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Abstract

We study finite orbits of non-elementary groups of automorphisms of compact projective surfaces. We prove that if the surface and the group are defined over a number field $\mathbf{k}$ and the group contains parabolic elements, then the set of finite orbits is not Zariski dense, except in certain very rigid situations, known as Kummer examples. Related results are also established when $\mathbf{k} = \mathbf{C}$. An application is given to the description of 'canonical vector heights' associated to such automorphism groups.

Contents

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1. Introduction

1.1 Setting

Let X be a complex projective surface. Denote by $\mathsf{Aut}(X)$ its group of automorphisms. The action of this group on the Néron–Severi group $\mathrm{NS}(X;\mathbf{Z})$ (respectively, on the cohomology group $H^2(X;\mathbf{Z})$) gives a linear representation $f \mapsto f^*$ from $\mathsf{Aut}(X)$ to $\mathsf{GL}(\mathrm{NS}(X;\mathbf{Z}))$ (respectively, $\mathsf{GL}(H^2(X;\mathbf{Z}))$). By definition, a subgroup Γ of $\mathsf{Aut}(X)$ is *non-elementary* if its image $\Gamma^* \subset \mathsf{GL}(\mathrm{NS}(X;\mathbf{Z}))$ (respectively, $\subset \mathsf{GL}(H^2(X;\mathbf{Z}))$) contains a free group of rank ≥ 2; equivalently, Γ^* does not contain any abelian subgroup of finite index (see [CD23c, CD23a] for details and examples).

Our purpose is to study the existence and abundance of finite (or 'periodic') orbits under such non-elementary group actions. Several possible scenarios can be imagined:

(a) a large (that is, dense or Zariski dense) set of finite orbits;
(b) finitely many finite orbits;
(c) no finite orbit at all.

For a cyclic group generated by a single automorphism, the situation is well understood: in many cases the set of periodic points is large (see [Can14] for an introduction to this topic and [Xie15] for the case of birational transformations). On the other hand, for non-elementary groups, we expect the existence of a dense set of periodic points to be a rare phenomenon; this expectation will be confirmed by our results.

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The only examples we know for situation (a) are given by abelian surfaces and their siblings, Kummer surfaces. Here, by *Kummer surface* we mean a smooth surface X which is a (not necessarily minimal) desingularization of the quotient A/G of an abelian surface $A = \mathbf{C}^2/\Lambda$ by a finite group $G \subset \mathsf{Aut}(A)$. If G is generated by the involution $(x, y) \mapsto (-x, -y)$ of A, we find the classical Kummer surfaces and their blow-ups (see [BHPV04]). Given a subgroup $\Gamma \subset \mathsf{Aut}(X)$, we say that the pair (X, Γ) is a *Kummer group* if X is a Kummer surface and Γ comes from a subgroup of $\mathsf{Aut}(A)$ which normalizes G; precise definitions are given in § 5.7. If Γ is a group of automorphisms of an abelian surface A fixing the origin $0 \in A$, then all torsion points are Γ-periodic. This implies that most Kummer groups have a dense set of finite orbits (see Proposition 4.5).

1.2 Main results

We first illustrate property (c) in the family of Wehler surfaces, that is for smooth surfaces $X \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined by a polynomial equation of degree $(2, 2, 2)$. Such an X is a K3 surface. Generically, the three projections $X \to \mathbb{P}^1 \times \mathbb{P}^1$ are 2-to-1 ramified covers and $\mathsf{Aut}(X)$ is generated by the corresponding involutions. These examples occupy a central position in the dynamical study of surface automorphisms, both from the ergodic and arithmetic points of view (see, e.g., [Sil91, Kaw06, McM02]).

Theorem A. *For a very general Wehler surface X, every orbit under $\mathsf{Aut}(X)$ is Zariski dense. In particular there is no finite orbit under the action of $\mathsf{Aut}(X)$.*

Unfortunately, the proof of this theorem has an obvious limitation: it does not single out any explicit example defined over $\overline{\mathbf{Q}}$ satisfying property (c).

Our main theorem concerns property (b). To state it, recall that there are three types of automorphisms, characterized by the behavior of the linear endomorphism f^* (see [Can14]). If f^* has finite order, then f is *elliptic*. It is *parabolic* if f^* has infinite order, but none of its eigenvalues has modulus >1; it is *loxodromic* if some eigenvalue $\lambda(f)$ of f^* has modulus $|\lambda(f)| > 1$ (in that case $\lambda(f)$ is unique and $\lambda(f) \in (1, +\infty)$). A non-elementary group of automorphisms contains a non-abelian free group all of whose non-trivial elements are loxodromic, and a group containing both loxodromic and parabolic elements is automatically non-elementary.

Theorem B. *Let X be a smooth projective surface, defined over some number field $\mathbf{k}$. Let Γ be a subgroup of $\mathsf{Aut}(X)$, also defined over $\mathbf{k}$, containing both parabolic and loxodromic automorphisms. If the set of finite orbits of Γ is Zariski dense in X, then (X, Γ) is a Kummer group.*

In the setting of the theorem, and more generally when Γ is any group of automorphisms of a complex projective surface X containing a loxodromic element, there is a maximal Γ-invariant curve D_Γ; more precisely, either Γ does not preserve any curve, or there exists a unique, maximal, Γ-invariant, Zariski-closed subset of pure dimension 1. This curve D_Γ can be contracted to yield a (singular) complex analytic surface X_0 and a Γ-equivariant birational morphism

$$\pi_0 \colon X \to X_0. \tag{1.1}$$

Moreover, *when Γ contains a parabolic automorphism, X_0 is projective* (see Proposition 3.9). Another important result towards Theorem B states that *any non-elementary subgroup $\Gamma \subset \mathsf{Aut}(X)$ contains a loxodromic element whose maximal invariant curve is equal to D_Γ* (see Theorem D in § 3). With this notation, Theorem B says that property (b) holds on X_0 if (X, Γ) is defined over a number field and is not a Kummer group.

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Let us stress that even if X and Γ are defined over $\mathbf{k}$, Theorem B deals with orbits of Γ in $X(\mathbf{C})$. This is different from the results of [Sil91, Kaw08], in which a finiteness theorem is obtained for periodic orbits of loxodromic automorphisms in $X(\mathbf{k}')$, where $\mathbf{k}'$ is some fixed number field containing $\mathbf{k}$ (it ultimately relies on Northcott's finiteness theorem).

Under the assumptions of Theorem B, we obtain the following corollaries (see Corollaries 6.1, and 6.2 and Proposition 4.5 for details).

– If Γ does not preserve any algebraic curve and X is not an abelian surface, then Γ admits at most finitely many finite orbits.
– If C is an irreducible curve containing infinitely many Γ-periodic points, then either C is Γ-periodic or (X, Γ) is a Kummer group and C comes from a translate of an abelian subvariety. If C has genus ≥ 2, it contains at most finitely many Γ-periodic points.
– If Γ has a Zariski-dense set of finite orbits, then its finite orbits are dense in $X(\mathbf{C})$ for the Euclidean topology; furthermore, if f_1 and f_2 are two loxodromic automorphisms in Γ, their periodic points coincide, except for at most finitely many of them which are located on Γ-invariant curves.

As we shall see in Remark 6.6, the last statement provides a partial answer to a question of Kawaguchi.

1.3 Proof strategy and extension to complex coefficients

Let us say a few words about the proof of Theorem B (a more detailed outline is given in § 5.1). Given two 'typical' loxodromic elements f, g in Γ, intuition suggests that $\mathrm{Per}(f) \cap \mathrm{Per}(g)$ cannot be Zariski dense unless some 'special' phenomenon happens. This situation has been referred to as an *unlikely intersection* problem in the algebraic dynamics literature (see, e.g., [Zan12, § 3.4]). Previous work on this topic suggests to handle this problem using methods from arithmetic geometry (see, e.g., [BD11, DF17]). In this respect a key idea would be to use arithmetic equidistribution (see [Yua08, BB10]) to derive an equality $\mu_f = \mu_g$ between the measures of maximal entropy of f and g. Unfortunately we do not know how to infer rigidity results directly from this equality, so the proof of Theorem B is not based on this sole argument. To reach a conclusion, we make use of the dynamics of the whole group Γ, in particular of the classification of Γ-invariant measures (see [Can01b, CD23b]), together with the classification of loxodromic automorphisms f whose measure of maximal entropy μ_f is absolutely continuous with respect to the Lebesgue measure (see [CD20, FT21]). The existence of parabolic elements in Γ is required at three important stages, including the arithmetic step; in particular, we are not able to prove Theorem B without assuming that Γ contains parabolic elements (see § 6.3 for a more precise discussion).

Even if arithmetic methods lie at the core of the proof of Theorem B, it is natural to expect that the assumption that X and Γ be defined over a number field is superfluous. We are indeed able to get rid of it when Γ has no invariant curve.

Theorem C. *Let X be a compact Kähler surface which is not a torus. Let Γ be a subgroup of $\mathsf{Aut}(X)$ which contains a parabolic element and does not preserve any algebraic curve. Then Γ admits only finitely many periodic points.*

The proof of Theorem C is based on specialization arguments, inspired notably by the approach of [DF17] (see § 7). It applies, for instance, to the action of $\Gamma = \mathsf{Aut}(X)$ on any unnodal Enriques surface X, and to the foldings of euclidean pentagons with generic side lengths (see [CD23a] for details on these examples).

