]) Finally, we show that if an algebra has a basis and a well-ordering on that basis which satisfies certain axioms which are natural for our theory of Grobner bases, then the algebra can be associated to a path algebra

We should point out that there is a general theory of Grobner bases for graded structures [12] of which our axioms are a particular case Other work on noncommutative Grobner bases that is of interest can be found in [1, 7, 13]

We begin the exposition with the notation of a Grobner generating set for a subspace of a vector space, given a well-ordered basis for the ambient space In section two, we isolate five properties shared by commutative polynomial rings, free associative algebras, and path algebras and show that algebras satisfying these properties have a theory of Grobner bases Section three considers finite Grobner bases The final section shows that if we regard the five relevant properties of the examples as axioms then an algebra satisfying them can be associated with a graph in such a way that the algebra is a homomorphic image of the corresponding path algebra with finite fibers Path algebras are inevitable

1 **Vector spaces and order.** Our approach to Grobner bases rests on some elementary observations about a vector space with a distinguished well-ordered basis Suppose, to begin with, that B is simply a well-ordered set (By this we mean that B is equipped with a total order $\leq$ satisfying the minimum condition) If C is a nonempty finite subset of B, we let TIP(C) denote the largest element in C The order on B induces a relation (which is also written $\leq$) on the collection Fin(B) of all finite subsets of B, according to the following inductive description

$C \leq D$ provided either

(1) $C = \emptyset$,

(2) $C \neq \emptyset$ and $\mathrm{TIP}(C) < \mathrm{TIP}(D)$, or

(3) $C \neq \emptyset$, $\mathrm{TIP}(C) = \mathrm{TIP}(D)$, and $C \setminus \{\mathrm{TIP}(C)\} \leq D \setminus \{\mathrm{TIP}(D)\}$

THEOREM 1 $\big(\mathrm{Fin}(B), \leq\big)$ *is a well-ordered set*

PROOF We leave it as an exercise in induction on cardinality to show that $\leq$ is a total ordering on Fin(B) and limit ourselves to a proof that $\leq$ satisfies the minimum condition Consider a descending chain

$$A_1^1 > A_2^1 > A_3^1 >$$

in Fin(B) Since $\emptyset$ is the unique minimal element in Fin(B), no A_j^1 is empty By definition

$$\mathrm{TIP}(A_1^1) \geq \mathrm{TIP}(A_2^1) \geq$$

Since B is well-ordered, this list stabilizes at $A_{\pi(1)}^1$ with tip x_1 For $n \geq \pi(1)$ set $A_n^2 = A_n^1 \setminus \{x_1\}$ Then

$$A_{\pi(1)}^2 > A_{\pi(1)+1}^2 > A_{\pi(1)+2}^2 >$$

https://doi.org/10.4153/CJM-1993-041-8 Published online by Cambridge University Press

Repeat the process. We have used Cantor diagonalization to construct a strictly descending sequence of elements in B,

$$x_1 > x_2 > x_3 > \cdots.$$

This contradicts the minimum condition on B. ∎

The next result says that if we "replace" one element of a finite set with elements that are smaller than this element, then the resulting set is less than the original. It is proved easily by induction.

LEMMA 2. *Let* $C, D, E, F \in \mathrm{Fin}(B)$.
(i) If $E \subseteq F$ *then* $E \leq F$.
(ii) If $\mathrm{TIP}(D) \in C$ *then* $(C \cup D) \setminus \mathrm{TIP}(D) < C$. ∎

Now suppose V is a vector space with the well-ordered basis B.

Every nonzero $v \in V$ can be written uniquely as a linear combination of members in B, each with nonzero coefficient. We call those elements of B which appear the *support* of v or $\mathrm{supp}(v)$. The support of the zero vector is $\emptyset$. If $v, w \in V$ we write $v < w$ when $\mathrm{supp}(v) < \mathrm{supp}(w)$. Finally, assuming v is nonzero, we use the abbreviation $\mathrm{TIP}(v)$ for $\mathrm{TIP}\big(\mathrm{supp}(v)\big)$. In the literature, $\mathrm{TIP}(v)$ is sometimes called the head of v.

LEMMA 3. *Suppose* v, $w \in V$ *and* $\mathrm{TIP}(w) \in \mathrm{supp}(v)$. *Let* λ *be the unique scalar such that* $\mathrm{TIP}(w) \notin \mathrm{supp}(v - \lambda w)$. *Then* $v - \lambda w < v$.

