Can. J. Math., Vol. XXXVII, No. 2, 1985, pp. 310-323

OPTIMIZATION AND α-DISFOCALITY FOR ORDINARY DIFFERENTIAL EQUATIONS

M. ESSÉN

For $f \in L^1(0, T)$, we define the distribution function

$$m(t, f) = |\{s \in (0, T):f(s) > t\}|,$$

where T is a fixed positive number and $|\cdot|$ denotes Lebesgue measure. Let $\Phi:[0, T] \to [0, m]$ be a nonincreasing, right continuous function. In an earlier paper [**3**], we discussed the equation

$$(0.1) \qquad y'' - qy = 0, y(0) = 1, y'(0) = \alpha, t \in (0, T),$$

when the coefficient q was allowed to vary in the class

$$\mathscr{F} = \mathscr{F}(\Phi) = \{q \in L^\infty(0, T):m(\cdot, q) = m(\cdot, \Phi)\}.$$

We were in particular interested in finding the supremum and infimum of $y(T)$ when q was in $\mathscr{F}$ or in the convex hull $\Omega(\Phi)$ of $\mathscr{F}(\Phi)$ (see below). Certain ideas of L.-E. Zachrisson were crucial in our proofs in [**3**] and will again be used in the present paper, the purpose of which is to discuss analogous questions for the equation

$$(0.2) \qquad y'' + qy = 0, y(0) = 1, y'(0) = \alpha, t \in (0, T),$$

where the nonnegative coefficient q varies in $\mathscr{F}$ or $\Omega(\Phi)$. It turns out that the machinery developed in [**3**] works well also in this case, provided that the solutions of (0.2) do not change sign in $[0, T]$.

As an example of how our general results can be used, we determine $\sup y(T)$ and $\inf y(T)$ when y is a positive solution of (0.2) and the coefficient q is allowed to vary in the class E_B of nonnegative functions on $[0, T]$ which are such that

$$\int_0^T q = B$$

(cf. Theorem 3).

An equation $y'' + qy = 0$ is *right disfocal* in $[a, b]$ if the (nontrivial) solutions of the equation which have $y'(a) = 0$ have no zeros in $[a, b]$. We shall say that the equation is *α-disfocal* in $[0, T]$ if the solution of (0.2) has no zero in $[0, T]$. It turns out that our optimization method is a good tool for studying α-disfocality.

Received November 25, 1983. Part of this work was done while the author was visiting McMaster University supported by NSERC Grant #A7322.

As applications of our results, we determine, for certain classes of coefficients q,

I. The maximal length of an interval J where all solutions of (0.2) are α-disfocal in J when q varies in the class: this is related to Lyapunov's inequality (cf. Corollaries 2 and 3 in Section 1);

II. The minimal length of an interval J where no solution of (0.2) is α-disfocal in J when q varies in the class: this is a generalization of a result of F. J. Tipler (cf. Theorem 4 in Section 4).

A solution of (0.2) is a function y such that y and y' are absolutely continuous, y'' is the a.e. existing derivative of y' and the equation is satisfied a.e. If y is a solution of (0.2) associated with $q \in \mathscr{F}(\Phi)$, we consider

Problem 1. Determine $\inf y(T)$, $q \in \mathscr{F}(\Phi)$.

Problem 2. Determine $\sup y(T)$, $q \in \mathscr{F}(\Phi)$.

To characterize the extremal elements, we shall use a kind of calculus of variations which does not work in $\mathscr{F}(\Phi)$: this class is too small. Therefore, following L.-E. Zachrisson, we shall introduce a larger class associated with the partial order $\prec$ introduced by Hardy, Littlewood and Polya. To $f \in L^1(0, T)$, we associate

$$f^*(s) = \sup_{m(t,f) > s} t, \quad s \in [0, T],$$

which is a nonincreasing, right continuous function on $\mathbf{R}$: we shall call it the decreasing rearrangement of f. It is easy to check that

(0.3) f and f^* have the same distribution functions,

$$(0.4) \qquad \int_0^t f \leqq \int_0^t f^*, \quad t \in [0, T],$$

$$(0.5) \qquad \int_0^T f = \int_0^T f^*.$$

If f and g are in $L^1(0, T)$, we shall say f *majorizes* g, written $g \prec f$, if

$$(0.6) \qquad \int_0^t g^* \leqq \int_0^t f^*, \quad t \in [0, T],$$

$$(0.7) \qquad \int_0^T g^* = \int_0^T f^*.$$

In a similar way, we can also define an increasing rearrangement f^{**} of f (the details are omitted). It has the following properties:

(0.3 a) f and f^{**} have the same distribution functions,

$$(0.4\text{ a}) \qquad \int_0^t f \geqq \int_0^t f^{**}, \quad t \in [0, T],$$

$$(0.5\text{ a}) \qquad \int_0^T f = \int_0^T f^{**}.$$

A crucial property of these rearrangements is that if f and g are nonnegative on $[0, T]$ with $f \in L^1(0, T)$ and $g \in L^\infty(0, T)$, then

$$(0.8) \qquad \int_0^T f^{**}g^* \leqq \int_0^T fg \leqq \int_0^T f^*g^*$$

(cf. [**4**], 10.2 and 10.13).

