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# ON PRIMORDIAL GROUPS FOR THE GREEN RING

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*Abstract* Consider the Mackey functor that assigns to each finite group $G$ the Green ring of finitely generated $kG$-modules, where $k$ is a field of characteristic $p > 0$. Thévenaz foresaw in 1988 that the class of primordial groups for this functor is the family of $k$-Dress groups. In this paper we prove that this is true for the subfunctor defined by the Green ring of finitely generated $kG$-modules of trivial source.

*Keywords:* Green ring; primordial group; trivial source module

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## 1. Introduction

For a field $k$, the Green ring of the category of finitely generated $kG$-modules, $a(kG)$, is, by definition, spanned over $\mathbb{Z}$ by elements $[M]$: one for each isomorphism class of finitely generated $kG$-modules and with structures given by $[M] + [N] = [M \oplus N]$ and $[M][N] = [M \otimes_k N]$. The subring generated by the $kG$-modules of trivial source (defined in § 2.1) is denoted by $a(kG, \mathrm{triv})$.

Assigning either of the two above-mentioned rings to each finite group $G$ defines a globally defined Mackey functor, as described by Bouc [**3**] and by Webb [**9**]. These functors are denoted by $a(k_{\_})$ and $a(k_{\_}, \mathrm{triv})$. Based on § 3 of [**9**], we suggest that the concept of primordial group can be defined for any globally defined Mackey functor. In terms of our two functors this concept is expressed in a familiar way: let $M$ be either $a(k_{\_})$ or $a(k_{\_}, \mathrm{triv})$. A group $G$ is then called primordial for $M$ if $M(G)/T(G) \neq 0$ with

$$T(G) = \sum_{\substack{H \hookrightarrow G \\ H \ncong G}} \mathrm{tr}_H^G M(H).$$

This definition also works for the Mackey functor $G_0(k_{\_})$, which assigns to $G$ the Grothendieck group of finitely generated $kG$-modules. We write $\mathrm{Prim}(M)$ for the class of primordial groups for $M$.

Primordial groups were first studied by Dress [**6**] in the context of Green functors for a finite group $G$. Thévenaz [**8**] proved that for such a functor $N$, the closure under conjugation and subgroups of the primordials for $N$ is the minimal set $\mathscr{D}$ of subgroups of $G$ satisfying $N(G) = \sum_{H\in\mathscr{D}} \mathrm{tr}_H^G N(H)$. Thévenaz also proved that if $k$ is a field of characteristic $p > 0$, the primordial groups for $\mathbb{Q} \otimes a(kG)$ are the $p$-hypoelementary subgroups of $G$ (see § 2 for definitions). Also, he conjectured that the primordials for $G_0(kG)$ were the $k$-elementary subgroups of $G$, which was proved in 1989 by Raggi [**7**], and that an analogue of the Brauer–Berman–Witt Theorem (see [**1**, Theorem 5.6.7]) should hold for $a(kG)$. The main result of this work, Theorem 3.2, states that this is true for the subring of trivial source modules.

Recent work in the area of induction can be found in the papers of Boltje [**2**] and Coşkun [**4**]. It is important to mention that some results about primordial groups can be generalized to the context of globally defined Mackey functors: Lemma 3.1, for example, shows the general behaviour of primordial groups of subfunctors.

## 2. Preliminaries

From this point on we assume that $k$ is a field of characteristic $p > 0$, that all modules are finitely generated, and that all groups are finite.

Recall that for a group $G$ and a prime $r$, $O^r(G)$ is defined as the smallest normal subgroup of $G$, such that $G/O^r(G)$ is an $r$-group, and $O_r(G)$ is the largest normal $r$-subgroup of $G$.

**Definition 2.1.** If $q$ and $r$ are primes, a group $H$ is called $q$-*hyperelementary* if $O^q(H)$ is cyclic, and $r$-*hypoelementary* if $H/O_r(H)$ is cyclic.

Observe that $H$ is $q$-hyperelementary if and only if $H = C \rtimes Q$, with $Q$ a $q$-group and $C$ a cyclic group of order prime to $q$, and it is $r$-hypoelementary if and only if $H = D \rtimes C$, where $D$ is an $r$-group and $C$ is cyclic of order prime to $r$. It is easy to prove that the classes of $q$-hyperelementary and $r$-hypoelementary groups are closed under subgroups and quotients.