PROOF. The statement is a disguised version of Lemma 2. ∎

For the remainder of this section, W will denote a subspace of V. Let $\mathrm{NONTIPS}(W)$ denote $\{e \in B \mid e \text{ is not the tip of any vector in } W\}$. In some sense, the next theorem is the real beginning of our theory.

THEOREM 4. $W \oplus \mathrm{span}\big(\mathrm{NONTIPS}(W)\big) = V$.

PROOF. Set $W' = \mathrm{span}\big(\mathrm{NONTIPS}(W)\big)$. By construction $W \cap W' = 0$.

If $V \neq W + W'$ we can choose $v \in V$ minimal (for $<$) with respect to $v \notin W + W'$. Since $v \notin W'$ there is an $e \in \mathrm{supp}(v)$ which is the tip of some element $w \in W$. Choose the scalar λ so that $e \notin \mathrm{supp}(v - \lambda w)$. By the lemma, $v - \lambda w < v$. Since $v - \lambda w \in W + W'$, we have $v \in W + W'$, a contradiction. ∎

The theorem exhibits a vector space splitting $0 \longrightarrow W \longrightarrow V \underset{\nu}{\overset{\pi}{\rightleftarrows}} V/W \longrightarrow 0$: if $v \in V$ then there is a unique linear combination, $\nu\pi(v)$, of basis elements in $\mathrm{NONTIPS}(W)$ so that $v \equiv \nu\pi(v)$ modulo W. We may refer to $\nu\pi(v)$ as the *normal form* for v modulo W. Its value lies in the fact that there is a simple algorithm for calculating the normal form, based on the proof of Theorem 4. If v, w and λ are as in Lemma 3, we will say that $v - \lambda w$ is a *simple* (vector space) *reduction* of v over w. A (vector space) *reduction* of v is a sequence of successive simple reductions. As a consequence of Lemma 3 and the minimum condition on B, after some finite number of simple reductions, a vector can be reduced no more.

A (vector space) *Gröbner generating set* for W is a nonempty subset $G \subseteq W$ such that any element of B which is a tip of a vector in W is the tip of at least one vector in G.

THEOREM 5 *If G is a Grobner generating set for W then every element in V can be reduced over elements in G to its normal form modulo W*

PROOF If $v \in V$, reduce it over G until it can be reduced no further First, the new vector is congruent to v modulo W Second, no tip of an element in W can appear in its support by Lemma 3 ∎

Since the normal form of a vector is zero if and only if it is in W, we see from Theorem 5 that a Grobner generating set for W spans W How can one decide whether a subset of W is a Grobner generating set? Unfortunately, not every set which spans W works Indeed, let $B = \{e_1, e_2, e_3\}$ where $e_1 > e_2 > e_3$ and let W be the span of e_1 and e_2 over $\mathbf{R}$ Then $H = \{e_1 + e_2, e_1 - e_2\}$ is not a Grobner generating set The crux of the problem is that e_1 reduces as completely as possible over H to both e_2 and $-e_2$ This lack of uniqueness can be finessed for vector spaces by adding the restriction that one member of the generating set cannot reduce over another, alas, there is no comparable good news for algebras

We go one step further A nonzero vector $w \in W$ is *sharp in W* ([4]) provided that the coefficient of its tip is 1 and no member of supp(w) other than TIP(w) is the tip of any vector in W Let SHARP(W) denote the collection of all sharp vectors in W

THEOREM 6 ([4]) SHARP(W) *is a Grobner generating set for W such that no member simply reduces over another*

PROOF Suppose e is the tip of some vector in W Let w be the smallest vector in W with tip e, after multiplying by a suitable scalar, w is sharp (We have applied Lemma 3 again) The only way that one sharp vector can reduce over another is if both have the same tip But in that case, their difference is a vector in W whose support lies in NONTIPS(W), a contradiction ∎

2 **Path algebras.** The path algebra $K\Gamma$ is a vector space over the field K concocted from a finite directed graph Γ with vertex set $\Gamma_0 = \{v_1, \ldots, v_V\}$ and arrow set $\Gamma_1 = \{a_1, \ldots, a_A\}$ Its distinguished basis B consists of all directed paths in Γ The length of a path is the number of arrows in that path, each vertex lies in B, regarded as a path of length zero The product on B is concatenation if the terminal vetex of the path p is the origin vertex of q then the path pq makes sense If the terminus and origin do not match up, the product is zero Hence we can identify a path of positive length with a word in the arrow symbols Given an order $v_1 < v_2 < \cdots < v_V$ on vertices and $a_1 < a_2 < \cdots < a_A$ on arrows, the length lexicographic order on B is a well ordering defined as follows $m < m'$ if either the length of m is less than the length of m' or the lengths are equal but m precedes m' in alphabetical order, reading left to right