A function $\sigma:[0, T] \to [0, T]$ is *measure-preserving* if, for each measurable set $E \subset [0, T]$, $\sigma^{-1}(E)$ is measurable and $|\sigma^{-1}(E)| = |E|$. Let Σ denote the class of such functions. If is known that to each $f \in L^1(0, T)$, there exists $\sigma \in \Sigma$ such that $f = f^* \circ \sigma$ (cf. [**6**], Lemma 2). In particular, we have

$$\mathscr{F}(\Phi) = \{q \in L^\infty(0, T):q(t) = \Phi \circ \sigma(t), \sigma \in \Sigma\}.$$

We can now define the convex hull $\Omega(\Phi)$ of the set $\mathscr{F}(\Phi)$ as the subset of $L^\infty(0, T)$ obtained by taking the weak* closure of the set of finite sums of the form $\Sigma c_i\Phi_i$ where $\{\Phi_i\}$ is a sequence in $\mathscr{F}(\Phi)$, $\{c_i\}$ is a finite sequence of nonnegative numbers and $\Sigma c_i = 1$.

From [**7**], we see that

$$\Omega(\Phi) = \{g \in L^1(0, T):g \prec \Phi\}$$

and that $\mathscr{F}(\Phi)$ is the set of extreme points of $\Omega(\Phi)$.

Problem 3. Determine $\inf y(T)$, $g \in \Omega(\Phi)$.

Problem 4. Determine $\sup y(T)$, $q \in \Omega(\Phi)$.

Our results concern cases where the extremal values are positive.

Remark. Condition (0.6) is formally similar to a condition used by Nehari (cf. (2.31) in [**8**]).

1. The existence of extremals. The main results on problems 3 and 4. Let $\{y_n\}$ be a sequence of solutions of (0.2) associated with a sequence $\{q_n\}$ in $\Omega(\Phi)$. Since Φ is bounded, we can use weak* compactness and find $q_0 \in \Omega(\Phi)$ which is such that $q_n \to q_0$ in the weak* sense in $L^\infty(0, T)$. Let y_0 be the solution of (0.2) with $q = q_0$. It is not difficult to prove that

$$\lim_{n\to\infty} y_n(T) = y_0(T).$$

Thus there exists at least one couple (q_0, y_0) which gives a solution of Problem 3 or Problem 4.

Our purpose is to discuss Problems 3 and 4 for equation (0.2) when $\alpha \neq 0$. To explain the method, we first look at the simpler case $\alpha = 0$, e.g., we consider

$$(1.1) \qquad y'' + qy = 0, y(0) = 1, y'(0) = 0, \quad t \in (0, T), q \in \Omega(\Phi).$$

THEOREM 1. *Let* (q_0, y_0) *be an extremal couple for the infimum problem for equation* (1.1). *We assume that all solutions of* (1.1) *are positive when* q *varies in* $\Omega(\Phi)$. *Then we have* $q_0 = \Phi$:q_0 *is the decreasing rearrangement.*

THEOREM 2. *Let* (q_0, y_0) *be an extremal couple for the supremum problem for equation* (1.1). *We assume that there exist solutions of* (1.1) *which are positive when* q *varies in* $\Omega(\Phi)$. *Then we have* $q_0 = \Phi^{**}$:q_0 *is the increasing rearrangement.*

Remark. An analogue of Theorem 2 holds when q may change sign. We shall discuss this after the proof in Section 2.

In Lyapunov's criterion, we start from the assumption that

$$\int_0^T q = B.$$

If more is known about the distribution of values of q, we can deduce stronger versions of this criterion:

COROLLARY 1. *Let* q *be a nonnegative function in* $L^\infty(0, T)$ *such that*

$$\|q\|_\infty \leqq n^2 \quad \text{and} \quad \int_0^T q(t)dt = B.$$

If we have $y(T) = 0$, *where* y *is the solution of* (1.1), *then*

$$B \geqq n \arctan(n/(n^2T - B)). \tag{1.2}$$

When $n \to \infty$, we obtain $B \geqq T^{-1}$, which is the criterion of Lyapunov (cf. Corollary 5.1 in **[5]**). More general results of this type are given in Corollary 2.