1.4 Canonical vector heights

Theorem B will be applied to answer a question of Baragar on the existence of certain canonical heights (see [Bar04, BvL07, Kaw13]).

Let X be a projective surface, defined over a number field $\mathbf{k}$. Denote by $\mathrm{Pic}(X)$ the Picard group of $X_{\overline{\mathbf{Q}}}$. The Weil height machine provides, for every line bundle L on X, a height function $h_L\colon X(\overline{\mathbf{Q}}) \to \mathbf{R}$, defined up to a bounded error $O(1)$. This construction is additive, $h_{aL+bL'} = ah_L + bh_{L'} + O(1)$ for all pairs $(L, L') \in \mathrm{Pic}(X)^2$ and all coefficients $(a, b) \in \mathbf{Z}^2$. When $L = \mathcal{O}_X(1)$ for some embedding $X \subset \mathbb{P}^N_{\mathbf{k}}$, then h_L coincides with the usual logarithmic Weil height.

If f is a regular endomorphism of X defined over $\mathbf{k}$ and L is an ample line bundle such that $f^*L = L^{\otimes d}$ for some integer $d > 1$, then $h_L \circ f = dh_L + O(1)$. Tate's renormalization trick

$$\hat{h}_L(x) := \lim_{n\to+\infty} \frac{1}{d^n} h_L(f^n(x)) \tag{1.2}$$

provides a *canonical height* for f and L, that is, a function $\hat{h}_L\colon X(\overline{\mathbf{Q}}) \to \mathbf{R}_+$ such that $\hat{h}_L = h_L + O(1)$ and $\hat{h}_L \circ f = d\hat{h}_L$ exactly, with no error term. This construction was extended to loxodromic automorphisms of surfaces by Silverman, Call, and Kawaguchi (see [Sil91, CS93, Kaw08]): in this case one obtains a pair of canonical heights $\hat{h}^{\pm}_f$ satisfying $\hat{h}^{\pm}_f \circ f^{\pm 1} = \lambda(f)^{\pm}\hat{h}^{\pm}_f$. (Here $\hat{h}^+_f$ and $\hat{h}^-_f$ are Weil heights associated to $\mathbf{R}$-divisors.)

If Γ is an infinite subgroup of $\mathsf{Aut}(X)$, also defined over $\mathbf{k}$, it is natural to ask whether a Γ-equivariant family of heights can be constructed. Specifically, one looks for a family of representatives $\hat{h}_L$ of the Weil height functions, i.e. $\hat{h}_L = h_L + O(1)$ for every L in $\mathrm{Pic}(X;\mathbf{R}) := \mathrm{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{R}$, depending linearly on L, and satisfying the exact relation

$$\hat{h}_L(f(x)) = \hat{h}_{f^*L}(x) \quad (\forall x \in X(\overline{\mathbf{Q}})) \tag{1.3}$$

for every pair $(f, L) \in \Gamma \times \mathrm{Pic}(X;\mathbf{R})$ (see § 8 for details). A prototypical example is given by the Néron–Tate height, when Γ is the group of automorphisms of an abelian surface preserving the origin. Such objects were named *canonical vector heights*[1] by Baragar in [Bar03]. He proved their existence when X is a K3 surface with Picard number 2, in which case $\mathsf{Aut}(X)$ is virtually cyclic. He also gave evidence for their non-existence on certain Wehler surfaces (see [BvL07]). In [Kaw13] Kawaguchi obtained a complete proof of this non-existence for an explicit family of Wehler surfaces; his argument relies on the study of Γ-periodic orbits. Extending Kawaguchi's methods and using Theorem B, we completely solve this existence problem for groups with parabolic elements: *let X be a smooth projective surface and Γ be a non-elementary subgroup of* $\mathsf{Aut}(X)$ *containing parabolic elements, both defined over a number field* $\mathbf{k}$*; if* (X, Γ) *possesses a canonical vector height, then X is an abelian surface and Γ has a finite orbit* (see Theorem E in § 8). The last assertion implies that, after conjugation by a translation, a finite index subgroup of Γ preserves the neutral element of the abelian surface X, in particular the Néron–Tate height provides a canonical vector height; we explain how all possible canonical vector heights are derived from the Néron–Tate height (see § 8.4).

1.5 Stationary measures

Another application, which was our primary source of motivation for this paper, concerns the classification of invariant and stationary measures. For simplicity, we suppose in this section, as

[1] The name 'vector height' comes from the following viewpoint. Assume $\mathrm{Pic}(X;\mathbf{R}) = \mathrm{NS}(X;\mathbf{R})$, and fix a basis L_i of $\mathrm{Pic}(X;\mathbf{R})$. For x in $X(\overline{\mathbf{Q}})$, consider the vector $(h_{L_i}(x)) \in \mathbf{R}^\rho$, where $\rho = \dim_{\mathbf{R}} \mathrm{Pic}(X;\mathbf{R})$. Property (1.3) can be phrased in terms of these vectors, hence, the terminology.

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in Theorem C, that X is not abelian, Γ contains both loxodromic and parabolic elements, and Γ does not preserve any algebraic curve.

With the classification and finiteness theorems for invariant Zariski-diffuse measures obtained in [CD23b], Theorem C implies that the set of Γ-invariant probability measures is a finite dimensional simplex, whose extremal points are given by:

- uniform counting measures on the finite orbits of Γ;
- measures μ for which there is a Γ-invariant totally real (possibly singular) surface Σ such that $\mathrm{Supp}(\mu) = \Sigma$ and μ is absolutely continuous with respect to any area form on Σ;
- measures μ such that $\mathrm{Supp}(\mu) = X$ and μ is absolutely continuous with respect to any volume form on X.

Note that some or all of these categories may be empty.

Now assume furthermore that X is a K3 or Enriques surface, that X and Γ are defined over $\mathbf{R}$, and consider the restriction of the action of Γ on the real part $X(\mathbf{R})$ of X, which we assume to be non-empty. Fix a probability measure ν on Γ, whose support is a finite set generating Γ. Then, applying the results of [CD23c], this yields a classification and finiteness theorem for ν-stationary measures on the real locus: *the only ν-stationary measures on $X(\mathbf{R})$ are convex combinations of the natural area forms of the components of $X(\mathbf{R})$, together with finitely many finite orbits.*

1.6 Related recent results

Let us discuss a few related works which appeared after this work (and [CD23c]) were released.

Theorem A plays an important role in our more recent work [CD22], as it leads to global hyperbolicity properties of the dynamics on an open and dense set of Wehler surfaces, for the euclidean topology.

In [FT23], Filip and Tosatti use a similar blend of dynamics, analysis, and diophantine geometry to construct canonical currents and heights on some K3 surfaces X; as for generic Wehler surfaces, they assume that $\mathsf{Aut}(X)$ induces a lattice in the group of isometries of $\mathrm{NS}(X;\mathbf{Z})$ and that the genus 1 fibrations on X have no reducible fiber.

A recent paper of Corvaja, Tsimerman, and Zannier [PT23] studies 'finite orbits' of the group generated by a pair (g, h) of parabolic automorphisms, with completely different methods. Their results are stronger than ours in some respects, since they conclude that there are only finitely many points x for which $\{g^n h^m(x); n, m \in \mathbf{Z}\}$ is finite; thus, only a small part of the $\langle g, h\rangle$-orbit is assumed to be finite. But their assumptions are stronger (for instance, they exclude isotrivial fibrations); in particular, their technique does not lead to a complete classification of the situations where finite orbits are Zariski dense, as in Theorem B. A hope would be to combine our results with [PT23] to classify all pairs (g, h) of parabolic automorphisms with a Zariski-dense set of common periodic points.

Finite $\mathsf{Aut}(X)$-orbits on Wehler surfaces (over arbitrary fields) are studied by Fuchs, Litman, Silverman, and Tran in [FLST23]. They study the structure of orbits over prime fields, and then construct complex examples with finite orbits of size as large as 288.

1.7 Organization of the paper

We prove Theorem A in § 2, which is independent of the rest of the paper. Section 3 studies invariant curves for loxodromic automorphisms and non-elementary groups; we obtain an effective bound for the degree of a curve invariant under a loxodromic automorphism (see Proposition 3.7) and prove Theorem D. In § 4 we briefly discuss the case of tori and review the Kummer construction. The core of the paper is § 5, in which we develop the arithmetic method outlined above and establish Theorem B. Section 6 is devoted to consequences of Theorem B, and

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related comments. We prove Theorem C in § 7. Finally, canonical vector heights are discussed in § 8, where we solve Baragar's problem. Some open problems and possible extensions of our results are discussed in §§ 6.3 and 7.4.

2. Very general Wehler surfaces

Consider the family of Wehler surfaces described in § 3 of [CD23c] and [Bar04, Can01a, Kaw13, McM02]. In this section we prove Theorem A. For convenience, let us recall the statement of the theorem.