The reader has undoubtedly noticed that path algebras are not merely vector spaces They are algebras with a multiplicative structure strongly tied to B We isolate five properties which will play a role in the theory of Grobner bases for algebras First, a definition If $a, b \in B$ we say that *a divides b*, or $a|b$ for short, provided we can find $u, v \in B$ such that $b = uav$ The well-ordered basis B of $K\Gamma$ satisfies

(M1) $B \cup \{0\}$ is closed under multiplication
(M2) "Divides" is reflexive on B.
(M3) For each $a \in B$, $\{b \mid b \text{ divides } a\}$ is finite.
(M4) If $a, b, u, v \in B$ and neither of the products below are zero then

$$a < b \Rightarrow uav < ubv.$$

(M5) If $a, b \in B$ then $a|b$ implies $a \leq b$.

Consider the following two examples:

POLYNOMIAL RINGS. The commutative polynomial algebra $K[x_1, \ldots, x_n]$ is a vector space over the field K which has a basis B consisting all monomials together with 1. If we order the variables $1 < x_1 < x_2 < \cdots < x_n$ then B can be well ordered by using degree and lexicographic ordering. That is, if $m = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and $m' = x_1^{b_1} \cdots x_n^{b_n}$ then $m < m'$ if either $\sum a_i < \sum b_i$ or if $\sum a_i = \sum b_i$ and there is a j so that $a_i = b_i$ for $i < j$ and $a_j < b_j$.

FREE ALGEBRAS. The free algebra $K\langle x_1, \ldots, x_n \rangle$ is a vector space over the field K which has a basis B consisting of all words in $x_1, \ldots, x_n$. If we order the letters $x_1 < x_2 < \cdots < x_n$ then B can be well ordered by the length-lexicographic ordering.

Again we see that properties (M1)–(M5) are satisfied for these examples. For this section, R will be a ring that satisfies (M1)–(M5) with a given distinguished well-ordered basis B.

LEMMA 7. *A. "Divides" is a partial order.*
B. For each $b \in B$, $\{(u, v, w) \in B^3 \mid uvw = b\}$ is finite.

PROOF. A. We need only check that "divides" is antisymmetric. If $a|b$ and $b|a$ then $a \leq b$ and $b \leq a$ by (M5). Hence $a = b$.

B. Suppose $b = uvw$ for $u, v, w \in B$. Since "divides" is reflexive, we can find $e_1, e_2, e_3, e_4 \in B$ such that $u = e_1 u e_2$ and $w = e_3 w e_4$. Thus

$$b = e_1 u e_2 v e_3 w e_4.$$

In particular, each of u, v and w divides b. According to (M3), there are only finitely many possibilities. ∎

Fix a nonzero two-sided ideal I of R. An algebra *Gröbner generating set* for I is a nonempty subset $G \subseteq I$ such that the tip of each nonzero element of I is divisible by the tip of some element in G. (In the literature, G is called a "Gröbner basis" for I.) We wish to connect this notion with the appropriate version of reduction for algebras.

Let $a \in R$ be nonzero. A *simple* (algebra) *reduction* ρ for a is determined by a 4-tuple (λ, u, f, v) where $\lambda \in K^*, f \in R \setminus \{0\}$, and $u, v \in B$. It satisfies

(1) $u\,\mathrm{TIP}(f)v \in \mathrm{supp}(a)$
(2) $u\,\mathrm{TIP}(f)v \notin \mathrm{supp}(a - \lambda ufv)$.

In this case $a - \lambda ufv$ is written $\rho(a)$ and referred to as the *reduction of a by ρ*. An algebra *reduction* for a is a sequence $\rho_1, \ldots, \rho_q$ such that ρ_1 is a simple reduction for a and

ρ_{j+1} is a simple reduction for $\rho_j \cdots \rho_2\rho_1(a)$ where $j = 1, \ldots, q-1$ Here we say that a reduces to $\rho_q\rho_{q-1} \cdots \rho_1(a)$ If ρ_i is determined by $(\lambda_i, u_i, f_i, v_i)$ we say that $f_1, \ldots, f_q$ are the *subdivisors* for the reduction When each subdivisor is a member of a subset S of R we declare that the reduction is *over S*