When $y'(0) = \alpha \neq 0$, the situation is more complicated. If $B > 0$ is given, let $E = E_B$ be the class of nonnegative functions $q \in L^1(0, T)$ which are such that

$$\int_0^T q = B.$$

If y is a solution of (0.2) we shall discuss the problem of determining $\inf y(T)$ when y is a solution of (0.2) and q varies in E_B.

THEOREM 3. *Assume that all solutions of* (0.2) *are positive when* q *varies in* E_B. *Then*

$$\inf_{q \in E_B} y(T) = 1 + (\alpha - B)T, \quad \alpha T \leqq 1, \tag{1.3}$$

$$\inf_{q \in E_B} y(T) = 1 + \alpha T - (1 + \alpha T)^2 B/(4\alpha), \quad \alpha T > 1. \tag{1.4}$$

COROLLARY 2. *Assume that* $q \in L^1(0, T)$, *that* $q_+ \in E_B$ *and that the solution of equation* (0.2) *has a zero in* $(0, T]$. *Then we have*

$$B \geqq \alpha + T^{-1}, \quad \alpha T \leqq 1, \tag{1.5}$$

$$B \geqq 4/(T + \alpha^{-1}), \quad \alpha T > 1. \tag{1.6}$$

These two criteria are also of Lyapunov type.

Let J_B be the class of nonnegative integrable functions q on $[0, \infty)$ which are such that

$$\int_0^\infty q = B.$$

Let T_α be the least upper bound for all $T > 0$ which are such that for all $q \in J_B$, equation (0.2) is α-disfocal on $[0, T]$ (i.e., the solution of (0.2) is positive on $[0, T]$).

COROLLARY 3.

$$T_\alpha = \begin{cases} (B - \alpha)^{-1}, & \alpha \leqq B/2, \\ 4B^{-1} - \alpha^{-1}, & \alpha > B/2. \end{cases}$$

2. The general extremum problems: a characterization of the extremal coefficients. The first steps in the solutions of Problems 3 and 4 are similar. By the change of variables $u = -y'/y$, equation (0.2) is changed into

$$(2.1) \qquad u' - u^2 = q,\ u(0) = -\alpha, \quad t \in (0, T).$$

If q_0 is an extremal coefficient for the infimum problem, let $q \in \Omega(\Phi)$ and define

$$q_\delta = (1 - \delta)q_0 + \delta q, \quad 0 \leqq \delta < 1,$$

$$(2.2) \qquad u'_\delta - u_\delta^2 = q_\delta,\ u_\delta(0) = -\alpha, \quad t \in (0, T),$$

$$(2.3) \qquad y''_\delta + q_\delta y_\delta = 0,\ y_\delta(0) = 1,\ y'_\delta(0) = \alpha, \quad t \in (0, T).$$

The function q_δ is also in the class $\Omega(\Phi)$.

Since the extremal solutions of (0.2) are always assumed to be positive on $[0, T]$, equation (2.2) will always have a solution on $[0, T]$ for all values of δ which are sufficiently close to 0. Forming the difference of (2.2) and (2.2) with $\delta = 0$, we have

$$(u'_\delta - u'_0) - (u_\delta + u_0)(u_\delta - u_0) = \delta(q - q_0), \quad (u_\delta - u_0)(0) = 0,$$

and two integrations will give us

$$\log(y_0(T)/y_\delta(T)) = \int_0^T (u_\delta - u_0)(t)dt = \delta \int_0^T (q - q_0)(s) \int_s^T \exp\left(\int_s^t (u_\delta + u_0)\right)dtds.$$

Dividing by δ and letting $\delta \to 0+$, we have

$$(2.4)\qquad \int_0^T (q - q_0)(s)Q(s)ds \leqq 0 \quad \text{for all } q \in \Omega(\Phi),$$

where

$$Q(t) = y_0(t)^2 \int_t^T y_0(s)^{-2}ds, \quad t \in [0, T].$$

Similarly, if (q_0, y_0) is an extremal couple for the supremum problem, we have

$$(2.5)\qquad \int_0^T (q - q_0)(t)Q(t)dt \geqq 0 \quad \text{for all } q \in \Omega(\Phi).$$

Let us first discuss (2.4). If $Q = Q^* \circ \sigma$, where $\sigma \in \Sigma$, we choose $q = q_0^* \circ \sigma$ in (2.4) and obtain

$$(2.6)\qquad \int_0^T Q^*q_0^* = \int_0^T Qq \leqq \int_0^T Qq_0 \leqq \int_0^T Q^*q_0^*.$$

In the last step, we used (0.8). Consequently, there is equality all the way in (2.6) and we have

$$(2.7)\qquad \int_0^T \left\{ \int_{\{Q(t)>s\}} q_0(t)dt \right\} ds = \int_0^T \left\{ \int_{\{Q^*(t)>s\}} q_0^*(t)dt \right\} ds.$$