Theorem 2.1. *If $X \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is a very general Wehler surface, then $\mathsf{Aut}(X)$ does not preserve any non-empty, proper, and Zariski-closed subset of X.*

Here, *very general* means that this property holds in the complement of a set of countably many hypersurfaces in the space of surfaces of degree $(2,2,2)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. The proof follows from an elementary but tedious parameter counting argument. As shown in § 2.5, such a statement fails if $\mathsf{Aut}(X)$ is replaced by a thin non-elementary subgroup.

2.1 Notation and preliminaries

We use the notation of [CD23c, § 3]: $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, with affine coordinates (x, y, z) (denoted (x_1, x_2, x_3) in [CD23c]), π_1, π_2, and π_3 are the projections on the first, second, and third factors, and π_{ij} is the projection (π_i, π_j) onto $\mathbb{P}^1 \times \mathbb{P}^1$. Then $L_i = \pi_i^*(\mathcal{O}(1))$, $L = L_1^2 \otimes L_2^2 \otimes L_3^2$, and $X \subset M$ is a member of the linear system $|L|$. In the affine coordinates (x, y, z), X is defined by a polynomial equation of degree $(2,2,2)$, which we write

$$P(x,y,z) = A_{222}x^2y^2z^2 + A_{221}x^2y^2z + \cdots + A_{100}x + A_{010}y + A_{001}z + A_{000}. \tag{2.1}$$

Thus, $H^0(M, L)$ is of dimension 27 and since the equation $\{P = 0\}$ is defined up to multiplication by a complex scalar, the family of Wehler surfaces X is 26-dimensional. Modulo the action of $G = \mathsf{PGL}(2, \mathbf{C})^3$ they form an irreducible family of dimension 17.

It was shown in [CD23c, § 3] that there exists a Zariski-open set $W_0 \subset |L|$ of surfaces $X \in |L|$ such that:

(i) X is a smooth K3 surface;
(ii) each of the three projections $(\pi_{ij})_X : X \to \mathbb{P}^1 \times \mathbb{P}^1$ is a finite map, that is, X does not contain any fiber of $\pi_{ij} : M \to \mathbb{P}^1 \times \mathbb{P}^1$.

From now on, we suppose that X belongs to W_0. Let i, j, and k be three indices with $\{i, j, k\} = \{1, 2, 3\}$. Denote by $\sigma_i : X \to X$ the involution of X that permutes the points in the fibers of the 2-to-1 branched covering $(\pi_{jk})_X : X \to \mathbb{P}^1 \times \mathbb{P}^1$. By [CD23c, § 3], the σ_i generate a non-elementary subgroup of $\mathsf{Aut}(X)$. This subgroup is isomorphic to $\mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/2\mathbf{Z}$, it preserves the subspace of $\mathrm{NS}(X; \mathbf{Z})$ generated by the Chern classes of the L_i, and its action on this subspace is given by explicit matrices. Then $f_{ij} = \sigma_i \circ \sigma_j$ is a parabolic automorphism of X, preserving the genus 1 fibration $\pi_k : X \to \mathbb{P}^1$. Moreover, if X is very general the L_i generate $\mathrm{NS}(X; \mathbf{Z})$.

2.2 Invariant curves

Proposition 2.2. *If $X \in W_0$, $\mathsf{Aut}(X)$ does not preserve any algebraic curve.*

This follows from the previous paragraph and the following more precise result.

Lemma 2.3. *Let X be a smooth Wehler surface. Assume that the three involutions σ_i induce a faithful action of the group $\mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/2\mathbf{Z}$. Then the group generated by the σ_i does not preserve any curve.*

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Proof. Assume that C is an invariant curve. Since no curve can be contained simultaneously in fibers of π_1, π_2, and π_3, without loss of generality, we may suppose that $\pi_1\colon C \to \mathbb{P}^1(\mathbf{C})$ is dominant. Then the automorphism $f_{23} = \sigma_2 \circ \sigma_3$ has finite order: indeed, on a general fiber F of π_1, it acts as a translation that preserves the non-empty finite set $F \cap C$. This contradicts the fact that f_{23} is parabolic and finishes the proof. □

Thus, to prove Theorem 2.1, we are left to prove the non-existence of periodic orbits, which is the purpose of the following paragraphs.

2.3 Elliptic curves

Here we study $(2, 2)$ curves in dimension 2. We keep notation as in § 2.1. Let us consider the line bundles $L_i = \pi_i^*(\mathcal{O}(1))$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and set $L = L_1^2 \otimes L_2^2$. Fix (affine) coordinates (x, y) on $\mathbb{P}^1 \times \mathbb{P}^1$, with x and y in $\mathbf{C} \cup \{\infty\}$. A curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ in the linear system $|L|$ is given by an equation of degree $(2, 2)$ in (x, y). Assume that C contains the points $(0, 0)$, $(\infty, 0)$, and $(0, \infty)$ and that it is smooth at the origin, with a tangent line given by $x + y = 0$. Then its equation reduces to the form

$$\alpha x^2y^2 + \beta x^2y + \gamma xy^2 + \delta xy + \varepsilon(x + y) = 0 \tag{2.2}$$

for some complex numbers α, β, γ, δ, and ε, with $\varepsilon \neq 0$. Denote this curve by $C_{(\alpha,\beta,\gamma,\delta,\varepsilon)}$. For a general choice of these parameters, C is a smooth curve of genus 1. We will need the following more precise result.

Lemma 2.4. *Fix $(\beta, \gamma, \delta, \varepsilon)$ with $\varepsilon \neq 0$. Then for general α, $C_{(\alpha,\beta,\gamma,\delta,\varepsilon)}$ is smooth.*

Proof. An explicit calculation shows that the points of C on $\{\infty\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{\infty\}$ are smooth unless $\alpha = \beta = \gamma = 0$. Thus, for $\alpha \neq 0$, C has no singular point at infinity. Now, viewing (2.2) as a quadratic equation in x depending on the variable y, we can consider its discriminant $\Delta_x = \Delta_x(y)$; it is a polynomial of degree 4 in y that detects fibers $\mathbf{C} \times \{y\}$ intersecting C at a single point (for those values y for which $C \cap \mathbf{C} \times \{y\}$ is contained in $\mathbf{C}^2$, that is, the polynomial in x is of degree 2). It is easy to check that if (x, y) is a singular point of C, then y must be a multiple root of Δ_x. Hence, if $y \mapsto \Delta_x(y)$ only has simple roots, C is smooth in $\mathbf{C}^2$. Thus, it is enough to check that if $(\beta, \gamma, \delta, \varepsilon)$ is an arbitrary 4-tuple such that $\varepsilon \neq 0$, Δ_x has only simple roots for general α. But $\Delta_x(y) = ay^4 + by^3 + cy^2 + \mathrm{d}y + e$, where only b depends on α, with $b(\alpha) = 2\gamma\delta - 4\alpha\varepsilon$, and $e = \varepsilon^2 \neq 0$. Now the discriminant of Δ_x, as a degree 4 polynomial in y, is a polynomial expression in (a, b, c, d, e), and as a polynomial in b it has a unique leading term $27b^4e^2$. Thus, $(\beta, \gamma, \delta, \varepsilon)$ being fixed, with $\varepsilon \neq 0$, this discriminant depends non-trivially on α; for a general α, this discriminant is not zero, thus Δ_x has four distinct roots, so that C is smooth, as was to be proved. □

There are two involutions σ_1 and σ_2 on C, respectively, permuting the points in the fibers of the projections $(\pi_2)_{|C}\colon C \to \mathbb{P}^1$ and $(\pi_1)_{|C}\colon C \to \mathbb{P}^1$; that is, σ_i changes the ith coordinate, while keeping the others unchanged. The composition $f = \sigma_1 \circ \sigma_2$ is a translation on C mapping $(0, \infty)$ to $(\infty, 0)$; in particular, f is not the identity.

Lemma 2.5. *Fix $(\beta, \gamma, \delta, \varepsilon)$ with $\varepsilon \neq 0$ and assume that the curve $C_{(0,\beta,\gamma,\delta,\varepsilon)}$ is smooth. Then the dynamics of the translation f on $C_{(\alpha,\beta,\gamma,\delta,\varepsilon)}$ varies non-trivially with α: it is periodic for a countable dense set of α's, and non-periodic for the other parameters.*

Proof. For α in the complement of a finite set, $C_\alpha := C_{(\alpha,\beta,\gamma,\delta,\varepsilon)}$ is a smooth curve of genus 1, and f acts as a translation on C_α. Let us analyze the orbit of $(0, \infty)$. Denote by $u, v \in \mathbf{C} \cup \{\infty\}$ the complex numbers such that $\sigma_2(\infty, 0) = (\infty, v)$ and $(u, v) = \sigma_1(\infty, v) = f^2(0, \infty)$. The translation

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f is periodic of period 2 if and only if $(u, v) = (0, \infty)$, if and only if (∞, ∞) is a point of C_α, if and only if $\alpha = 0$. Hence, f is periodic of period 2 on C_0 but after perturbation it is not of period 2 anymore. For small α we can write $C = \mathbf{C}/\Lambda_\alpha$ for some lattice $\Lambda_\alpha = \mathbf{Z} + \mathbf{Z}\tau(C_\alpha)$ and $f(z) = z + t(C_\alpha)$, with $t(C_\alpha)$ and $\tau(C_\alpha)$ depending holomorphically on the parameters α. If we further decompose $t(C_\alpha) = a(C_\alpha) + b(C_\alpha)\tau(C_\alpha)$, where a and b are two real analytic functions with values in $\mathbf{R}$, then both a and b must be non-constant. Indeed if one of them were constant, then the other would be a non-constant real holomorphic function, which is impossible (see [Can01b, Proposition 2.2] for a similar argument). The result follows. □