LEMMA 8 (SEE LEMMA 3) *If $h, g \in R$ and g reduces to h, then $h < g$*

PROOF We need only check the lemma when $h = \rho(g)$ for a simple reduction ρ Suppose ρ is determined by (λ, u, f, v) If $e \in \text{supp}(f)$ and $e \neq \text{TIP}(f)$, then $uev < u\,\text{TIP}(f)v$ by (M4) The result now follows from Lemma 2 ■

By transplanting the proof of Theorem 5 we obtain

THEOREM 9 *If G is an (algebra) Grobner generating set for I then every element in R can be algebra reduced over G to its (vector space) normal form modulo I* ■

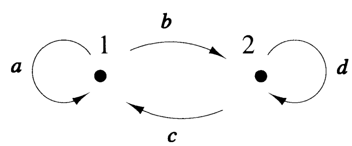
This time we have the immediate corollary that I is generated as a two-sided ideal by a Grobner generating set There is also a refinement of sharpness A sharp vector in I is *minimal sharp* provided its tip is divides-minimal among tips of vectors in I The collection of all such algebra elements is denoted MINSHARP(I) In the commutative theory, this set has been called the *reduced Grobner basis*

THEOREM 10 (SEE THEOREM 6) MINSHARP(I) *is a Grobner generating set for I such that no member reduces over another*

PROOF Let e be the tip of some element in I Among all elements in I, there is one whose tip, e', is divides-minimal with respect to $e'|e$ (Apply (M3)) The sharp vector with tip e' is minimal sharp This proves MINSHARP(I) is a Grobner generating set The second part follows from sharpness and the definition of minimal sharp ■

We are left with the issue of finding sufficient, easily verifiable conditions which guarantee that a generating set for the ideal I is a Grobner generating set The assumption that no member algebra reduces over another, is not enough

Consider the path algebra where Γ is the graph



The order is induced by $1 < 2 < a < b < c < d$ Set $f = abcabca - a^2$ and let I be the ideal generated by f Then $\{f\}$ is not a Grobner generating set for I Indeed, I contains

$$fbca - abcf = abca^2 - a^2bca$$

Its tip, $abca^2$, is certainly not divisible by TIP(f) What has occurred is that a difference of multiples of f has lost its tip because f "overlapped" itself,

$$\underline{abc\overline{abca}}\overline{bca}$$

https://doi.org/10.4153/CJM-1993-041-8 Published online by Cambridge University Press

More formally, if $u, v \in B$, a (u, v)-*overlap* occurs when one can factor $u = u_1 w$, $v = wv_1$ where $u_1 \neq u$ and $v_1 \neq v$. Different factorizations in B describe different overlaps. Suppose $f, g \in R$ such that $\mathrm{TIP}(f) = u$ appears in f with coefficient α and $\mathrm{TIP}(g) = v$ appears in g with coefficient β. When a (u, v)-overlap occurs, we say that f and g overlap and that $\beta f v_1 - \alpha u_1 f$ is the *overlap difference* for the factorization. To examine f and g for all overlaps is to consider every (u, v)- and (v, u)-overlap.

We need one more concept before providing a description of when a set is MINSHARP(I). We say that two (nonzero) elements b_1 and b_2 in B are *uniform-equivalent* provided that for all $x, y \in B$

$$xb_1y = 0 \Leftrightarrow xb_2y = 0.$$

Notice that uniform-equivalence is compatible with multiplication in B: if $u, v \in B$ and b_1 is uniform-equivalent to b_2 then ub_1v is uniform-equivalent to ub_2v assuming both products are nonzero.

If R is the polynomial ring or the free associative algebra and B is the set of the monomials then any two elements are uniform-equivalent. On the other hand, if R is the path algebra $K\Gamma$, then two paths in B are uniform-equivalent if and only if they have the same origin vertices and the same terminus vertices.

A nonzero element of $f \in R$ is *uniform* provided all of the elements in its support are uniform-equivalent to each other. If f is an arbitrary nonzero element of R then we can decompose it uniquely

$$f = f_1 + f_2 + \cdots + f_n$$

where the supports of the f_j belong to distinct (uniform) equivalence classes. We refer to $f_1, \ldots, f_n$ as the *uniform projections* of f. We say that an ideal I of R is *uniform-homogeneous* if whenever a nonzero element lies in I then so do all of its uniform projections.