We know that

$$|\{Q(t) > s\}| = |\{Q^*(t) > s\}|$$

and that for all real s,

$$\int_{\{Q(t)>s\}} q_0(t)dt \leqq \int_{\{Q^*(t)>s\}} q_0^*(t)dt.$$

It follows from (2.7) that for all real s, we have

$$(2.8)\qquad \int_{\{Q(t)>s\}} q_0(t)dt = \int_{\{Q^*(t)>s\}} q_0^*(t)dt,$$

$$(2.9)\qquad \operatorname*{ess\,inf}_{\{Q(t)>s\}} q_0(t) \geqq \operatorname*{ess\,sup}_{\{Q(t)\leqq s\}} q_0(t).$$

It follows from (2.9) that if we work in an interval J where Q is strictly increasing, then q_0 must also be increasing in J if we avoid a set of measure zero. Redefining q_0 on this set, we can without changing anything essential assume that q_0 is increasing on J. Similarly, if Q is strictly decreasing on an interval, we can assume that q_0 is decreasing on the interval. If these relations hold, we shall say that the function Q and q_0 are *co-dependent*.

The discussion of the supremum problem is similar. If $Q = Q^* \circ \sigma$, where $\sigma \in \Sigma$, we choose $q = q_0^{**} \circ \sigma$ in (2.5) and see that

$$(2.10)\qquad \int_0^T Q^*q_0^{**} = \int_0^T Qq \geqq \int_0^T Qq_0 \geqq \int_0^T Q^*q_0^{**}.$$

In the last step, we used (0.8). Consequently, there is equality all the way in (2.10) and we obtain

$$(2.11) \qquad \int_0^T \left\{ \int_{\{Q(t)>s\}} q_0(t)dt \right\} ds = \int_0^T \left\{ \int_{\{Q^*(t)>s\}} q_0^{**}(t)dt \right\} ds.$$

For all real s, we know that

$$|\{Q(t) > s\}| = |\{Q^*(t) > s\}|$$

and that

$$\int_{\{Q(t)>s\}} q_0(t)dt \geqq \int_{\{Q^*(t)>s\}} q_0^{**}(t)dt.$$

It follows from (2.11) that for all real s, we have

$$(2.12) \qquad \int_{\{Q(t)>s\}} q_0(t)dt = \int_{\{Q^*(t)>s\}} q_0^{**}(t)dt,$$

$$(2.13) \qquad \operatorname*{ess\,sup}_{\{Q(t)>s\}} q_0(t) \leqq \operatorname*{ess\,inf}_{\{Q(t)\leqq s\}} q_0(t).$$

Arguing as above, we see (after possibly having changed q_0 on a set of measure zero) that if Q is strictly increasing on an interval J, q_0 must be decreasing on J. If Q is strictly decreasing on J, q_0 must be increasing on J. If these two relations hold, we shall say that the functions Q and q_0 are *contra-dependent*.

Summing up, we have proved that if q_0 is an extremal coefficient for the infimum problem, Q and q_0 will be co-dependent and that if q_0 is an extremal coefficient for the supremum problem, Q and q_0 will be contra-dependent.

It is now convenient to rewrite the original differential equation as a system of first-order equations. It is well-known that

$$y_1(t) = y_0(t) \int_t^T y_0(s)^{-2} ds, \quad t \in (0, T),$$

is also a solution of the equation

$$(2.14) \qquad y'' + q_0 y = 0, \quad t \in (0, T),$$

and we have the boundary conditions $y_1(T) = 0$, $y_1'(T) = -y_0(T)^{-1}$. If

$$\xi = ((y_0'/y_0) - y_1'/y_1))/2, \quad \eta = -((y_0'/y_0) + (y_1'/y_1))/2,$$

we find that

$$(2.15) \qquad \begin{aligned} \xi' &= 2\xi\eta \\ \eta' &= q_0 + \xi^2 + \eta^2, \\ \xi(0) &= \left(2 \int_0^T y_0(s)^{-2} ds\right)^{-1}, \\ &\qquad \eta(0) = \left(2 \int_0^T y_0(s)^{-2} ds\right)^{-1} - \alpha. \end{aligned}$$

We note that

$$(2.16) \qquad \xi(t)Q(t) = \xi(0)Q(0), \quad t \in [0, T).$$

This is clear since

$$Q = y_0 y_1 \quad \text{and} \quad (\xi Q)' = (y_0' y_1 - y_0 y_1')'/2 = 0.$$

We also note that ξ will be positive on $[0, T)$.