2.4 Proof of Theorem 2.1

2.4.1 *From finite orbits to fixed points.* Let us form the universal family $\mathcal{X} \subset W_0 \times M$, where $W_0 \subset |L|$ is the open set defined in § 2.1: the fiber of the projection $\mathcal{X} \to W_0$ above $X \in W_0$ is precisely the surface $X \subset M$. The group $\mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/2\mathbf{Z}$ acts by automorphisms on $\mathcal{X}$, preserving each fiber of $\mathcal{X} \to W_0$: the generators of the first, second, and third $\mathbf{Z}/2\mathbf{Z}$ factors give rise to involutions $\widetilde{\sigma}_1$, $\widetilde{\sigma}_2$, and $\widetilde{\sigma}_3$ which, when restricted to a fiber X, correspond to the automorphisms $\sigma_i \in \mathsf{Aut}(X)$. These involutions $\widetilde{\sigma}_i$ extend to birational involutions of the Zariski closure $\overline{\mathcal{X}} \subset |L| \times M$.

Remark 2.6. If $X \in |L|$ is smooth and contains a fiber $V = \{(x_0, y_0)\} \times \mathbb{P}^1 \subset X$ of π_{12}, the curve V is contained in the indeterminacy locus of $\tilde{\sigma}_3$ (one may consult [CO15] for further results: see Theorem 3.3 and the proof of its third and fourth assertions).

Consider the group $\mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/2\mathbf{Z}$ acting on $\mathcal{X}$. Its restriction to the fiber X gives a subgroup Γ of $\mathsf{Aut}(X)$. Let d be a positive integer. There are only finitely many homomorphisms from $\mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/2\mathbf{Z}$ to groups of order $\leq d!$, and the intersection of the kernels of these homomorphisms is a normal subgroup of finite index. Denote by Γ_d the corresponding subgroup of $\mathsf{Aut}(X)$. If Γ has an orbit of cardinality $\leq d$ on some surface X, then this orbit is fixed pointwise by Γ_d. Let us introduce the subvariety

$$\mathcal{Z}_d = \{(X, x);\ x \in X \text{ and } \forall f \in \Gamma_d,\ f(x) = x\} \subset \mathcal{X}. \tag{2.3}$$

Since $\mathcal{X} \to W_0$ is proper, from this discussion we obtain the following.

Lemma 2.7. *The following properties are equivalent:*

(1) *for a very general surface $X \in |L|$, every orbit of Γ in X is infinite;*
(2) *for every $d \geq 1$, the projection $\mathcal{Z}_d \to W_0$ is not surjective;*

2.4.2 *Preparation.* According to Lemma 2.7, to prove Theorem 2.1 it suffices to show that the projection of $\mathcal{Z}_d \subset \mathcal{X}$ onto W_0 is a proper subset for every $d \geq 1$. Thus, let us assume that there is an integer d for which $\mathcal{Z}_d$ surjects onto W_0 and seek for a contradiction. Pick a small open subset $U \subset W_0$ for the Euclidean topology, over which one can choose a holomorphic section $s\colon X \mapsto s_X$ of $\mathcal{X} \to W_0$ such that s_X is fixed by Γ_d; equivalently, the image of s is contained in $\mathcal{Z}_d$.

The group $G = \mathsf{PGL}_2(\mathbf{C}) \times \mathsf{PGL}_2(\mathbf{C}) \times \mathsf{PGL}_2(\mathbf{C})$ acts on M and on $|L|$, preserving W_0. Recall that modulo the action of this group, the space of Wehler surfaces is irreducible and of dimension 17.

2.4.3 *Case 1.* Let us first assume that we can find U such that s_X is not fixed by $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, nor $\tilde{\sigma}_3$. As in Lemma 2.4 this implies that for each pair of indices $i \neq j$, the fiber C of $(\pi_i)_X\colon X \to \mathbb{P}^1$ through s_X is smooth near s_X and $s_X \in C$ is not a ramification point of the

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projection $(\pi_j)_{|C}\colon C\to\mathbb{P}^1$. As in § 2.1, fix coordinates (x,y,z) on $M=(\mathbb{P}^1)^3$ with x, y, and z in $\mathbf{C}\cup\{\infty\}$. Modulo the action of G, we may assume that for every X in U:

(a) the point s_X is the point $(0,0,0)$ in $(\mathbb{P}^1)^3$;
(b) X contains $(\infty,0,0)$, $(0,\infty,0)$, and $(0,0,\infty)$;
(c) the tangent plane to X at the origin is given by the equation $x+y+z=0$;
(d) the coefficients of $x^2y^2z^2$ and x, y and z in the equation of X are all equal to the same complex number.

Note that assumption (a) can be achieved by a single translation, assumption (b) can be obtained by transformations of the form $(x,y,z)\mapsto(x/(x-\alpha),y/(y-\beta),z/(z-\gamma))$, assumption (c) is achieved by the action of diagonal maps (note that by our assumption, the tangent plane to X at the origin $s_X=(0,0,0)$ cannot be one of the coordinate planes), and then we obtain assumption (d) by the action of homotheties. After such a conjugation, the equation of X is of the form

$$\begin{aligned}
&Ax^2y^2z^2+Bx^2y^2z+B'x^2yz^2+B''xy^2z^2+Cx^2yz+C'xy^2z+C''xyz^2\\
&\quad+Dx^2y^2+D'x^2z^2+D''y^2z^2+Exyz\\
&\quad+Fx^2y+F'x^2z+F''xy^2+F'''y^2z+F^{iv}xz^2+F^{v}yz^2\\
&\quad+Gxy+G'xz+G''yz+A(x+y+z)=0.
\end{aligned}\tag{2.4}$$

Since this equation is defined up to multiplication by an element of $\mathbf{C}^*$, we are left with 19 parameters. The automorphism $f_{12}=\sigma_1\circ\sigma_2$ preserves the genus 1 fibration $(\pi_3)_{|X}\colon X\to\mathbb{P}^1$. The fiber of $(\pi_3)_{|X}$ through $(0,0,0)$ is a curve $C\subset\mathbb{P}^1\times\mathbb{P}^1$ given by the equation

$$Dx^2y^2+Fx^2y+F''xy^2+Gxy+A(x+y)=0.\tag{2.5}$$

Two cases need to be considered, depending on the smoothness of this curve.

– If this curve is singular, by Lemma 2.4 the coefficients in (2.5) satisfy a non-trivial relation of the form $P_3(D,F,F'',G,A)=0$.
– If it is smooth, consider an iterate f_{12}^m of f_{12} in Γ_d, with $1\le m\le d!$; then f_{12}^m is a translation of the genus 1 curve C that fixes s_X, so that it fixes C pointwise. From Lemma 2.5, the coefficients in (2.5) satisfy a relation of the form $Q_3(D,F,F'',G,A)=0$.

In both cases we get a relation of the form $R_3(D,F,F'',G,A)=0$ (with $R_3=P_3$ or Q_3) that depends non-trivially on the first factor. Similarly, looking at the dynamics of $f_{23}=\sigma_2\circ\sigma_3$ and $f_{31}=\sigma_3\circ\sigma_1$, we obtain two further relations of the form $R_1(D'',F'',F^v,G'',A)=0$ and $R_2(D',F',F^{iv},G',A)=0$.

We claim that the subset defined by these 3 constraints is of codimension 3: indeed, if we look at the subvariety cut out by the equations $R_i=0$, $i=1,2,3$ and slice it by a 3-plane corresponding to the coordinates D, D', and D'', then by Lemmas 2.4 and 2.5 and the independence of variables, this slice is reduced to a point. This shows that the image of the section $X\mapsto s_X$ has dimension $\le$16, which contradicts the fact that W_0/G is of pure dimension 17. Thus, our hypothesis on $\mathcal{Z}_d$ cannot be true and Case 1 does not hold.

2.4.4 *Case 2.* If Case 1 does not hold, every point $(X,(x,y,z))$ of $\mathcal{Z}_d$ has the property: *$(x,y,z)\in X$ is a ramification point for at least one of the three projections $(\pi_i)_{|X}$.* Equivalently, every point of the finite orbit $F=\Gamma_d(s_X)\subset X$ is fixed by at least one of the three involutions σ_i. This case is simpler, since a direct count of parameters will lead to a contradiction.

• If a point of F were a ramification point of each $(\pi_i)_{|X}$, this point would be a singularity of X, and X would not be in W_0. Thus, each point of F is a ramification point for at least one and at most 2 of the projections.