**Notation 2.2.** We write $\mathbb{Z}_m^*$ for the smallest non-negative representatives of the multiplicative group of units modulo $m$ (which we denote by $(\mathbb{Z}/m\mathbb{Z})^*$).

**Definition 2.3.** Suppose $H = C \rtimes Q$ is a $q$-hyperelementary group with $C = \langle x\rangle$ of order $m$ prime to $p$. The group $H$ is called $k$-*elementary* if the action of every $y \in Q$ on $x$ is given by $yxy^{-1} = x^a$ with $a \in I_m(k)$, where $I_m(k) \subseteq \mathbb{Z}_m^*$ is the set of the smallest non-negative representatives of the image of $\mathscr{G}\mathrm{al}(k(\omega)/k)$ under the injective morphism

$$\mathscr{G}\mathrm{al}(k(\omega)/k) \to (\mathbb{Z}/m\mathbb{Z})^*,$$
$$\sigma \mapsto \bar{a}$$

if $\sigma(\omega) = \omega^a$ with $1 \leqslant a \leqslant m-1$, $(a, m) = 1$ and $\omega$ is a primitive $m$th root of unity.

We write $\mathcal{E}_k$ for the class of $k$-elementary groups.

Note that in the latter definition we can replace $I_m(k)$ by $I_n(k)$, where $n$ is any multiple of $m$.

**Definition 2.4.** For a prime $q$, a group $H$ is called $q$-*Dress* if $O^q(H)$ is $p$-hypoelementary.

It is easy to see that $O^q(H)$ is $p$-hypoelementary if and only if $H/O_p(H)$ is $q$-hyperelementary. If, in addition to this, $H/O_p(H)$ is $k$-elementary, then $H$ is called *$k$-Dress* for $q$.

The class of $q$-Dress groups is closed under subgroups and quotients, and it is denoted by $\mathrm{Dr}_q$. The class of groups which are $k$-Dress for some prime will be denoted by $\mathrm{Dr}_k$. We write $\mathrm{Dr}_p^*$ for the class of $k$-Dress groups such that $p$ divides the order of $H/O_p(H)$; that is, $H/O_p(H) = C \rtimes D$ is $k$-elementary with $D$ being a non-trivial $p$-group.

**Notation 2.5.** Let $G$ be a group and $H$ a subgroup of $G$. If $M$ is a $kH$-module, we will write $M\uparrow_H^G$ for the induced module $kG \otimes_{kH} M$. If $N$ is a $kG$-module, the restriction of $N$ to $kH$ will be denoted by $N\downarrow_H^G$.

### 2.1. Trivial source modules

The following facts about trivial source modules are well known and can be found in [**1**] and [**5**].

Recall that for a $kG$-module it is equivalent to be $(G, H)$-projective and to be a direct summand of $L\uparrow_H^G$ for some $kH$-module $L$. If $M$ is an indecomposable $kG$-module, there exists a subgroup $D$ for which $M$ is $(G, D)$-projective and $D \leqslant_G H$ for any subgroup $H$ for which $M$ is $(G, H)$-projective; such a group $D$ is called a *vertex* of $M$. In this case, if $L$ is an indecomposable $kD$-module such that $M$ is a direct summand of $L\uparrow_D^G$, then $L$ is called a *source* of $M$. The module $M$ is said to have *trivial source* if the field $k$ is a source of $M$, which is equivalent to saying that $M$ is a direct summand of a permutation module.

It can be proved that any two vertices of $M$ are conjugate in $G$, and that any two sources of $M$ are conjugate by an element in $N_G(D)$. Since we are assuming that $k$ is a field of characteristic $p$, a vertex of $M$ is a $p$-subgroup of $G$.

We prove the following property of trivial source modules, which will be used later.