We shall also need

LEMMA 1. *Assume that* $Q(t) = (2c)^{-1}$, $t \in [a, b]$. *Let*

$$A = \int_a^T y_0(s)^{-2} ds,$$

where y_0 *is the positive function used in the definition of* Q. *Then*

$$(2.17) \qquad y_0(t) = (2Ac)^{-1/2} e^{c(t-a)}, \quad t \in [a, b].$$

If in particular q_0 *is nonnegative,* Q *can not be constant on an interval.*

The proof is an immediate consequence of the formula

$$Q(t)^{-1} = -(d/dt)(\log \int_t^T y_0(s)^{-2} ds).$$

3. Proof of theorems 1, 2 and 3. We begin with Theorem 1. Since Q and q_0 are co-dependent, it follows from (2.16) that ξ and q_0 are contra-dependent. From (2.15), we see that $\eta(0) > 0$ and that η' is nonnegative and thus that ξ' is nonnegative. We conclude that ξ is strictly increasing and that Q and q_0 are decreasing. Choosing $q = \Phi$ in the variational equation (2.4), we apply an integration by parts and find that

$$0 \geqq \int_0^T (\Phi - q_0)(s) Q(s) ds$$

$$= \int_0^T (-Q'(s)) \int_0^s (\Phi - q_0) \geqq 0.$$

In the last step, we used (0.4), (0.5) and Ryff's characterization of the class $\Omega(\Phi)$.

It follows that

$$Q'(s) \int_0^s (\Phi - q_0) = 0, \quad s \in [0, T).$$

According to Lemma 1, this is possible only if the second factor vanishes on $(0, T)$, i.e., we have $q_0 = \Phi$, and we have proved Theorem 1.

To prove Theorem 2, we first note that it follows from (2.15) that η and thus also ξ' is positive. Hence ξ is strictly increasing, Q will be strictly decreasing and the contradependence of Q and q_0 shows that q_0 is increasing. In the variational equation (2.5), we choose $q = \Phi^{**}$. An argument similar to that given in the proof of Theorem 1 will now show that $q_0 = \Phi^{**}$, and we have proved Theorem 2.

Remark. There is an analogue of Theorem 2 also in the case when the given decreasing function Φ may take negative values (we still assume that $\Phi \in L^{\infty}$). We first note that (2.13) holds also in this case. If we wish to work with positive functions in (2.10)-(2.13), we choose a positive constant c such that $q + c$ is positive for all $q \in \Omega(\Phi)$ and replace q, q_0 and q_0^{**} in (2.10) by $q + c$, $q_0 + c$ and $q_0^{**} + c$. In the final statement (2.13), we can delete c. Thus Q and q_0 are contra-dependent also in this case. It follows that ξ and q_0 are co-dependent. We claim that a solution η of the system (2.15) will be nonnegative. If this is not true, η must have a negative minimum in $(0, T)$ since $\eta(0) > 0$ and $\eta(t) \to \infty$, $t \to T-$. Hence there exists an interval $[a, b] \subset (0, T)$ such that for some $c > 0$, we have

$$\eta(t) \leqq \eta(a) < 0, \quad t \in [a, a + c]$$

$$\eta(t) < 0, t \in [a, b), \eta(b) = 0.$$

We conclude that ξ^2 and q_0 are decreasing in $[a, b]$ and that

$$\begin{aligned} 0 \geqq \eta(t) - \eta(a) &= \int_a^t (\xi^2 + q_0) + \int_a^t \eta^2 \\ &\geqq (t - a)(\xi(t)^2 + q_0(t) + \eta(a)^2), \quad t \in (a, a + c). \end{aligned}$$

Consequently, we have

$$\begin{aligned} \eta'(t) &= (q_0 + \xi^2 + \eta^2)(t) \\ &\leqq (q_0 + \xi^2)(a+) + \eta^2(t) \leqq -\eta(a)^2 + \eta(t)^2. \end{aligned}$$

Here $q_0(a+) = \lim q_0(t)$, $t \to a+$. Thus $\eta \leqq \varphi$ on $[a, b]$, where

$$\varphi' = -\eta(a)^2 + \varphi^2, \varphi(a) = \eta(a), \quad t \in [a, b].$$

But the last equation implies that $\varphi = \eta(a)$ on $[a, b]$. Hence

$$0 = \eta(b) \leqq \varphi(b) = \eta(a) < 0,$$

which is a contradiction. We have proved our claim, i.e., η is nonnegative on the interval $[0, T]$.