• Now, assume that *every point of F is a ramification point for exactly 2 of the projections.* Choose a local section s_X of $\mathcal{Z}_d$ above a small open set $U \subset W_0$ (for the Euclidean topology), as in § 2.4.3. Permuting the coordinates and using a translation in G, we assume that $s_X = (0,0,0)$ and s_X is fixed by σ_2 and σ_3. After this normalization, with notation as in (2.1), we have $A_{010} = A_{001} = A_{000} = 0$. Let $s'_X = \sigma_1(s_X)$; this point is not equal to s_X because otherwise X would be singular at s_X. Thus, we may use a transformation of the form $x \mapsto x/(x-\alpha)$ in G to assume that $s'_X = (\infty, 0, 0)$ (i.e. $A_{200} = 0$). Now by our assumption, this second point must be fixed by σ_2 and σ_3, which imposes two more constraints ($A_{201} = A_{210} = 0$). Now, consider the curve $C_1 \subset X$ defined by the equation $x = 0$. Using elements of G acting on y and z by $y \mapsto y/(y-\beta)$ and $z \mapsto z/(z-\gamma)$, we may assume that $(0,\infty,\infty)$ is on C_1 and is a ramification point for $(\pi_2)_{|C_1}$. With such a choice, the coefficients of y^2z^2 and y^2z vanish. At this stage we did not use the diagonal action of $(\mathbf{C}^*)^3$, which stabilizes $(0,0,0)$, $(\infty,0,0)$, and $(0,\infty,\infty)$. With this we can impose for instance the same non-zero coefficients for the terms xy, yz, and zx, so we end up with 17 coefficients, hence at most 16 free parameters. Again this contradicts the fact that $\dim(W_0) = 17$.

• Now, assume that *one of the points of the finite orbit F is fixed by σ_3 but not by σ_1 and σ_2.* The analysis is similar to that of the previous case. We may choose this point to be s_X, and using the group G, we can arrange that $s_X = (0,0,0)$, $\sigma_1(s_X) = (\infty,0,0)$, and $\sigma_2(s_X) = (0,\infty,0)$; with the notation from (2.1), this means $A_{000} = A_{200} = A_{020} = 0$. In addition $A_{001} = 0$ because $(0,0,0)$ is fixed by σ_3. By our hypothesis, $(\infty,0,0)$ is fixed by σ_2 or σ_3 (or both). This implies that at least one of A_{210} or A_{201} vanishes. Likewise $A_{120}A_{021} = 0$. Now consider the curve $C_2 \subset X$ given by $y = 0$. Given the constraints already listed, the equation of C_2 can be written as

$$\alpha x^2 z^2 + \beta x^2 z + \gamma x z^2 + \delta x z + \varepsilon z^2 + \iota x = 0. \tag{2.6}$$

There are 4 ramification points for $(\pi_1)_{|C_2}$, counting with multiplicities, and none of them satisfies $z = 0$. Thus, using $z \mapsto z/(z-\gamma)$ and $x \to \lambda x$ we may put one of them at $(1,0,\infty)$. This imposes $\alpha + \gamma + \varepsilon = 0$ and $\beta + \delta = 0$.

Finally, we may still use the subgroup $\{\mathrm{Id}\} \times \mathbf{C}^* \times \mathbf{C}^* \subset G$, which fixes the four points $(0,0,0)$, $(\infty,0,0)$, $(0,\infty,0)$, and $(1,0,\infty)$, to assume that the non-zero coefficients in front of yz, xz, and z^2 are equal. In conclusion, under our assumption we have found at least 10 independent linear constraints on the coefficients of the Wehler surface so again at most 16 free parameters remain.

Thus, in all cases we get a contradiction, and the proof of Theorem 2.1 is complete.

2.5 Examples

2.5.1 Consider the subgroup H of $\mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/2\mathbf{Z}$ generated by f_{23}^m and f_{31}^m, for some large positive integer m (as above, $f_{23} = \sigma_2 \circ \sigma_3$, $f_{31} = \sigma_3 \circ \sigma_1$). The automorphism f_{23} preserves the fibers of the projection $(\pi_1)_{|X}$ and its periodic points form a dense set of fibers (see [Can01b, CD23b] or § 3.1.1 here). The intersection number between a fiber of $(\pi_1)_{|X}$ and a fiber of $(\pi_2)_{|X}$ is equal to 2. Thus, if m is big enough, f_{23}^m and f_{31}^m share a common fixed point (in fact, $\simeq cm^4$ common fixed points, for some $c > 0$ as m goes to $+\infty$). If $X \in W_0$, $\langle f_{23}^m, f_{31}^m \rangle$ is non-elementary because the class $c_1 \in \mathrm{NS}(X;\mathbf{Z})$ of the invariant fibration of f_{23} is not fixed by f_{31}, and *vice versa* (see also Lemma 3.13). Taking a surface $X \in W_0$ that is defined over $\mathbf{Q}$, we get, in particular, the following.

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Proposition 2.8. *For every integer $N \geq 0$, there is a smooth Wehler surface X defined over $\mathbf{Q}$ and a non-elementary subgroup Γ of $\mathsf{Aut}(X_{\mathbf{Q}})$ with at least N fixed points.*

Remark 2.9. If $X \in W_0$ and $m \geq 1$, *the group $\langle f_{23}^m, f_{31}^m \rangle$ has infinite index in* $\mathsf{Aut}(X)$. Indeed, the index of $\langle (\sigma_2 \circ \sigma_3)^m, (\sigma_3 \circ \sigma_1)^m \rangle$ in $\mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/2\mathbf{Z}$ is infinite.

2.5.2 Let us construct smooth Wehler surfaces for which the subgroup Γ of $\mathsf{Aut}(X)$ generated by the three involutions σ_i has a finite orbit. Using affine coordinates (x, y, z) for $(\mathbb{P}^1)^3$, set $V = \{(\varepsilon_1, \varepsilon_2, \varepsilon_3);\ \varepsilon_i \in \{0, \infty\}\ \forall i = 1, 2, 3\}$. Observe that if V is contained in a Wehler surface X, then V is an orbit of Γ of size 8. Now, writing the equation of X as in (2.1), $V \subset X$ if and only if $A_{ijk} = 0$ for all triples $(i, j, k) \in \{0, 2\}^3$. The corresponding linear system has no base points, except from the points of V. If $(A_{100}, A_{010}, A_{001}) \neq (0, 0, 0)$, the surface is smooth at the base point $(0, 0, 0)$. The smoothness at the other 7 points of V is determined by similar constraints on the coefficients A_{ijk}: for instance, the smoothness at the point $(\infty, 0, 0)$ is equivalent to $(A_{210}, A_{201}, A_{100}) \neq (0, 0, 0)$. Thus, the theorem of Bertini shows that a general member of the family of Wehler surfaces containing V is smooth, and the result follows.

3. Non-elementary groups and invariant curves

The main purpose of this section is to establish the following.

Theorem D. *Let X be a compact Kähler surface and let Γ be a subgroup of $\mathsf{Aut}(X)$ containing a loxodromic element. Then there exists a loxodromic element f in Γ such that every f-periodic curve is Γ-periodic.*

Along the way, some results of independent interest will be obtained: Proposition 3.7, which will be used in § 7, gives an effective bound for the degree of a periodic curve under a loxodromic automorphism; Proposition 3.9 provides a singular model of (X, Γ) without Γ-periodic curves, and discusses ampleness properties of some line bundles. This model will be crucial for the study of the dynamical heights in § 5.

3.1 Preliminaries

Let X be a compact Kähler surface. By the Hodge index theorem, the intersection form $\langle \cdot | \cdot \rangle$ is non-degenerate and of signature $(1, h^{1,1}(X) - 1)$ on $H^{1,1}(X; \mathbf{R})$; its isotropic cone is the set of vectors $u \in H^{1,1}(X; \mathbf{R})$ with $\langle u | u \rangle = 0$. Fix a Kähler form κ_0 on X, with $\int_X \kappa_0 \wedge \kappa_0 = 1$, denote its class by $[\kappa_0]$, and define the positive cone in $H^{1,1}(X; \mathbf{R})$ to be the set

$$\mathrm{Pos}(X) = \{u \in H^{1,1}(X; \mathbf{R});\ \langle u | u \rangle > 0 \text{ and } \langle [\kappa_0] | u \rangle > 0\}. \tag{3.1}$$

Equivalently, $\mathrm{Pos}(X)$ is the connected component of $\{u \in H^{1,1}(X; \mathbf{R});\ \langle u | u \rangle > 0\}$ containing Kähler forms; in particular, its definition does not depend on κ_0. This cone $\mathrm{Pos}(X)$ contains one of the two connected components, denoted $\mathbb{H}_X$, of the hyperboloid $\{u \in H^{1,1}(X; \mathbf{R});\ \langle u | u \rangle = 1\}$; we can identify $\mathbb{H}_X$ with its projection $\mathbb{P}(\mathbb{H}_X)$ in the projective space $\mathbb{P}(H^{1,1}(X; \mathbf{R}))$, and in doing so we get $\mathbb{H}_X \simeq \mathbb{P}(\mathbb{H}_X) = \mathbb{P}(\mathrm{Pos}(X))$. Via this identification, the Hilbert metric on $\mathbb{H}_X$ coincides with the hyperbolic metric induced by the intersection form (see [CD23c, § 2]), and the boundary $\partial \mathbb{H}_X$ is identified to the projection of the isotropic cone in $\mathbb{P}(H^{1,1}(X; \mathbf{R}))$.