**Lemma 2.6.** *The tensor induction of a trivial source module is a trivial source module.*

**Proof.** We denote the tensor induction from $H$ to $G$ by $\uparrow_H^{G\otimes}$. Let $B$ be a $kH$-module of trivial source, then it is a direct summand of a permutation module, say $\bigoplus_{a\in[H/K]} k = B \oplus A$. From the proof of Proposition 3.15.2 (iii) in [**1**], it is easy to see that the tensor induction of a permutation module is a permutation module. By the same proposition, on the right-hand side we obtain

$$B{\uparrow_H^{G}}^{\otimes} \oplus A{\uparrow_H^{G}}^{\otimes} \oplus X,$$

where $X$ is a sum of modules induced from proper subgroups of $G$. □

The following corollary to the Green Indecomposability Theorem [**5**, 19.22] will be used in the following sections, as will the lemma that follows. Recall that $k$ is a field of characteristic $p$.

**Corollary 2.7 (Curtis and Reiner [5, 19.23]).** *Suppose that $G$ is a $p$-group and $H$ is an arbitrary subgroup. If $L$ is an absolutely indecomposable $kH$ module (i.e $k' \otimes L$ is indecomposable for every $k'$ field extension of $k$), then $L \uparrow_H^G$ is an absolutely indecomposable $kG$-module.*

**Lemma 2.8.** *Let $U$ be an indecomposable $kH$-module of trivial source with vertex containing $O_p(H)$, and let $M$ be an indecomposable $kG$-module for $G \leqslant H$ such that $U$ is a summand of $M \uparrow_G^H$. Suppose also that if $|H/O_p(H)|$ is divisible by $p$, we have $M$ of trivial source. Then*

(i) *$O_p(H)$ is contained in a vertex of $M$ and it acts trivially on $M$;*

(ii) *any indecomposable summand $V$ of $M \uparrow_G^H$ has trivial source, $O_p(H)$ is contained in a vertex of $V$ and it acts trivially on $V$.*

**Proof.** Let $D$ be a vertex of $U$ that contains $O_p(H)$. If $D_1$ and $S$ are a vertex and a source of $M$, respectively, then $O_p(H) \subseteq D_1 \subseteq G$ (because $U$ is a direct summand of $S \uparrow_{D_1}^H$). If $|H/O_p(H)|$ is a $p'$-number, then $O_p(H)$ is a vertex of $U$ and $D_1 \subseteq O_p(H)$, so we have $O_p(H) = D_1$. This implies that $S$ is a source of $U$ and that $M$ has trivial source. With this we obtain that $V$ is of trivial source, which is the first part of (ii).

Since $M$ is a summand of $k \uparrow_{D_1}^G$, then $M \downarrow_{O_p(H)}^G$ is a direct summand of $k \uparrow_{D_1}^G \downarrow_{O_p(H)}^G$. By the Mackey formula, the latter is isomorphic to $\bigoplus_a k$, where $a$ runs over $[G/D_1]$, so $M \downarrow_{O_p(H)}^G$ is isomorphic to a sum of $k$. With this we prove (i), and by the same argument we prove that $O_p(H)$ acts trivially on $V$.

Finally, if $A$ is a vertex of $V$, then $V \downarrow_{O_p(H)}^H \cong \bigoplus k$ is a summand of $k \uparrow_A^H \downarrow_{O_p(H)}^H$, which is isomorphic to $\bigoplus_b k \uparrow_{O_p(H)\cap A}^{O_p(H)}$. By Corollary 2.7, we have that $k \uparrow_{O_p(H)\cap A}^{O_p(H)}$ is indecomposable. For some $b$ we should then have $k \cong k \uparrow_{O_p(H)\cap A}^{O_p(H)}$, so $O_p(H)$ is contained in $A$. □

## 3. Primordial groups

**Lemma 3.1.**

(i) $\mathrm{Prim}(a(k_{\_}))$ *and* $\mathrm{Prim}(a(k_{\_}, \mathrm{triv}))$ *are closed under subgroups and quotients.*

(ii) $\mathcal{E}_k \subseteq \mathrm{Prim}(a(k_{\_})) \subseteq \mathrm{Prim}(a(k_{\_}, \mathrm{triv})) \subseteq \bigcup_q \mathrm{Dr}_q$.