From (2.15), we see that ξ' is nonnegative and thus that ξ is increasing on $[0, T]$. Hence Q is decreasing and q_0 is increasing. In the variational equation (2.5), we choose $q = \Phi^{**}$. An integration by parts shows that

$$Q'(s) \int_0^s (\Phi^{**} - q_0) = 0, \quad s \in [0, T].$$

When q_0 may be negative, there can exist intervals where $Q'(s) = 0$ (cf. Lemma 1). Let

$$\Gamma(s) = \int_0^s \Phi^{**}(t)dt \quad \text{and} \quad P(s) = \int_0^s q_0(t)dt.$$

The graphs of Γ and P are convex curves (both integrands are increasing). Since $q_0 \in \Omega(\Phi)$, we have

$$P(s) - \Gamma(s) \geqq 0, s \in [0, T], P(0) = \Gamma(0) = 0, P(T) = \Gamma(T).$$

It follows from Lemma 1 that

$$\{s \in [0, T]: P(s) - \Gamma(s) > 0\} \subset \{s \in [0, T]: q_0(s) < 0\}.$$

We have $P(s) = \Gamma(s)$ if $q_0(s) \geqq 0$ which means that $q_0(s) = \Phi^{**}(s)$ for these values of s.

More work is needed to study the set for which the graph of P is not on the boundary of the admissible domain. Further information is given in [**3**].

These methods will not give us an analogue of Theorem 1 for the case when Φ changes sign. In this case, ξ and q_0 are contra-dependent, and the argument which proves that η is nonnegative does not work. The situation is better understood for equation (0.1) with a nonnegative coefficient q and with $\alpha = 0$. For this equation, the supremum problem can be handled by our optimization method (cf. [**3**]). The infimum problem has also been solved, but the method is different: it involves rearrangements of the coefficient q (cf. [**1**, Theorem 5.2], [**2**]). One possible way to extend Theorem 1 to the case of coefficients of varying sign is to try find an analogue of the rearrangement method used in the treatment of equation (0.1).

Proof of Corollary 1. We define

$$\Phi(t) = \begin{cases} n^2, & t \in [0, n^{-2}B), \\ 0, & t \in [n^{-2}B, T]. \end{cases}$$

If $q \in E_B$ and $\|q\|_\infty \leqq n^2$, we have $q \prec \Phi$ and thus that $q \in \Omega(\Phi)$. If y_0 is the solution of (1.1) with $q = \Phi$, it follows from Theorem 1 that $y_0(T) \leqq y(T) = 0$. We obtain (1.2), since

$$y_0(T) = \cos(B/n) - n \sin(B/n)(T - n^{-2}B).$$

Proof of Theorem 3. Let

$$\Phi(t) = \Phi_{B,m}(t) = \begin{cases} m, & t \in (0, B/m), \\ 0, & t \in [B/m, T). \end{cases}$$

We shall first solve the infimum problem in the class $\mathscr{F}(\Phi_{B,m})$. Here we can use the general theory from the beginning of Section 2. We note that if $q \in E_B$ and $\|a\|_\infty \leqq m$, we have $q \prec \Phi_{B,m}$. When $m \to \infty$, the solution of the infimum problem over the class $\mathscr{F}(\Phi_{B,m})$ will tend to the solution of the infimum problem over the class E_B.

We shall first deduce the general form of the extremal configuration for the infimum problem over E_B.

Let η be defined by (2.15). There are two cases to discuss, corresponding to the two possibilities $\eta(0) \geqq 0$ and $\eta(0) < 0$. If $\eta(0) \geqq 0$, we can argue exactly as in the proof of Theorem 1, and it follows that $q_0 = \Phi_{B,m}$, where q_0 is the extremal coefficient for the class $\mathscr{F}(\Phi_{B,m})$. When $m \to \infty$, we find that q_0 tends to $B\delta$ (where δ is the Dirac functional at the origin), and the corresponding value of $y(T)$ is given by (1.1). If $\eta(0) < 0$, we note that η is nondecreasing and that $\eta(t) \to \infty$ as $t \to T-$. Consequently, there exists $a = a(m) \in (0, T)$ such that η is negative on $(0, a)$ and positive on (a, T) and that ξ is strictly decreasing in $(0, a)$ and strictly increasing in (a, T). Since $\xi = Q^{-1}$ and Q and q_0 are co-dependent, Q and q_0 are increasing on $(0, a)$ and decreasing on (a, T).

In the variational equation (2.4), we choose

$$q(t) = \begin{cases} m, & t \in I, \\ 0, & t \in (0, T)\setminus I. \end{cases} \tag{3.1}$$

Here I is an interval of length B/m which is chosen in such a way that

$$\int_0^a q = \int_0^a q_0 = \gamma B, \quad \int_a^T q = \int_a^T q_0 = (1 - \gamma)B,$$

where $\gamma \in [0, 1]$ is determined by $q_0 \in \mathscr{F}(\Phi_{B,m})$. Using these relations, we see that

$$\begin{aligned} 0 &\geqq \int_0^T (q - q_0)(s)Q(s)ds \\ &= \int_0^a Q'(s) \int_s^a (q - q_0) + \int_a^T (-Q'(s)) \int_a^s (q - q_0) \geqq 0. \end{aligned}$$

