J. Austral. Math. Soc. (*Series A*) **50** (1991), 243–247

ABELIAN GROUPS WITH SMALL COTORSION IMAGES

RÜDIGER GÖBEL

(Received 10 May 1989; revised 17 January 1990)

Communicated by R. Lidl

Abstract

Epimorphic images of compact (algebraically compact) abelian groups are called cotorsion groups after Harrison. In a recent paper, Ph. Schultz raised the question whether "cotorsion" is a property which can be recognized by its small cotorsion epimorphic images: If G is a torsion-free group such that every torsion-free reduced homomorphic image of cardinality $2^{\aleph_0}$ is cotorsion, is G necessarily cortorsion? In this note we will give some counterexamples to this problem. In fact, there is no cardinal κ which is large enough to test cotorsion.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*) (1985 *Revision*): 20 K 30.

1. Introduction

In a recent paper [8, Problem 3], Ph. Schultz raised the following problem in abelian group theory.

If G is a torsion-free group such that every torsion-free reduced homomorphic image of cardinality at most 2^{ω} is torsion, is G necessarily cortorsion?

In this note we will give some counterexamples to this problem. In fact, there is no cardinal κ which is large enough to test cotorsion in the above sense. In order to state the theorem, recall D. K. Harrison's notion that a group G is *cotorsion* if and only if G is an epimorphic images of algebraically compact abelian group. Moreover algebraically compact abelian groups are direct summands of those groups which permit a compact Hausdorff topology. Hence cotorsion groups are the epimorphic images of groups with a compact topology. These groups are "well-known" and may be characterized by cardinal invariants. A group G is *cotorsion-free* if G has no non-trivial cotorsion subgroup, see [5, 6] or [2]. For abelian groups this is equivalent to saying that G is torsion-free (that is, G does not contain cyclic groups of prime order), G does not contain copies of the rationals $\mathbb{Q}$ and of the p-adic integer J_p for any prime p (cf. [6] or [2]). Then we have

THEOREM 1. *Suppose λ, κ are infinite cardinals with $\kappa^\omega = \kappa$ and $\lambda^\kappa = \lambda$. Then there exists a cotorsion-free group G of cardinality λ such that all its torsion-free epimorphic image of cardinality at most κ are cotorsion.*

The proof is similar to that in [1] but simpler. Amalgamating into the proof ideas from [1], we can also prescribe any cotorsion-free ring of cardinality at most λ as the endomorphism ring of this G; for example, we may also assume that $\operatorname{End} G = R$ is a fixed subring $R \subset \mathbb{Q}$. Hence the nucleus $R(G)$ $(= \operatorname{nuc} G)$ of G is R; see [7, 8].

Fixing κ, there is a class, not a set of cardinals λ as in Theorem 1. If $\kappa = 2^{\aleph_0}$, we obtain our smallest counterexample of size $2^{2^{\aleph_0}}$ of Schultz' Problem 3. Observe that $|G| > 2^{\aleph_0}$ is also necessary.

2. Preliminaries

In this section we sketch some combinatorial tools taken from [1], but simplified. In order to see the similarity with [1], we will use the same names and letters for the present setting.

Let κ, λ be two infinite cardinals with $\kappa^\omega = \kappa$ and $\lambda^\kappa = \lambda$. Then $\operatorname{cf}(\lambda) > \kappa > \omega$ from König's Lemma. We take $T = \lambda \times \omega$ and call subsets $v = \{p\} \times \omega \subset T$ $(p \in \lambda)$ branches of the tree T. Moreover v can be viewed as the canonical map $v : \omega \to T$ $(n \to p \times n)$ and $\operatorname{Br}(T)$ denotes all such maps.

Let $\tau = 1\tau$ be the free generator of an infinite cyclic group $\mathbb{Z}_\tau$ and suppose all $\tau \in T$ are independent. If I is a subset of T, then we denote $B_I = \bigoplus_{i \in I} i\mathbb{Z}$ and write $B = B_T$ for the free "basic" group B.