**Proof.** (i) The proof is the same for both functors, so $M$ represents either of them. First, let $G$ be a primordial group for $M$, and let $H$ be a subgroup of $G$. By Lemma 2.6, tensor induction provides a map from $M(H)$ to $M(G)$, and clearly it sends the class of the field $k$ to itself. If we suppose that $k$ can be written as a linear combination of modules induced from proper subgroups of $H$, then, by (iii) and (iv) of Proposition 3.15.2 in [**1**], its image is a linear combination of modules induced from proper subgroups of $G$. This contradicts that $G$ is primordial for $M$.

Now we take $G/K$, a quotient of $G$. Consider the inflation from $M(G/K)$ to $M(G)$. Again, the class of $k$ is invariant under inflation, so if it could be written as a linear

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combination of modules induced from proper subgroups of $G/K$, then these could be seen as modules induced form proper subgroups of $G$, which is a contradiction.

(ii) In order to prove the inclusions

$$\mathcal{E}_k \subseteq \mathrm{Prim}(a(k_-)) \subseteq \mathrm{Prim}(a(k_-, \mathrm{triv})),$$

we recall that for every group $H$ we have the following morphisms:

$$a(kH) \to G_0(kH) \quad \text{and} \quad a(kH, \mathrm{triv}) \hookrightarrow a(kH).$$

The first of these sends the class of $T$ in $a(kH)$ to the class of $T$ in $G_0(kH)$. These are morphisms of unitary algebras and commute with induction. To represent any of them we write $f_H \colon M(H) \to N(H)$. Now suppose that $H$ is primordial for $N$. Given the properties of $f_H$, if $k$ can be written as a linear combination of modules induced from proper subgroups of $H$ in $M(H)$, then that can also be made in $N(H)$, which is a contradiction. So $H$ is primordial for $M$. Recall that $\mathrm{Prim}(G_0(k_-)) = \mathcal{E}_k$, as mentioned in § 1.

To prove the inclusion $\mathrm{Prim}(a(k_-, \mathrm{triv})) \subseteq \bigcup_q \mathrm{Dr}_q$, we will write $M$ for $a(k_-, \mathrm{triv})$ and $\mathcal{D}$ for $\mathrm{Prim}(a(k_-, \mathrm{triv}))$. Let $G$ be any group. We will use the following facts.

(a) Using the Dress Induction Theorem for the Burnside ring, as stated in Yoshida's paper [**10**], we obtain a generalization of this theorem for the Mackey functor $M$. If we denote the class of $p$-hypoelementary groups by $p$-Hypo, we have

$$M(G) = \sum_{\substack{K \leqslant G \\ K \in \mathcal{H}(p\text{-Hypo})}} \mathrm{tr}_K^G M(K) + \bigcap_{\substack{L \leqslant G \\ L \in p\text{-Hypo}}} \ker(\mathrm{res}_L^G),$$

where $\mathrm{res}_L^G$ is the restriction map from $M(G)$ to $M(L)$ and

$$\mathcal{H}(p\text{-Hypo}) = \{H \text{ group} \mid \exists q \text{ prime with } O^q(H) \in p\text{-Hypo}\}.$$

It is not hard to prove that $\mathcal{H}(p\text{-Hypo}) = \bigcup_q \mathrm{Dr}_q$.

(b) Observe that the Mackey functor $a(k_-)$ satisfies the Frobenius reciprocity formulae

$$\mathrm{tr}_K^G(m \cdot \mathrm{res}_K^G n) = (\mathrm{tr}_K^G m) \cdot n \quad \text{and} \quad \mathrm{tr}_K^G((\mathrm{res}_K^G n) \cdot m) = n \cdot (\mathrm{tr}_K^G m)$$

for all $m$ in $a(kK)$ and $n$ in $a(kG)$, with $K \leqslant G$.

(c) Note also that $\mathcal{D}$ is the smallest class of groups that is closed under subgroups and quotients such that, for every group $G$,

$$M(G) = \sum_{\substack{K \leqslant G \\ K \in \mathcal{D}}} \mathrm{tr}_K^G M(K).$$

A proof of this is a slight modification of Thévenaz's in [**8**].