Forgetting about torsion, $H^2(X; \mathbf{Z}) \cap H^{1,1}(X; \mathbf{R})$ can be identified with the Néron–Severi group $\mathrm{NS}(X; \mathbf{Z})$; thus, we shall consider $\mathrm{NS}(X; \mathbf{R})$ as a subspace of $H^{1,1}(X; \mathbf{R})$.

An automorphism of X has a *type* (elliptic, parabolic, or loxodromic) according to the type of its induced action on $\mathbb{H}_X$. Given a subgroup $\Gamma \leq \mathsf{Aut}(X)$, we denote by Γ_{par} (respectively, Γ_{lox}) the set of parabolic (respectively, loxodromic) automorphisms in Γ.

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3.1.1 *Parabolic automorphisms.* If g is parabolic, it permutes the fibers of a genus-1 fibration $\pi_g\colon X \to B_g$, and induces an automorphism $\overline{g}$ of the curve B_g. The induced automorphism $\overline{g}$ has finite order, except maybe when X is a torus $\mathbf{C}^2/\Lambda$ (see [CF03, Proposition 3.6]).

If $\overline{g}$ is the identity, then g preserves each fiber of π_g, acting as a translation on each smooth fiber. If U_0 is a disk in B_g that does not contain any critical value of π_g, the universal cover of $\pi_g^{-1}(U_0)$ is holomorphically equivalent to $U_0 \times \mathbf{C}$, with its fundamental group $\mathbf{Z}^2$ acting by $(x, y) \in U_0 \times \mathbf{C} \mapsto (x, y + a + b\tau(x))$ for every $(a, b) \in \mathbf{Z}^2$, where $\tau\colon U_0 \to \mathbf{C}$ is a holomorphic function taking its values in the upper half plane. In these coordinates g lifts to a diffeomorphism $\tilde{g}(x, y) = (x, y + t(x))$ for some holomorphic function $t\colon U_0 \to \mathbf{C}$. The mth iterate g^m fixes pointwise a fiber $\{x\} \times \mathbf{C}/(\mathbf{Z} \oplus \mathbf{Z}\tau(x))$ if and only if $mt(x) \in \mathbf{Z} \oplus \mathbf{Z}\tau(x)$. The union of such fibers, for all $m \geq 1$, form a dense subset of X. This comes from the fact that 't varies independently from τ', a property which implies also that the differential of g^m at a fixed point is, except for finitely many fibers, a 2×2 upper triangular matrix with ones on the diagonal and a non-trivial lower left coefficient. We refer to [Can01b, Can14, CD23b], and to the proof of Theorem 5.15 for a slightly different viewpoint on this property, using real-analytic coordinates.

The induced action g^* on $H^{1,1}(X; \mathbf{R})$ admits a simple description: if F is any fiber of π_g, its class $[F] \in H^{1,1}(X; \mathbf{R})$ is fixed by g^*, the ray $\mathbf{R}_+[F]$ is contained in the isotropic cone, and $(1/n^2)(g^n)^*w$ converges towards a positive multiple of $[F]$ for every $w \in \mathrm{Pos}(X)$. In particular, the class $[F]$ is nef. Regarding the induced action on $\mathbb{H}_X$, $\mathbb{P}([F])$ is the unique fixed point of the parabolic map g^* on $\mathbb{H}_X \cup \partial\mathbb{H}_X$ (see [Can01b, Can14, CD23b]).

Recall that a linear endomorphism of a vector space is unipotent if all its eigenvalues $\alpha \in \mathbf{C}$ are equal to 1; it is *virtually unipotent* if some of is positive iterates is unipotent. Since the topological entropy of g vanishes, Lemma 2.6 of [CP21] gives the following.[2]

Lemma 3.1. *Let X be a complex projective surface. If $g \in \mathsf{Aut}(X)$ is parabolic, then g^* is virtually unipotent, both on* $\mathrm{NS}(X; \mathbf{R})$ *and on* $H^2(X; \mathbf{R})$.

3.1.2 *Loxodromic automorphisms.* The dynamics of a loxodromic automorphism f is much richer (see [Can14]). The isolated periodic points of f of period m equidistribute towards a probability measure μ_f as m goes to $+\infty$, the topological entropy of f is positive, and μ_f is the unique ergodic, f-invariant probability measure of maximal entropy.

We denote by $\lambda(f)$ the spectral radius of the induced automorphism f^* on $H^{1,1}(X)$, which is larger than 1. Then $\lambda(f)$ and $1/\lambda(f)$ are eigenvalues of f^* with multiplicity 1, with respective nef eigenvectors θ_f^+ and θ_f^- which are isotropic and generate an f^*-invariant plane $\Pi_f \subset H^{1,1}(X; \mathbf{R})$ (see Figure 1). Their projectivizations are the two fixed points on $\partial\mathbb{H}_X$ of the induced loxodromic isometry of $\mathbb{H}_X$. The remaining eigenvalues have modulus 1. We normalize the eigenvectors $\theta_f^{\pm}$ by imposing

$$\langle \theta_f^+ | [\kappa_0] \rangle = \langle \theta_f^- | [\kappa_0] \rangle = 1, \tag{3.2}$$

where κ_0 is the Kähler form introduced at the beginning of § 3.1 (recall that $\langle [\kappa_0] | [\kappa_0] \rangle = 1$). We set $m_f = \frac{1}{2}(\theta_f^+ + \theta_f^-)$. With such a choice, $\langle m_f | m_f \rangle = \frac{1}{2}\langle \theta_f^+ | \theta_f^- \rangle > 0$.

Remark 3.2. Denote by $\mathrm{Ang}_{\kappa_0}(\theta_f^+, \theta_f^-)$ the visual angle between the boundary points $\mathbb{P}(\theta_f^+)$ and $\mathbb{P}(\theta_f^-)$, as seen from $[\kappa_0]$ (or $\mathbb{P}([\kappa_0])$). Then

$$\langle m_f | m_f \rangle = \left(\sin\left(\tfrac{1}{2}\mathrm{Ang}_{\kappa_0}(\theta_f^+, \theta_f^-)\right)\right)^2 = \tfrac{1}{2}(1 - \cos(\mathrm{Ang}_{\kappa_0}(\theta_f^+, \theta_f^-))), \tag{3.3}$$

[2] This can also be proved directly. For instance, on $\mathrm{NS}(X; \mathbf{R})$ this follows from the fact that the intersection form is negative definite on $[F]^\perp/\mathbf{R}[F]$ and the lattice $\mathrm{NS}(X; \mathbf{Z})$ is g^*-invariant.

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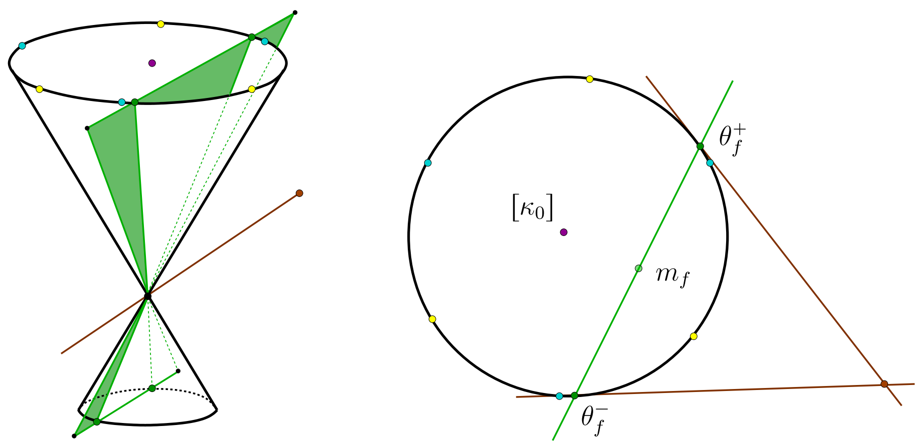

FIGURE 1. Left: a picture of $\mathrm{NS}(X;\mathbf{R})$ in case $\rho(X)=3$. The highlighted plane is Π_f, it intersects the isotropic cone along the lines $\mathbf{R}\theta_f^+$ and $\mathbf{R}\theta_f^-$; the line outside the cone is $\Pi_f^\perp$, the central point is $[\kappa_0]$. If f preserves a curve E, its class lies on $\Pi_f^\perp$. Right: a projective view of the same picture, where the two tangent lines are the projectivizations of the planes $(\theta_f^+)^\perp$ and $(\theta_f^-)^\perp$.

so, in particular, $0<\langle m_f|m_f\rangle\leq 1$, and the right-hand inequality is an equality if and only if $m_f=[\kappa_0]$.[3] In $\mathbb{H}_X$, the geodesic joining $\mathbb{P}(\theta_f^-)$ and $\mathbb{P}(\theta_f^+)$ is the curve $\mathsf{Ax}(f)$ parametrized by $s\theta_f^+ + t\theta_f^-$ with $s\in\mathbf{R}_+^*$ and $st=\langle\theta_f^+|\theta_f^-\rangle^{-1}$. The projection of $[\kappa_0]$ on $\mathsf{Ax}(f)$ is $(\sqrt{2}/\langle\theta_f^+|\theta_f^-\rangle^{1/2})m_f$ and, by [BC16, Lemma 6.3], we have

$$\cosh(\mathrm{d}_{\mathbb{H}}([\kappa_0],\mathsf{Ax}(f)))=\frac{\sqrt{2}}{\langle\theta_f^+|\theta_f^-\rangle^{1/2}}. \tag{3.4}$$

3.1.3 *Non-elementary subgroups of* $\mathsf{Aut}(X)$*.* See [CD23c, § 2.3] for more details about the results of this paragraph. By definition, a subgroup $\Gamma\subset\mathsf{Aut}(X)$ is non-elementary if it acts on $\mathbb{H}_X$ as a non-elementary group of isometries or, equivalently, if it contains a non-abelian free group, all of whose elements $f\neq\mathrm{id}$ are loxodromic. Such a group $\Gamma\subset\mathsf{Aut}(X)$ preserves a unique subspace $\Pi_\Gamma\subset H^{1,1}(X;\mathbf{R})$ on which: (i) Γ acts strongly irreducibly and (ii) the intersection form induces a Minkowski form. Moreover, $\Pi_\Gamma=\Pi_{\Gamma_0}$ for any finite index subgroup of Γ.