For instance, when B is closed under multiplication (*e.g.*, for the free algebra or polynomial algebra), all ideals are uniform-homogeneous. In the case of path algebras, if $\{e_\alpha \mid \alpha \in \mathcal{A}\}$ is the collection of vertices regarded as idempotents in $K\Gamma$, then for $f \in K\Gamma$

$$f = \sum_{\alpha,\beta \in \mathcal{A}} e_\alpha f e_\beta$$

is the decomposition of f into uniform projections. It follows that every ideal in a path algebra is uniform-homogeneous.

The following observation is immediate.

LEMMA 11. *A sharp element for a uniform-homogeneous ideal of R is uniform.* ■

The next result is essential to our generalization to path algebras.

LEMMA 12. *Suppose that S is a set of uniform elements of R and $u, v \in B$. If $r \in R$ is a uniform element which reduces to 0 over S then urv reduces to 0 over S.*

PROOF. Suppose $f \in R$ is uniform and $u, v \in B$ are such that $ufv \neq 0$. Then $uxv \neq 0$ for each x in the support of f. Let $\rho = (\lambda, a, g, b)$ be a simple reduction for f with $g \in S$.

By the compatibility of uniform-equivalence with multiplication, λagb is uniform Next let $u\rho v$ denote the simple reduction (λ, ua, g, bv) Notice that $ua \neq 0$ and $bv \neq 0$ because $ua\,\mathrm{TIP}(g)bv \in u(\mathrm{supp} f)v$ Moreover, $u\rho v$ is a simple reduction for ufv and

$$(u\rho v)(ufv) = u\big(\rho(f)\big)v$$

Finally, $\rho(f)$ is uniform since $a\,\mathrm{TIP}(g)b$ is uniform-equivalent to members of both the support of f and of λagb

The lemma follows by induction on the number of simple reductions it takes to reduce r to 0 ∎

Notice that if r is not uniform and r is reduced over a set of uniform elements then each simple reduction in the sequence perturbs exactly one uniform projection As a consequence, if r is reduced to zero over a set of uniform elements, then so are all of its uniform projections It is easy to find examples of sets of non-uniform elements in a path algebra where the conclusion of Lemma 12 is false

THEOREM 13 *Let S be a subset of nonzero uniform elements in R which generates the ideal I Assume that*

(i) the coefficient of the tip of each member of S is 1,

(ii) no member of S reduces over any other, and

(iii) every overlap difference for two (not necessarily distinct) members of S always reduces to zero over S

Then $S = \mathrm{MINSHARP}(I)$

PROOF According to Bergman's Diamond Lemma ([2]), there is exactly one outcome which can be obtained from an element of R by applying any maximal sequence of simple reductions over S, there is a unique complete reduction (In his language, (ii) implies that all inclusion ambiguities are resolvable vacuously and (iii) states that overlap ambiguities are resolvable) As a consequence, the map which sends an element to its complete reduction is additive Let

$$J = \{r \in R \mid r \text{ reduces to 0 over } S\} \cup \{0\}$$

By the remark following Lemma 12, J is closed under taking uniform projections If we apply Lemma 12 to a uniform element $r \in J$ then we see that urv reduces to 0 for every $u, v \in B$ Hence J is an ideal of R Since every element of S trivially reduces to 0 over S, we see that $I \subseteq J$ On the other hand, $J \subseteq I$ by the nature of reduction over S

Thus $I = J$

We claim that S is a Grobner generating set for I Any $0 \neq f \in I$ reduces to 0 over S, at some point in the reduction, the tip of f disappears Thus $\mathrm{TIP}(f)$ is divisible by the tip of some member of S

Suppose $s \in S$ and $e \mid \mathrm{TIP}(s)$ where $e = \mathrm{TIP}(f)$ for some $f \in I$

Since S is a Grobner generating set we can find $s' \in S$ such that $\mathrm{TIP}(s') \mid e$ Hence $\mathrm{TIP}(s') \mid \mathrm{TIP}(s)$ By condition (i), $s' = s$, and so $e = s$ (Lemma 7A) Thus $\mathrm{TIP}(s)$ is divides-minimal among tips of elements in I. A similar argument shows that s is sharp. Therefore $S \subseteq \text{MINSHARP}(I)$.

If $g \in \text{MINSHARP}(I)$ then $\text{TIP}(s) | \text{TIP}(g)$ for some $s \in S$. But s and g are both minimal sharp. Thus $s = g$. We have shown that $S = \text{MINSHARP}(I)$. ∎

We conclude this section with some remarks on constructing a Gröbner generating set. When it is known that there is a finite Gröbner generating set (see the next section), the work of Mora [6] can easily be adapted to provide an algorithm for constructing MINSHARP(I). If there is no assurance of finiteness, the problem of constructing better and better approximations of a Gröbner generating set is quite subtle and will be addressed in a future paper.