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We consider the inclusion $a(kG, \mathrm{triv}) \subseteq \mathbb{Q} \otimes a(kG)$ and we will write $N$ for $\mathbb{Q} \otimes a(k_{-})$. From an article of Thévenaz [**8**] we have that $\mathrm{Prim}(N)$ is the class $p$-Hypo and that

$$N(G) = \sum_{\substack{K \leqslant G \\ K \in p\text{-Hypo}}} \mathrm{tr}_K^G N(K).$$

So we have

$$1_{M(G)} = 1_{N(G)} = \sum_{\substack{K \leqslant G \\ K \in p\text{-Hypo}}} \mathrm{tr}_K^G n_K,$$

where $1_{M(G)}$ represents the unity of $M(G)$ and $n_K$ is an element of $N(K)$. From the formula in (a), if $m$ is any element $m$ of

$$\bigcap_{\substack{L \leqslant G \\ L \in p\text{-Hypo}}} \ker(\mathrm{res}_L^G),$$

we have

$$\begin{aligned} m = 1_{M(G)} \cdot m &= \sum_{\substack{K \leqslant G \\ K \in p\text{-Hypo}}} (\mathrm{tr}_K^G n_K) \cdot m \\ &= \sum_{\substack{K \leqslant G \\ K \in p\text{-Hypo}}} \mathrm{tr}_K^G (n_K \cdot \mathrm{res}_K^G m) \quad \text{by (b)} \\ &= \sum_{\substack{K \leqslant G \\ K \in p\text{-Hypo}}} \mathrm{tr}_K^G (n_K \cdot 0) = 0. \end{aligned}$$

So we have

$$M(G) = \sum_{\substack{K \leqslant G \\ K \in \mathcal{H}(p\text{-Hypo})}} \mathrm{tr}_K^G M(K).$$

Since $\mathcal{H}(p\text{-Hypo})$ is closed under subgroups and quotients, using (c) we obtain

$$\mathrm{Prim}(a(k_{-}, \mathrm{triv})) \subseteq \bigcup_q \mathrm{Dr}_q.$$

□

**Theorem 3.2.** $\mathrm{Prim}(a(k_{-}, \mathrm{triv})) = \mathrm{Dr}_k$.

The proof is given by Propositions 3.3 and 3.5.

From Proposition 3.5 we will conclude that every primordial group for $a(k_{-})$ has to be $k$-Dress for some prime. On the other hand, the following proposition shows that every $k$-Dress group for a prime different from $p$ is primordial for $a(k_{-})$. As for the general case, the main difficulty arises from the fact that the techniques we use (namely Lemma 2.8) are ineffective for non-trivial source modules.

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**Proposition 3.3.**

(i) $\mathrm{Dr}_k \setminus \mathrm{Dr}_p^* \subseteq \mathrm{Prim}(a(k_-))$.

(ii) $\mathrm{Dr}_p^* \subseteq \mathrm{Prim}(a(k_-, \mathrm{triv}))$.

**Proof.** We prove (i) and (ii) simultaneously by contradiction. Let $H$ be a $k$-Dress group. We have two cases.

- $H/O_p(H)$ is not divisible by $p$; in this case we suppose that $H$ is not primordial for $a(k_-)$ to prove (i).
- $H/O_p(H)$ is divisible by $p$, so we assume that $H$ is not primordial for $a(k_-, \mathrm{triv})$ to prove (ii).

In both cases, $k$ can be written as a linear combination of modules induced from proper subgroups of $H$:

$$k \oplus \left( \bigoplus_i M_i \uparrow_{L_i}^H \right) \cong \bigoplus_j N_j \uparrow_{T_j}^H .$$

We show that in the first case we can assume that $M_i$ and $N_j$ are trivial source modules. Notice that if an indecomposable module of trivial source is a direct summand of one $M_i \uparrow_{L_i}^H$ (or $N_j \uparrow_{T_j}^H$), then, by Lemma 2.8, $M_i$ (or $N_j$) and all the indecomposable summands are trivial source modules. Therefore, by the Krull–Schmidt Theorem, we can assume that all of the $M_i$ and $N_j$ are of trivial source. Since in the second case we already assume that they are trivial source modules, the following arguments are valid for both cases. From 2.8, we have that $O_p(H)$ acts trivially on $N_j$ and $M_i$ and that they have a vertex containing $O_p(H)$. We take the quotients

$$k \oplus \left( \bigoplus_i M_i \uparrow_{L_i/O_p(H)}^{H/O_p(H)} \right) \cong \bigoplus_j N_j \uparrow_{T_j/O_p(H)}^{H/O_p(H)} .$$