Each of the following conditions implies that Γ is non-elementary:

- Γ contains a pair of loxodromic elements (f,g) with $\{\theta_f^+,\theta_f^-\}\cap\{\theta_g^+,\theta_g^-\}=\emptyset$;
- Γ contains two parabolic elements associated to different fibrations; equivalently, Γ contains a parabolic and a loxodromic element.

If $\mathsf{Aut}(X)$ contains a non-elementary group Γ, then X is automatically projective and Π_Γ is contained in the Néron–Severi group $\mathrm{NS}(X;\mathbf{R})$ (see [CD23a]). If, in addition, Γ contains a parabolic element, then Π_Γ is defined over $\mathbf{Q}$ with respect to the lattice $\mathrm{NS}(X;\mathbf{Z})$ (see [CD23c, Lemma 2.9]).

The limit set $\mathrm{Lim}(\Gamma)\subset\partial\mathbb{H}_X$ is the closure of the set of fixed points of loxodromic elements in $\mathbb{P}(\Pi_\Gamma)$, or equivalently the smallest closed invariant subset in $\partial\mathbb{H}_X$. The following lemma is well known (see [Kap01, Lemma 3.24]).

[3] This can be obtained from elementary Euclidean geometry in the hyperplane $\langle\cdot|[\kappa_0]\rangle=1$ by fixing coordinates in which the quadratic form associated to the intersection product becomes $x_0^2-x_1^2-\cdots-x_n^2$ and $[\kappa_0]=(1,0,\ldots 0)$.

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Lemma 3.3. *If* Γ *is non-elementary,* $\{(\mathbb{P}(\theta_f^+), \mathbb{P}(\theta_f^-));\ f \in \Gamma_{\mathrm{lox}}\}$ *is dense in* $\mathrm{Lim}(\Gamma)^2$.

3.2 Periodic curves of loxodromic automorphisms

Fix a loxodromic automorphism $f\colon X \to X$. Our purpose in this paragraph is to bound the degree of the periodic curves of f.

Lemma 3.4. *Let* e *be an element of* $H^{1,1}(X;\mathbf{R})$ *such that* e *is orthogonal to* m_f *and* $\langle [\kappa_0] | e\rangle = 1$. *Then* $\langle e|e\rangle < 0$ *and*

$$-\langle e|e\rangle \geq \frac{\langle m_f | m_f\rangle}{1-\langle m_f|m_f\rangle} = \left(\tan\left(\frac{1}{2}\mathrm{Ang}_{\kappa_0}(\theta_f^+, \theta_f^-)\right)\right)^2.$$

Note that under the assumption of the lemma m_f cannot be equal to $[\kappa_0]$, so $0 < \langle m_f|m_f\rangle < 1$ by Remark 3.2.

Proof. Write $m_f = [\kappa_0] + v$ and $e = [\kappa_0] + w$ where v and w are in the orthogonal complement $[\kappa_0]^\perp$. Then, $\langle e|m_f\rangle = 0$, so $\langle v|w\rangle = -1$, and the Cauchy–Schwarz inequality gives $1 \leq (-\langle v|v\rangle)(-\langle w|w\rangle)$ because the intersection form is negative definite on $[\kappa_0]^\perp$. This inequality is equivalent to $1 \leq (1-\langle m_f|m_f\rangle)(1-\langle e|e\rangle)$ and the result follows. □

If $C \subset X$ is a curve, define its *degree* (with respect to κ_0) to be

$$\deg(C) = \int_C \kappa_0 = \langle [C] | [\kappa_0]\rangle, \tag{3.5}$$

and similarly define the degree of an automorphism $g \in \mathsf{Aut}(X)$ by

$$\deg(g) = \int_X \kappa_0 \wedge g^*\kappa_0 = \langle [\kappa_0] | g^*[\kappa_0]\rangle. \tag{3.6}$$

In the following lemma, K_X denotes the canonical bundle of X.

Lemma 3.5. *Let* $c_X \geq 0$ *be a constant such that* $\langle K_X|\cdot\rangle \leq c_X\langle [\kappa_0]|\cdot\rangle$ *on the effective cone. If* $f \in \mathsf{Aut}(X)$ *is loxodromic and* E *is a reduced, connected, and* f*-periodic curve, then*

$$\langle \theta_f^+|\theta_f^-\rangle \deg(E) \leq 2(1+c_X).$$

If E *is not connected, then* E *has at most* $\rho(X)-2$ *connected components, thus*

$$\langle \theta_f^+|\theta_f^-\rangle \deg(E) \leq 2(\rho(X)-2)(1+c_X) \leq 2(b_2(X)-2)(1+c_X),$$

where $\rho(X)$ *is the Picard number of* X *and* $b_2(X)$ *is its second Betti number.*

If E is f-invariant, then $[E]$ is orthogonal to Π_f for the intersection form, so the Hodge index theorem implies that $[E]^2 < 0$. Thus, if E is irreducible, it is determined by its class $[E]$, and Lemma 3.5 shows that f has only finitely many irreducible periodic curves; this finiteness result strengthens [Kaw08, Proposition B] (see also [Can01a] and [Can14, Proposition 4.1]). We denote by D_f the union of these irreducible f-periodic curves.

Example 3.6. We can take $c_X = 0$ when X is a K3, Enriques, or abelian surface.

Proof of Lemma 3.5. Assume first that E is connected. Set $e = [E]/\deg(E)$ so that $\langle e|[\kappa_0]\rangle = 1$. Since E is reduced and connected, its arithmetic genus $\langle K_X + E|E\rangle + 2$ is non-negative (see [BHPV04, § II.11]), so

$$-\langle E|E\rangle \leq 2 + \langle K_X|E\rangle \leq 2 + c_X \deg(E). \tag{3.7}$$

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On the other hand, Lemma 3.4 implies

$$-\langle E|E\rangle = -\deg(E)^2\langle e|e\rangle \geq \frac{\langle m_f|m_f\rangle}{1-\langle m_f|m_f\rangle}\deg(E)^2. \tag{3.8}$$

Putting these two inequalities together we get

$$\deg(E)^2 \leq \frac{1-\langle m_f|m_f\rangle}{\langle m_f|m_f\rangle}(2+c_X\deg(E)). \tag{3.9}$$

Solving for the corresponding quadratic equation in $\deg(E)$, and applying the inequality $t(1-t)\leq 1/4$ with $t=\langle m_f|m_f\rangle$ finally gives

$$\langle m_f|m_f\rangle\deg(E) \leq (1-\langle m_f|m_f\rangle)c_X + 1/\sqrt{2} \leq c_X+1. \tag{3.10}$$

For the second assertion, write E as a union of disjoint connected components E_i. The classes $[E_i]$ are pairwise orthogonal, and are contained in $(\theta_f^+)^\perp\cap(\theta_f^-)^\perp$, a subspace of codimension 2 in the Néron–Severi group of X. This implies that there are at most $\rho(X)-2$ connected components. □

Proposition 3.7. *Let X be a compact Kähler surface with a reference Kähler form κ_0 such that $\int\kappa_0^2=1$. If $f\in\mathsf{Aut}(X)$ is loxodromic and E is an f-invariant curve, then*

$$\deg(E)\leq 2^{54}(\rho(X)-2)(1+c_X)\deg(f)^{56},$$

where the degrees are relative to κ_0 and c_X is as in Lemma 3.5.

Proof. As in Remark 3.2, denote by $\mathrm{d}_{\mathbb{H}}$ the hyperbolic distance on $\mathbb{H}_X$ and let $\mathsf{Ax}(f)$ be the axis of the loxodromic isometry f^*. Lemma 4.8 in [BC16] implies that[4]

$$\mathrm{d}_{\mathbb{H}}([\kappa_0],\mathsf{Ax}(f))\leq 28\,\mathrm{d}_{\mathbb{H}}([\kappa_0],f^*[\kappa_0]) = 28\cosh^{-1}(\deg(f)).$$

Then, using the formula (3.4) for the distance $\mathrm{d}_{\mathbb{H}}([\kappa_0],\mathsf{Ax}(f))$ together with the elementary inequality $\cosh(kx)\leq 2^{k-1}\cosh(x)^k$, we obtain

$$\frac{2}{\langle\theta_f^+|\theta_f^-\rangle} = \cosh(\mathrm{d}_{\mathbb{H}}([\kappa_0],\mathsf{Ax}(f)))^2 \leq 2^{54}(\deg(f))^{56}. \tag{3.11}$$

The result now follows from Lemma 3.5. □

3.3 Γ-periodic curves, singular models, and ampleness

Denote by $\Pi_\Gamma^\perp$ the orthogonal complement of Π_Γ with respect to the intersection form.