Since we know the signs of Q' in the intervals $(0, a)$ and (a, T) and the integrals of $q - q_0$ are nonnegative, it follows that

$$Q'(s) \int_a^s (q - q_0)(t)dt = 0 \quad s \in (0, T).$$

According to Lemma 1, we can divide by Q' and it follows that $q_0 = q$ defined by (3.1). Letting $m \to \infty$, we find that q_0 tends to $B\delta_a$ where

$$a = \lim_{m\to\infty} a(m)$$

(we may have to take a subsequence). Thus the extremal coefficient for the class E_B is in this case of the form $q_0 = B\delta_a$, where $a \in [0, T]$ and δ_a is a Dirac functional at the point a. To prove Theorem 3, we now solve the equation

$$y'' + B\delta_a y = 0, y(0) = 1, y'(0) = \alpha, t \in (0, T), \tag{3.2}$$

interpreted as the limit of equations with coefficients in $\mathscr{F}_{B,m}$, compute $y(T)$ which depends on the parameter $a \in [0, T]$ and choose a in such a way that $y(T)$ becomes minimal. The solution of (3.2) is

$$y(t) = 1 + a\alpha + (t - a)(\alpha - B(1 + a)), \quad t \in [a, T],$$

and the minimum of $y(T)$ when a varies over $[0, T]$ is (1.3) when $\alpha T \leqq 1$

and (1.4) when $\alpha T > 1$. We have proved Theorem 3.

Proof of Corollary 2. We first note that if w is defined by

$$w'' + q_+ w = 0, \; w(0) = 1, \; w'(0) = \alpha, \; t \in (0, T),$$

then w must have a zero in $(0, T)$. But if

$$\alpha T \leqq 1 \quad \text{and} \quad B < \alpha + T^{-1},$$

the right hand member of (1.3) will be positive and the assumptions of Theorem 3 are fulfilled. In particular, w will be positive which is a contradiction. Similarly when

$$\alpha T > 1 \quad \text{and} \quad B < 4/(T + \alpha^{-1}),$$

w will again be positive, we obtain a contradiction and we have proved Corollary 2.

Proof of Corollary 3. Let us call the expression in the right hand member T'_α. If $T < T'_\alpha$, the right hand members of (1.3) and (1.4) are positive. Hence $T_\alpha \geqq T'_\alpha$. If $T > T'_\alpha$, it is clear from the proof of Theorem 3 that we can find $q \in J_B$ such that the corresponding solution of (0.2) will change sign on $[0, T]$. Hence $T_\alpha \leqq T'_\alpha$, and the corollary is proved.

4. An extension of a result of F. J. Tipler. Changing our definition slightly we shall say that a nonnegative, locally integrable function q on $[0, \infty)$ is in E_B if

$$\int_0^T q(t)dt = B.$$

We consider the equation

(4.1) $$y'' + qy = 0, \; y(0) = 1, \; y'(0) = \alpha, \; t > 0.$$

I am grateful to A. Källström for showing me the following result of F. J. Tipler (cf. [**9**, Theorem 7]):

THEOREM A. *Let* $q \in E_B$ *be continuous on* $[0, \infty)$. *If* $\alpha = 0$, *the solution of* (4.1) *has a zero in the interval* $[0, T + B^{-1}]$.

As a generalization, we prove

THEOREM 4. *If* $q \in E_B$ *and if* $-T^{-1} < \alpha < 0$, *the solution of* (4.1) *has a zero in*

(4.2) $$[0, T + (1 + \alpha T)/(B(1 + \alpha T) - \alpha)].$$

Remark 1. We note that the solution of (4.1) has a zero in

(4.3) $$[0, T + (B - \alpha)^{-1}], \quad 0 \leqq \alpha < B,$$

(4.4) $$[0, -\alpha^{-1}], \quad 1 + \alpha T < 0.$$

https://doi.org/10.4153/CJM-1985-019-3 Published online by Cambridge University Press

There are easy proofs of (4.3) and (4.4) avoiding our general machinery.

Remark 2. If $\alpha \geqq B$ and $q \in E_B$ vanishes on $[T, \infty)$, the solution of (4.1) has no positive zero. This is an immediate consequence of Theorem 1.

Remark 3. Our method has the advantage that we do not need to guess the form of the expression in (4.2): it follows from our computations.

Proof of Theorem 4. We consider the class

$$\mathscr{H}_n = \{q \in E_B: \|q\|_\infty \leqq n^2\}.$$

Without changing too much, we can approximate (4.1) by an equation for which the coefficient q is in $\mathscr{H}_n$ (n is large). Thus it suffices to solve our problem when $q \in \mathscr{H}_n$ and let $n \to \infty$.