This isomorphism turns into an equality in $a(k(H/O_p(H)), \mathrm{triv})$, which is contained in $a(k(H/O_p(H)))$. Since $H/O_p(H)$ is $k$-elementary, Lemma 3.1 yields a contradiction. □

**Lemma 3.4.** *If $H$ is of the smallest order that is $q$-Dress but not $k$-Dress, then $H$ is of the form*

$$H = \langle x \rangle \rtimes \langle y \rangle, \quad \text{where } |\langle x \rangle| = r,\ |\langle y \rangle| = q^n,$$

*with $r$ and $q$ different primes and $yxy^{-1} = x^a$ with $a \in \mathbb{Z}_r^* \setminus I_r(k)$.*

**Proof.** Being a quotient of $H$, $H/O_p(H)$ is $q$-Dress but, since it is not $k$-elementary and $O_p(H/O_p(H)) = 1$, $H/O_p(H)$ is not $k$-Dress. Therefore, the minimality of $H$ implies $O_p(H) = 1$. Hence, we have $H = C \rtimes Q$ with $C = \langle s \rangle$ cyclic of order $m$ and $Q$ a $q$-group such that $m$ is not divisible by $p$ and $q$. Now, as $H$ is not $k$-elementary, there exists $y \in Q$ such that $ysy^{-1} = s^a$ with $a \in \mathbb{Z}_m^* \setminus I_m(k)$, so $H = C \rtimes C_{q^n}$, where $C_{q^n}$ is the cyclic group of order $q^n$ generated by $y$.

Now we write $C = \prod_i C_{r_i}$, where $C_{r_i}$ is the $r_i$-Sylow subgroup of $C$ and the $r_i$ are all the primes that divide $m$, so

$$C \rtimes C_{q^n} = \prod_i (C_{r_i} \rtimes C_{q^n}).$$

In addition, if $\omega$ is a primitive $m$th root of unity, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{G}\mathrm{al}(k(\omega)/k) & \hookrightarrow & (\mathbb{Z}/m\mathbb{Z})^* \\ \cong\downarrow & & \downarrow\cong \\ \prod_i \mathcal{G}\mathrm{al}(k(\omega^{m/r_i^{\alpha_i}})/k) & \hookrightarrow & \prod_i (\mathbb{Z}/r_i^{\alpha_i}\mathbb{Z})^* \end{array}$$

where $\alpha_i$ is the largest positive integer such that $r_i^{\alpha_i}$ divides $m$. Thus, there exists $i$ such that $C_{r_i} \rtimes C_{q^n}$ is not $k$-elementary. Rewriting $C_{r_i} = C_{r^\alpha}$ we have $H = C_{r^\alpha} \rtimes C_{q^n}$, where $C_{r^\alpha} = \langle x_o \rangle$, $C_{q^n} = \langle y \rangle$ and $y x_o y^{-1} = x_o^a$ with $a \in \mathbb{Z}_{r^\alpha}^* \smallsetminus I_{r^\alpha}(k)$.

Finally, we take $\zeta$ to be a primitive $r^\alpha$th root of unity. We have

$$\mathcal{G}\mathrm{al}(k(\zeta)/k) \cong \mathcal{G}\mathrm{al}(k(\zeta)/k(\zeta^{r^{\alpha-1}})) \times \mathcal{G}\mathrm{al}(k(\zeta^{r^{\alpha-1}})/k)$$

and

$$(\mathbb{Z}/r^\alpha\mathbb{Z})^* \cong A_{r^{\alpha-1}} \times A_{r-1},$$

where these groups have order $r^{\alpha-1}$ and $r-1$, respectively. The morphism

$$\mathcal{G}\mathrm{al}(k(\zeta)/k) \hookrightarrow (\mathbb{Z}/r^\alpha\mathbb{Z})^*$$

takes $\mathcal{G}\mathrm{al}(k(\zeta)/k(\zeta^{r^{\alpha-1}}))$ into $A_{r^{\alpha-1}}$ and $\mathcal{G}\mathrm{al}(k(\zeta^{r^{\alpha-1}})/k)$ into $A_{r-1}$, which is isomorphic to $(\mathbb{Z}/r\mathbb{Z})^*$, so we have the commutative diagram

$$\begin{array}{ccc} \mathcal{G}\mathrm{al}(k(\zeta)/k) & \hookrightarrow & (\mathbb{Z}/r^\alpha\mathbb{Z})^* \\ \downarrow & & \downarrow \\ \mathcal{G}\mathrm{al}(k(\zeta^{r^{\alpha-1}})/k) & \hookrightarrow & (\mathbb{Z}/r\mathbb{Z})^* \end{array}$$