3. **Finite generation.** Much of the literature is devoted to algebras whose ideals are, *a priori*, known to have a finite Gröbner generating set. We continue with the notation and hypothesis of the previous sections: R is an algebra which satisfies (M1)–(M5).

The project which inspired this paper is concerned with finite dimensional algebras.

LEMMA 14. *Let S be a finitely generated semigroup and let T be a nonempty subset such that $ST \subseteq T$. If $S \setminus T$ is finite then there exists a finite set $W \subseteq T$ such that $T = W \cup SW$.*

PROOF. Let V be the finite generating set for S. If $a \in S$ let $l(a)$ denote the smallest length of a product of generators equal to a. Set $N = \max\{l(z) \mid z \notin T\}$ and $W = \{b \in T \mid l(b) \leq N + 1\}$. Since N and V are finite, W is finite.

We claim that $T = W \cap SW$. If not, choose $x \in T$ of minimal length such that x does not lie in the union. Certainly $l(x) > 1$. Write $x = v_1 v_2 \cdots v_l$ where $l = l(x)$ and $v_i \in V$.

By the minimal choice of x, we have $v_2 \cdots v_l \notin T$. Moreover $l(v_2 \cdots v_l) = l - 1$; hence $l - 1 \leq N$. Therefore $l(x) \leq N + 1$. By definition, $x \in W$—a contradiction. ∎

THEOREM 15. *Assume that R is a finitely generated algebra and I is a nonzero ideal of R. If R/I is finite dimensional then I has a finite Gröbner generating set.*

PROOF. It is not difficult to see that the finite algebra generation of R implies the finite generation of $B \cup \{0\}$ as a semigroup. Set

$$T = \{0\} \bigcup \{b \in B \mid b = \text{TIP}(g) \text{ for some } g \in I\}.$$

Properties (M2) and (M4) show that $ST \subseteq T$. Also, $B \setminus T = \text{NONTIPS}(I)$ which, by Theorem 4, is finite. Now the reflexivity of "divides" and the lemma imply that there is a finite set $W \subseteq T$ such that

$$T = BWB \cup \{0\}.$$

Suppose $f \in \text{MINSHARP}(I)$. Then there exists $w \in W \subseteq T$ such that $w | \text{TIP}(f)$. The definition of minimal sharpness yields $w = \text{TIP}(f)$.

Thus the finiteness of W forces MINSHARP(I) to be finite. ∎

Significantly more is true. Every finite dimensional algebra over an algebraically closed field has the "same" representation theory as a basic algebra ([10, Chapter 2,

§2 1]) Any such basic algebra is a finite dimensional homomorphic image of a path algebra so that the kernel contains all sufficiently long paths ([10, Chapter 2, §2 1])

Suppose that C is a semigroup ideal of $B \cup \{0\}$ (In the presence of (M2) this means $0 \in C$ and $BCB \subseteq C$) If KC is the linear span of C then KC is a two-sided ideal of R and NONTIPS(KC) $= B \setminus C$ Recall the vector space splitting

$$0 \longrightarrow KC \longrightarrow R \underset{\nu}{\overset{\pi}{\rightleftarrows}} R/KC \longrightarrow 0$$

It is not difficult to see that, via the identification ν, R/KC satisfies (M1)–(M5) with $\leq$ the restriction of the original well-ordering to $B \setminus C$

THEOREM 16 *Let C be semigroup ideal of $B \cup \{0\}$ If I is an ideal of R such that $C \subseteq I$ then*

$$\text{MINSHARP}(I) = \nu\big(\text{MINSHARP}(\pi I)\big) \cup \big(C \cap \text{MINSHARP}(I)\big)$$

PROOF Suppose $f \in$ MINSHARP(I) If TIP(f) $\in C$ then $f \in C$ since sharp vectors are minimal among vectors in I with the same tip (Theorem 6) If TIP(f) $\notin C$ then no elements of C are in its support $\nu\pi(f) = f$ We have shown the inclusion of MINSHARP(I) in the union The reverse inclusion follows from the following simple observation Because C is a semigroup ideal, if $b \in B \setminus C$ and $b'|b$ then $b' \in B \setminus C$ ∎