If we assume furthermore that $q \in \mathscr{H}_n$ where n is large, the conclusion of Theorem 4 becomes

$$\text{(4.2 a)} \quad [0, T + (\beta_n + n^{-1}\alpha \tan(B/n))/(n\beta_n \tan(B/n) - \alpha)],$$
$$0 < \beta_n + n^{-1}\alpha \tan(B/n), \ \alpha < 0.$$

Here $\beta_n = 1 + \alpha(T - n^{-2}B)$.

The extremal configuration will be described during the proof. If $n \to \infty$ in (4.2 a), we obtain (4.2).

From now on, we assume that $q \in \mathscr{H}_n$ for some large fixed n. Let $\mathscr{H}_n'$ be the subclass of $\mathscr{H}_n$ which is such that if $q \in \mathscr{H}_n'$, then we have

$$\inf_{[0,T]} y(t) > 0,$$

where y is the solution of (4.1). If $q \in \mathscr{H}_n \backslash \mathscr{H}_n'$, there is nothing to prove.

If $q \in \mathscr{H}_n'$ and y is a solution of (4.1), let $[T, T_1)$ be a maximal interval such that $y(t) > 0$, $t \in [T, T_1)$. We have

$$y(t) \leqq y(T) + y'(T)(t - T), \quad t \in [T, T_1].$$

If $y'(T) < 0$, it follows that y has a zero in the interval $[0, T + \sigma]$, where

$$\sigma = \sup(-y(T)/y'(T)), \quad q \in \mathscr{H}_n'.$$

To determine $\sup y'(T)/y(T)$, $q \in \mathscr{H}_n'$ (which is $-\sigma^{-1}$), we use the same optimization technique as in Section 2. It is clear that there exists an extremal couple (q_0, y_0) such that the supremum is assumed. Using the same notation as in Section 2, we put $u = -y'/y$ and find that

$$(u_\delta - u_0)(T) = \delta \int_0^T (q - q_0)(s) \exp(\int_s^T (u_\delta + u_0)ds.$$

Dividing by δ and letting $\delta \to 0+$, we find that

$$(4.5) \qquad \int_0^T (q - q_0)(s) y_0(s)^2 ds \geqq 0 \quad \text{for all } q \in \mathscr{K}_n.$$

This conclusion is correct only if

$$\inf_{[0,T]} y_0(t) > 0.$$

This will give us the condition on the parameters in (4.2 a).

Since $\alpha < 0$, we have $y_0' \leqq 0$ on $[0, T]$. In (4.5), we choose

$$(4.6) \qquad q(t) = \begin{cases} 0, & t \in [0, T - n^{-2}B), \\ n^2, & t \in [T - n^{-2}B, T]. \end{cases}$$

The function $q \in \mathscr{K}_n$ has been chosen in such a way that

$$\int_0^t (q - q_0)(s) ds \leqq 0, \quad t \in [0, T].$$

Applying an integration by parts to (4.5), we obtain

$$0 \leqq \int_0^T (-2y_0 y_0')(s) \int_0^s (q - q_0)(t) dt ds \leqq 0.$$

The integrand is non-positive and we must have

$$y_0'(t) \int_0^t (q - q_0)(s) ds = 0, \quad t \in [0, T].$$

If $\alpha < 0$, $y_0'(t)$ is negative in $[0, T]$ and we must have $q = q_0$ defined by (4.6). We can now compute y_0 and deduce that

$$y_0(T) = (1 + \alpha(T - n^{-2}B)) \cos(B/n) + n^{-1}\alpha \sin(B/n).$$

We see that $\inf_{[0,T]} y_0(t)$ is positive if and only if $y_0(T) > 0$ which gives the condition in (4.2 a).

Rererences

1. M. Essén, *The* cos $\pi\lambda$ *theorem*, Springer Lecture Notes in Mathematics *467* (Springer-Verlag, 1975).

2. ——— *A theorem on convex sequences*, UUDM Report 1981:12. Analysis *2* (1982), 231-252.

3. ——— *Optimization and rearrangements of the coefficient in the operator $d^2/dt^2 - p(t)^2$ on a finite interval*, UUDM Report 1983:3. To appear, Journal of Mathematical Analysis and Applications.

4. G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities* (Cambridge University Press, 1934).

5. P. Hartman, *Ordinary differential equations* (J. Wiley, 1964).

6. J. Ryff, *Orbits of L^1-functions under doubly stochastic transformations*, Trans. of the Amer. Math. Soc. *117* (1965), 92-100.

7. ——— *Majorized functions and measures*, Indag. Math. *30* (1968), 431-437.

8. C. A. Swanson, *Comparison and oscillation theory of linear differential equations* (Academic Press, 1968).

9. F. J. Tipler, *General relativity and conjugate differential equations*, Journal of Differential Equations *30* (1978), 165-174.

University of Uppsala,
Uppsala, Sweden