Now, we have $a^{q^n} \equiv 1 \bmod r^\alpha$, thus $r$ does not divide the order of $a$ modulo $r^\alpha$, and we have $a \in \mathbb{Z}_r^*$. Since $a$ is not in $I_{r^\alpha}(k)$, we have $a \in \mathbb{Z}_r^* \smallsetminus I_r(k)$. Taking $x = x_o^{r^{\alpha-1}}$ we have the result. □

**Proposition 3.5.** $\mathrm{Prim}(a(k_-, \mathrm{triv})) \subseteq \mathrm{Dr}_k$.

**Proof.** The proof is by contradiction. We suppose there is a group $H$ of smallest order in $\mathrm{Prim}(a(k_-, \mathrm{triv}))$ that is not $k$-Dress. Observe that Lemma 3.4 is also valid if $H$ satisfies a property that is preserved under subgroups and quotients and if $H$ is $q$-Dress and satisfies a property that is preserved under subgroups and quotients. Since this is the case for the property of being primordial for $a(k_-, \mathrm{triv})$, the lemma gives us $H = C \rtimes Q$

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with $C = \langle x \rangle$ of order $r$ and $Q = \langle y \rangle$ of order $q^n$ with $r$ and $q$ different primes and $yxy^{-1} = x^a$ with $a \in \mathbb{Z}_r^* \smallsetminus I_r(k)$.

We write $1_H$ to identify the field $k$ as a $kH$-module. We shall prove that $1_H$ is a sum of modules induced from proper subgroups of $H$, contradicting the assumption of primordiality.

The image of $1_Q$ under the induction morphism $1_Q \uparrow_Q^H$ is isomorphic, as a vector space, to $\bigoplus_{i=0}^{r-1} kx^i \otimes_Q 1_Q$. If $\omega$ is a primitive $r$th root of unity, then the $k(\omega)H$-module $k(w) \otimes 1_Q \uparrow_Q^H$ as a $k(w)$-vector space has the basis

$$\{x^i \otimes_Q 1_Q \mid i = 0, \dots, r-1\}.$$

We can define another basis $y_t := \sum_{i=0}^{r-1} \omega^{-ti}(x^i \otimes_Q 1_Q)$ for $t = 0, \dots, r-1$. To prove that it is a basis, observe that the matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \cdots & \omega^{-j} & \cdots & \omega^{-(r-1)} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(r-1)} & \cdots & \omega^{-j(r-1)} & \cdots & \omega^{-(r-1)^2} \end{pmatrix}$$

has determinant equal to $\prod_{i \neq j}(\omega^{-i} - \omega^{-j})$, which is different from 0 since $\omega$ is an $r$th primitive root of unity.

$H$ acts on this basis in the following way:

$$\begin{aligned} xy_t &= \sum_{i=0}^{r-1} \omega^{-ti}(x^{i+1} \otimes_Q 1_Q) \quad \text{if } j = i+1 \\ &= \sum_{j=0}^{r-1} \omega^{-t(j-1)}(x^j \otimes_Q 1_Q) \\ &= \omega^t y_t, \end{aligned}$$

$$\begin{aligned} yy_t &= \sum_{i=0}^{r-1} \omega^{-ti}(yx^i \otimes_Q 1_Q) \\ &= \sum_{i=0}^{r-1} \omega^{-ti}(x^{ai} \otimes_Q 1_Q) \\ &= \sum_{i=0}^{r-1} \omega^{-tbi}(x^i \otimes_Q 1_Q) \\ &= y_{t'}, \end{aligned}$$

where $b \in \mathbb{Z}_r^*$ is such that $\bar{b}\bar{a} = 1$ in $(\mathbb{Z}/r\mathbb{Z})^*$, and $0 \leqslant t' \leqslant r-1$ satisfies $t' \equiv tb \bmod r$. From these relations we see that $y_0$ is fixed under the action of $H$, so $k(\omega)y_0$ is $k(\omega)H$-isomorphic to $k(\omega)$ and we have

$$k(\omega) \otimes 1_Q \uparrow_Q^H \cong k(\omega) \oplus \left( \sum_{t=1}^{r-1} k(\omega) y_t \right).$$