If $\hat{B}$ denotes the $\mathbb{Z}$*-adic completion* of B, then every element $g \in \hat{B}$ can be expressed as a convergent sum

$$g = \sum_{\tau \in T} g_\tau \tau \quad \text{with } g_\tau \in \hat{\mathbb{Z}}.$$

The *support* of g is defined to be $[g] = \{\tau \in T : g_\tau \neq 0\}$, which is at most countable. Particular elements $v^k \in \hat{B}$ derive from branches $v \in \mathrm{Br}(T)$. If

$$v^k = \sum_{n \geq k} \frac{n!}{k!} v(n),$$

then

$$v^k \in \hat{B}, \qquad k \cdot v^k - v^{k-1} = v(k-1) \in B \quad \text{for all integers } k \geq 1$$
$$\text{and } [v^k] \subseteq [v^0] = v.$$

The support of a subset $X \subseteq \hat{B}$ will be $[X] = \bigcup_{x \in X}[x]$ which is of cardinality at most $|X| \cdot \aleph_0$. A subset $P \subseteq T$ is a *canonical subset* of T provided $P = P' \times \omega$ for some subset $P' \subseteq \lambda$ of cardinality at most κ, where the maximal element of P', $\max P' \in P'$, exists. We call $\|P'\| = \max P'$, and $\|P'\| \times \omega$ is the *maximal branch* of P. The abelian group $\hat{B}_P$ is a *canonical summand* of $\hat{B}$ associated with the canonical subset P of T. If $X \subseteq \hat{B}$, then $\|X\|$ denotes the smallest $\|P\|$ such that X is contained in the canonical summand $\hat{B}_P$. If P does not exist, then $\|X\| = \infty$.

In order to work with partial homomorphisms and elements in $\hat{B}$ with special support, we will also need an easy

Definition 2.1. Let λ, κ and B be as above. A *trap* is a pair (P, φ) where P is a canonical subset of T and $\varphi : \hat{B}_P \to \hat{B}_P/\ker(\varphi)$ is an epimorphism.

The proof of Theorem 1 rests on an easy "Black Box", which we call *Small Black Box*. Its proof is a counting argument, which follows by substantial simplification of Shelah's Black Box proved in [1, pages 476–479].

The Small Black Box 2.2. For some ordinal λ^* $(< \lambda^+)$ there exists a transfinite sequence of traps $(P_\alpha, \varphi_\alpha)$ $(\alpha \in \lambda^*)$ such that, for $\alpha, \beta < \lambda^*$,

$$\beta < \alpha \Rightarrow \|P_\beta\| < \|P_\alpha\|.$$

For any subset $X \subseteq \hat{B}$ with $|X| \leq \kappa$ and any homomorphism φ from $\hat{B}$ onto $\hat{B}/\ker\varphi$ there exists $\alpha < \lambda^*$ such that $X \subseteq \hat{B}_{P_\alpha}$, $\|X\| < \|P_\alpha\|$ and $\varphi \restriction \hat{B}_{P_\alpha} = \varphi_\alpha$.

We will abbreviate $B_{P_\alpha} = B_\alpha$ and $v_\alpha = \|P_\alpha\| \times \omega \subseteq T$.

3. Cortorsion-free groups with only cotorsion small images

We want to apply the Small Black Box to prove Theorem 1.

Proof of Theorem 1. Let λ, κ and $B = \bigoplus_{\tau \in T} \mathbb{Z}\tau$ be as in (2.1). We apply (2.2) to the following inductive construction G. The group G will be the union of an ascending, continuous chain of subgroups $G_\alpha \subset \hat{B}$ $(\alpha < \lambda^*)$

https://doi.org/10.1017/S1446788700032729 Published online by Cambridge University Press

with $G_0 = B$. Suppose G_β has been constructed for all $\beta \leq \alpha$ subject to the following condition on non-limit stages β (in place of α):

$$G_{\alpha+1} = \langle G_\alpha, g_\alpha \rangle_* \subseteq \hat{B}$$

for some particular $g_\alpha \in \hat{B}$.

Recall that $\langle - \rangle_*$ denotes the pure subgroup generated by $-$.

Next we want to specify our choice of g_α. Consider g_α satisfying

$$(*) \qquad g_\alpha = x_\alpha + v_\alpha^0 \quad \text{where } \|x_\alpha\| < \|v_\alpha\| \text{ and } x_\alpha \in \hat{B}_\alpha.$$

If we can find g_α with $(*)$ such that

$$(**) \qquad g_\alpha \varphi_\alpha \in B_\alpha \varphi_\alpha \backslash (G_\alpha \cap \hat{B}_\alpha)\varphi_\alpha$$

then we will choose this one. If g_α with $(*)$ and $(**)$ does not exist, let $g_\alpha = 0$ hence $G_{\alpha+1} = G_\alpha$ by purity of G_α. The construction of $G = \bigcup_{\alpha<\lambda^*} G_\alpha$ is now complete.