Let us return to finite dimensional algebras R is a path algebra and I is an ideal of R which contains $C(N)$, the semigroup ideal consisting of 0 and all paths of length N or greater Every member of $C(N) \cap$MINSHARP(I) is a path of length N Therefore, if π_{N+1} and ν_{N+1} give the vector space splitting for $R/KC(N+1)$ then

$$\text{MINSHARP}(I) = \nu_{N+1}\big(\text{MINSHARP}(\pi_{N+1} I)\big)$$

In other words, to calculate MINSHARP(I) whenever a path of length $N+1$ or greater appears in the support of an element, we can replace that path with 0

4 **Graphs.** In the preceding discussion, we took some pains to isolate those properties of our examples which appeared to be fundamental In this section we look more closely at these attributes as axioms and show that the path algebra can be viewed as a universal example

Let S be a semigroup with 0 and assume that $B = S \setminus \{0\}$ is a well-ordered set satisfying (M1)–(M5)

LEMMA 17 *If $a = uv \in B$ then $u|a$ and $v|a$*

PROOF Since "divides" is reflexive, we can find b and c in B such that $u = buc$ Then $a = bucv$ Thus $u|a$ Similarly, $v|a$ ∎

If $a \in B$ we let its *origin* set be

$$O(a) = \{r \in B \mid a = ras \text{ for some } s \in B\}$$

and its *terminus* set be

$$\mathcal{T}(a) = \{s \in B \mid a = ras \text{ for some } r \in B\}.$$

LEMMA 18. *$\mathcal{O}(a)$ and $\mathcal{T}(a)$ are finite semigroups inside B.*

PROOF. Both sets are nonempty by the reflexivity of "divides". If $a = r_1as_1 = r_2as_2$ then $r_1r_2as_2s_1 = a$. Hence $\mathcal{O}(a)$ and $\mathcal{T}(a)$ are each semigroups. Finiteness follows from Lemma 17: the union of $\mathcal{O}(a)$ and $\mathcal{T}(a)$ consists of divisors of a and there are only finitely many of them. ∎

THEOREM 19. *$\mathcal{O}(a)$ and $\mathcal{T}(a)$ each consist of a single idempotent.*

PROOF. We present the argument for $\mathcal{O}(a)$. Suppose $u, v, w \in \mathcal{O}(a)$. Since $v|uvw$, we have that $v \leq uvw$. If $v < uvw$ then $v < uvw < u^2vw^2 < u^3vw^3 < \cdots$. (No product is zero by Lemma 18.) But there cannot be infinitely many elements in the semigroup $\mathcal{O}(a)$. It follows that $v = uvw$ for all $u, v, w \in \mathcal{O}(a)$. Obviously

(1) $u = u^3$ for all $u \in \mathcal{O}(a)$.

Hence $u = (u^2)(u)(u)$ implies

(2) $u = u^2$ for all $u \in \mathcal{O}(a)$.

Now if $u, v \in \mathcal{O}(a)$ then $u = vu^2$, so $u = vu$. Similarly, $v = v^2u$ implies $v = vu$. Therefore

(3) $u = v$ for all $u, v \in \mathcal{O}(a)$. ∎

Observe that if $e \in B$ happens to be an idempotent then $e = \mathcal{O}(e) = \mathcal{T}(e)$. We next relate idempotents to each other.

THEOREM 20. *If e and f are distinct idempotents then $ef = 0$.*

PROOF. Suppose $ef \neq 0$. If $e < f$ then $(e)(e)(f) < (e)(f)(f)$, a contradiction. Likewise, if $f < e$ we reach the contradiction $(e)(f)(f) < (e)(e)(f)$. ∎

LEMMA 21. *Let e be an idempotent in B. If $x \in B$ and $ex \neq 0$ then $e = \mathcal{O}(x)$. If $xe \neq 0$ then $e = \mathcal{T}(x)$.*

PROOF. Write $x = axb$ where $a = \mathcal{O}(x)$ and $b = \mathcal{T}(x)$. Since $ex \neq 0$ we have $ea \neq 0$. Apply the previous theorem. ∎

THEOREM 22. *If e is an idempotent in B then its only divisor is itself.*

PROOF. Suppose $uv = e$. Then $(vuvu)^2 = v(uv)^3u = v(uv)u$. That is, $f = vuvu$ is also an idempotent.