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It is clear that $\mathcal{G} = \mathcal{G}\mathrm{al}(k(\omega)/k)$ acts on $k(\omega) \otimes 1_Q \uparrow_Q^H$. Taking the fixed points of this action in the isomorphism above gives us

$$1_Q \uparrow_Q^H \cong 1_H \oplus \left( \sum_{t=1}^{r-1} k(\omega) y_t \right)^{\mathcal{G}}.$$

Let $\sigma$ be in $\mathcal{G}$. We write $b_\sigma \in I_r(k)$ for the integer through which $\sigma$ is defined. We have

$$\sigma y_t = \sum_{i=0}^{r-1} \sigma(\omega^{-ti})(x^i \otimes 1) = \sum_{i=0}^{r-1} \omega^{-tb_\sigma i}(x^i \otimes 1) = y_s,$$

where $0 \leqslant s \leqslant r-1$ and $s \equiv tb_\sigma \bmod r$. If $u = \sum_{t=1}^{r-1} \lambda_t y_t$ is in $(\sum_{t=1}^{r-1} k(\omega) y_t)^{\mathcal{G}}$, then for each $t \in \mathbb{Z}_r^*$ we must have $\sigma(\lambda_t) = \lambda_s$, where $s \in \mathbb{Z}_r^*$ and $s \equiv tb_\sigma \bmod r$. From this we define the vector spaces

$$M_l := \left\{ \sum_{\sigma \in \mathcal{G}} \sigma(\lambda_l) y_s \;\middle|\; s \equiv lb_\sigma \bmod r,\ \lambda_l \in k(\omega) \right\}$$

for each $l \in \mathbb{Z}_r^*$. Observe that $M_{l_1} = M_{l_2}$ if and only if $l_1 \equiv l_2 b_\sigma \bmod r$ for some $\sigma \in \mathcal{G}$; this implies that

$$\left( \sum_{t=1}^{r-1} k(\omega) y_t \right)^{\mathcal{G}} = \bigoplus_{l \in \mathbb{Z}_r^*/I_r(k)} M_l.$$

We shall prove that the right-hand side of this equality is a sum of modules induced from proper subgroups of $H$. We have $xM_l = M_l$ and $yM_l = M_{l'}$ with $l' \in \mathbb{Z}_r^*$ and $l' \equiv lb \bmod r$. Since $a$ does not belong to $I_r(k)$, neither does $b$, so $M_l$ is never fixed under the action of $y$. Then $M_l$ is not a $kH$-module. Taking the orbits of the action of $y$ we have

$$\begin{aligned}\left( \sum_{t=1}^{r-1} k(\omega) y_t \right)^{\mathcal{G}} &= \bigoplus_{l \in \frac{\mathbb{Z}_r^*/I_r(k)}{\sim}} \left( \bigoplus_{z \in [H/A]} z M_l \right) \\ &= \bigoplus_{l \in \frac{\mathbb{Z}_r^*/I_r(k)}{\sim}} (M_l \uparrow_A^H),\end{aligned}$$

where $A = \mathrm{Stab}_H(M_l)$, and $\sim$ represents the action of $y$.

It is clear that $A$ is a proper subgroup of $H$. Finally, observe that every $M_l$ is a trivial source $kA$-module. If $A = C_r \rtimes \langle y^d \rangle$, then $M_l$ is a direct summand of the induced module

$$\left( \sum_{\sigma \in \mathcal{G}} k y_s \right) \uparrow_{\langle y^d \rangle}^A,$$

where $s \in \mathbb{Z}_r^*$ and $s \equiv lb_\sigma \bmod r$. Since $ky_s \cong k$, $M_l$ has trivial source. □

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