We have to verify the conditions stated in Theorem 1. Since $|\lambda^*| = \lambda$ and $|T| = \lambda$ also $|G| = \lambda$ is immediate from the construction. If $u \in G$, then $|\{x \in G : [x] = [u]\}| = \aleph_0$ by our choice of generators g_α, τ $(\alpha < \lambda^*,\ \tau \in T)$ and it follows that the group G is cotorsion-free (cf. [1, Lemma 6.2]).

Suppose $\varphi: G \to H$ is an epimorphism with H torsion-free not cotorsion and $|H| \leq \kappa$. If $H = H' \oplus D$ is a decomposition of H into a reduced part H' and a divisible summand D, then $\varphi': G \to H \to H'$ which is φ followed by the canonical projection onto H' is an epimorphism and H' is torsion-free not cotorsion. Hence $\varphi': G \to H'$ satisfies the same assumption as φ. Hence we may also assume that H is reduced. The $\mathbb{Z}$-adic topology on H is now a Hausdorff topology and H is a dense, pure and proper subgroup of its $\mathbb{Z}$-adic completion $\hat{H}$. From $|H| \leq \kappa$ and the size of canonical subsets we can find such a canonical subset I of T with $(G \cap \hat{B}_I)\varphi = H$ and clearly $\hat{B}_I \varphi = \hat{H}$ by completeness and density. From the Small Black Box we find $\alpha < \lambda^*$ such that

$$I \subset P_\alpha, \quad \varphi \restriction B_\alpha = \varphi_\alpha \quad \text{and} \quad \|B_I\| < \|B_\alpha\|.$$

Clearly $(G \cap \hat{B}_\alpha)\varphi_\alpha = H$ as $G\varphi = H$, and $(G \cap \hat{B})\varphi = H$. Moreover $\hat{B}_I \varphi_\alpha = \hat{H}$ and H is a proper subgroup of $\hat{H}$. There exists $x_\alpha \in \hat{B}_I$ such that $x_\alpha \varphi_\alpha \in \hat{H} \backslash H$. If $v_\alpha \in \mathrm{Br}(T)$ satisfies $v_\alpha^0 \varphi_\alpha \in \hat{H} \backslash H$, then $G_\alpha \neq G_{\alpha+1}$ by $(*)$ and $(**)$. If $v_\alpha^0 \varphi_\alpha \in H$, then $\|x_\alpha\| < \|v_\alpha\|$ from $\|B_I\| < \|B_\alpha\|$ and $x_\alpha + v_\alpha^0$ was a candidate for g_α in the construction. Hence $G_\alpha \neq G_{\alpha+1}$ in any case, and there exists $g_\alpha \in \hat{B}_\alpha$ such that $G_{\alpha+1} = \langle G_\alpha, g_\alpha \rangle_*$ and $g_\alpha \varphi_\alpha \notin (G_\alpha \cap \hat{B}_\alpha)\varphi_\alpha$. Since $\varphi_\alpha = \varphi \restriction \hat{B}_\alpha$, we derive

$$g_\alpha \varphi \notin (G_\alpha \cap \hat{B}_\alpha)\varphi = (G \cap \hat{B}_\alpha)\varphi = H.$$

However $g_\alpha \in G$ contradicts $\varphi: G \to H$, and H must be cotorsion.

References

[1] A. L. S. Corner and R. Göbel, 'Prescribing endomorphism algebras, a unified treatment', *Proc. London Math. Soc.* (3) **50** (1985), 447–479.
[2] P. Eklof and A. Mekler, *Almost free modules, set theoretic methods*, (North-Holland, 1990).
[3] L. Fuchs, *Infinite abelian groups*, Vols. I, II, (Academic Press, New York, 1970, 1973).
[4] R. Göbel and S. Shelah, 'On semirigid classes of torsion-free abelian groups', *J. Algebra* **93** (1985), 136–150.
[5] R. Göbel, 'On stout and slender groups', *J. Algebra* **35** (1975), 39–55.
[6] R. Göbel and B. Wald, 'Wachstumstypen und schlanke Gruppen', *Symposia Mathematica* **23** (1979), 201–239.
[7] J. Hausen, 'Automorphismengesättigte Klassen abzählbarer abelscher Gruppen', *Studies on Abelian Groups*, pp. 146–181, (Springer-Verlag, Berlin 1968).
[8] Ph. Schultz, *Self-splitting groups*, (Report 23, August 1988, University of Western Australia), to appear.

Universität Essen GHS
Universitätsstr. 3
D4300 Essen 1
Germany