It follows from Lemma 21 that $f = \mathcal{O}(v) = \mathcal{T}(u)$. Thus $e = uv = ufv$. Consequently $f|e$. But $e|f$ by direct inspection. We conclude that $e = f$. In particular, $e = uev$. Therefore $u = \mathcal{O}(e)$ and $v = \mathcal{T}(e)$. By the remark preceding Theorem 20, $u = e = v$. ∎

We shall say that a nonidempotent element b in B is *irreducible* if its only divisors are $\mathcal{O}(b)$, $\mathcal{T}(b)$ and b itself.

https://doi.org/10.4153/CJM-1993-041-8 Published online by Cambridge University Press

THEOREM 23 *Every nonidempotent in B is a product of irreducible elements*

PROOF Let u be a minimal counterexample It cannot be an irreducible element or an idempotent Hence we can find $w \notin \{O(u), \mathcal{T}(u), u\}$ such that $u = awb$ for some $a, b \in B$ By Lemma 17, each of a, w and b is a divisor of u In this case, (M5) asserts that

$$a \leq u, \ w \leq u, \text{ and } b \leq u$$

"Induction" tells us that there must be at least one equality By assumption $w \neq u$ If $a = u$ then

$$u = uwb = O(u)u(wb),$$

so $wb = \mathcal{T}(u)$ By Theorem 22, $w = \mathcal{T}(u)$, a contradiction Similarly, $b \neq u$ ■

The theorems we have proved can be summarized quite neatly We have constructed a graph $\Gamma(S)$ whose vertices are the nonzero idempotents of S and whose arrows are its irreducible elements The origin and terminus of an arrow are determined in the obvious fashion Each member of $S \setminus \{0\}$ can be identified with at least one path, according to Theorem 23

Let $\Gamma(S)^*$ denote the path semigroup with zero derived from $\Gamma(S)$ Then $\Gamma(S)^*$ may differ from S in that two distinct paths in $\Gamma(S)$ may represent the same element of S or a path in $\Gamma(S)$ may represent 0 in S Nonetheless, we have already established the following omnibus result

THEOREM 24 *Let S be a semigroup with zero which satisfies (M1)–(M5) Then there is a graph* $\Gamma(S)$ *and a semigroup homomorphism* π $\Gamma(S)^* \longrightarrow S$ *(sending 0 to 0) such that the inverse image of an idempotent (or an irreducible) consists of a single idempotent (resp irreducible)* ■

By Theorem 23, the map π is surjective Also, as a consequence of Theorem 22, the graph $\Gamma(S)$ is finite whenever S is finitely generated

In some sense we have explained the natural appearance of path algebras in the theory of Grobner bases (Of course, a free algebra is a path algebra whose underlying graph has a single vertex and one loop for each free generator) The polynomial algebra $K[x_1, \quad , x_n]$ can be obtained from the free algebra $K\langle x_1, \quad , x_n\rangle$ by collapsing the finitely many monomials with the same number of occurrences of each letter, to a single monomial This reflects a general phenomenon

THEOREM 25 *Assume the notation of Theorem 24 If* $s \in S$ *is not zero then* $\pi^{-1}(s)$ *is finite*

PROOF Suppose not Choose $s \neq 0$ minimal in S subject to $\pi^{-1}(s)$ being infinite Set

$$A = \{x \in \Gamma(S)^* \mid \text{ there exists } y \in \pi^{-1}(s) \text{ with } x|y \text{ and } x \neq y\}$$

We claim, first, that A is infinite By the fidelity of π on irreducible elements, s is not irreducible Thus we can factor $y \in \pi^{-1}(s)$ as $y = x_1x_2x_3$ with $x_1, x_2, x_3 \in A$ Consequently, if A is finite so is $\pi^{-1}(s)$

We next argue that $A \cap \pi^{-1}(s) = \emptyset$. Indeed, if z lies in the intersection then there is a $y \in \pi^{-1}(s)$ such that $z \neq y$ and $z|y$. Write $y = uzv$. Then $s = \pi(z) = \pi(u)\pi(z)\pi(v)$. By the characteristic property of idempotents, $\pi(u) = O(s)$ and $\pi(v) = \mathcal{T}(s)$. The fidelity of π on idempotents implies that u and v are idempotents in $\Gamma(S)^*$, and so $y = z$. This is a contradiction.

For each $x \in A$ we have $\pi(x)|s$ and $\pi(x) \neq s$ But s has only finitely many divisors. That is, $\pi(A)$ is a finite set consisting of nonzero members of S, each less than s. By the minimal choice of s, we conclude that A is finite. However, we have already established that A is infinite. ∎

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