Lemma 3.8. *Let $\Gamma\subset\mathsf{Aut}(X)$ be a non-elementary subgroup.*

(i) *A curve $C\subset X$ is Γ-periodic if and only if $[C]\in\Pi_\Gamma^\perp$.*
(ii) *If Γ contains a parabolic element, and C is irreducible, then C is Γ-periodic if and only if C is contained in a fiber of π_g for every $g\in\Gamma_{\mathrm{par}}$.*

Proof. For assertion (i), we note that since the intersection form is negative definite on $\Pi_\Gamma^\perp$, Γ acts on this space as a group of Euclidean isometries. Thus, if $c\in\Pi_\Gamma^\perp$ is an integral class, then $\Gamma^*(c)$ is a finite set. Since Π_Γ is generated by nef classes, $[C]$ belongs to $\Pi_\Gamma^\perp$ if and only if each of its irreducible components does, so it is enough to prove the result for an irreducible curve. Now an irreducible curve C with negative self-intersection is uniquely determined by its class $[C]$; so if $[C]$ is contained in $\Pi_\Gamma^\perp$, we conclude that C is Γ-periodic. Conversely, if C is Γ-periodic,

[4] It was stated for birational transformations of $\mathbb{P}^2$ in [BC16] but the estimate holds in our setting with the same proof (actually an easier one since here we work in a finite-dimensional hyperbolic space).

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a finite index subgroup $\Gamma' \subset \Gamma$ preserves C. If $f \in \Gamma'_{\rm lox}$, then $\langle \theta_f^+ \,|\, [C]\rangle = 0$ because f preserves the intersection form. But $\mathrm{Vect}(\theta_f^+, f \in \Gamma'_{\rm lox})$ is a Γ'-invariant subspace of Π_Γ, hence by strong irreducibility it coincides with Π_Γ (see § 3.1.3). Thus, $[C] \in \Pi_\Gamma^\perp$, and in particular $[C]^2 < 0$.

Let us prove the second assertion. If $[C] \in \Pi_\Gamma^\perp$ and $g \in \Gamma_{\rm par}$, $[C]$ intersects trivially the class $[F]$ of the general fiber of π_g; this implies that C is contained in a fiber of π_g, and is an irreducible component of a singular fiber since $[C]^2 < 0$. Now, denote by S the set of irreducible curves which are contained in some fiber of π_g for all $g \in \Gamma_{\rm par}$; it remains to prove that each $C \in S$ is Γ-periodic. Since Γ is non-elementary, $\Gamma_{\rm par}$ contains two elements g_1 and g_2 with distinct fixed points on the boundary of $\mathbb{H}_X$; these fixed points are given by the classes $[F_1]$ and $[F_2]$ of any smooth fiber of π_{g_1} and π_{g_2}, respectively; hence, π_{g_1} and π_{g_2} cannot share any smooth fiber. This shows that elements of S are contained in singular fibers of π_{g_1} and, in particular, S is finite. Moreover, S is Γ-invariant, because $\Gamma_{\rm par}$ is invariant under conjugacy, thus every $C \in S$ is a Γ-periodic curve. □

Proposition 3.9. *Let $\Gamma \subset \mathsf{Aut}(X)$ be a non-elementary subgroup containing parabolic automorphisms. There is a birational morphism $\pi_0 \colon X \to X_0$ onto a normal projective surface X_0 and a homomorphism $\tau \colon \Gamma \to \mathsf{Aut}(X_0)$ such that:*

(1) *π_0 contracts all Γ-periodic curves and only them;*
(2) *π_0 is τ-equivariant, that is, $\pi_0 \circ f = \tau(f) \circ \pi_0$ for every $f \in \Gamma$;*
(3) *there is an ample line bundle A on X_0 such that $\pi_0^* A$ is a big and nef line bundle, whose class belongs to Π_Γ.*

This proposition shows that examples as in [CD20, § 11] do not appear for non-elementary groups containing parabolic automorphisms. Before starting the proof, recall that a line bundle M on X is semi-ample if and only if mM is globally generated (or equivalently base-point free) for some $m > 0$ (see [Laz04, § 2.1.B]); here we use the additive notation mM for the line bundle $M^{\otimes m}$. If M is semi-ample and $m \geq 1$ is sufficiently divisible, the line bundle mM determines a morphism

$$\Phi_{mM} \colon X \to X_{mM} \subset \mathbb{P}(H^0(X, mM)^\vee), \tag{3.12}$$

onto a projective (possibly singular) normal variety X_{mM}. According to Theorem 2.1.27 in [Laz04], there is an algebraic fibre space $\Phi \colon X \to Y$ such that:

(1) Y is a normal projective variety (see Example 2.1.15 in [Laz04]);
(2) $X_{mM} = Y$ and $\Phi_{mM} = \Phi$ for sufficiently divisible integers $m \geq 1$;
(3) there is an ample line bundle A on Y such that $\Phi^* A = \ell M$ (for some $\ell \geq 1$).

Example 3.10. To each $g \in \mathsf{Aut}(X)_{\rm par}$ corresponds a semi-ample line bundle L_g such that (i) the members of $|L_g|$ are given by the fibers of π_g and (ii) $\pi_g \colon X \to B_g$ coincides with the fibration $\Phi \colon X \to Y$ determined by L_g. The ray $\mathbf{R}_+[L_g] \subset H^{1,1}(X; \mathbf{R})$ determines the unique fixed point of g^* in $\partial \mathbb{H}_X$, and L_g is nef (see § 3.1).

Proof of Proposition 3.9. By Lemma 3.8, we can fix a finite number of parabolic elements $g_i \in \Gamma$, with $1 \leq i \leq k$ for some $k \geq 2$, such that the set of irreducible and Γ-periodic curves $C \subset X$ is exactly the set of irreducible curves which are contained in fibers of π_{g_i} for $i = 1, \ldots, k$. The line bundle $M = \sum_i L_{g_i}$ is semi-ample, because the L_{g_i} are; it is big because $M^2 > 0$; and its class belongs to Π_Γ because the classes $[L_{g_i}]$ belong to the limit set of Γ (see [CD23c, § 2.3.6]). Since M is big, the fibration $\Phi = \Phi_{mM} \colon X \to Y$ defined, as above, by some sufficiently divisible multiple of M, is a birational morphism (a generically finite fibration is a birational morphism since its

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fibers are, by definition, connected). By construction Φ contracts exactly the periodic curves of Γ. Thus, setting $\pi_0 = \Phi$ and $X_0 = Y$, we obtain a birational morphism that contracts all periodic curves, and only them. Since Γ permutes these curves, it induces a group of automorphisms on X_0. Moreover, we know that there is an ample line bundle A on X_0 such that $\pi_0^* A = \ell M$ for some $\ell \geq 1$; this proves the third assertion. $\square$

Remark 3.11. In the proof of Proposition 3.9, one may add extra parabolic automorphisms $g_j \in \Gamma_{\mathrm{par}}$, say with $k+1 \leq j \leq \ell$, and replace M by $\sum_{i=1}^{\ell} m_i L_i$ for any choice of integers $m_i > 0$, while getting the same conclusion. After multiplication by $\mathbf{Q}_+^*$, the classes constructed in this way form a dense subset of the convex cone

$$\left\{ \sum_{i=1}^{\ell} \alpha_i c_1(L_{g_i}); \ell \geq 1,\ g_i \in \Gamma_{\mathrm{par}},\ \text{and } \alpha_i \in \mathbf{R}_+^* \text{ for all } i \right\}. \tag{3.13}$$

This cone is Γ-invariant, its closure is the smallest convex cone whose projectivization contains the limit set $\mathrm{Lim}(\Gamma)$, and it spans Π_Γ because Π_Γ is the smallest vector space containing $\mathrm{Lim}(\Gamma)$. Thus, the classes of the form $\alpha c_1(\pi_0^* A)$, where A runs over the set of ample line bundles on X_0 and α runs over $\mathbf{Q}_+^*$, is a dense subset of this cone.

3.4 Proof of Theorem D

Let us first deal with the case where Γ is elementary. By [Can14, Theorem 3.2] there is a loxodromic element $f \in \Gamma$ such that $(f^*)^{\mathbf{Z}}$ has finite index in Γ^*. If $\mathsf{Aut}(X)^0$ is non-trivial, then X is a torus and then f has no invariant curve (see [Can14, Remark 3.3] and [CF03]). Otherwise, the kernel of the homomorphism $\Gamma \to \Gamma^*$ is finite, $f^{\mathbf{Z}}$ has finite index in Γ, and therefore a curve is Γ-periodic if and only if it is f-periodic, so we are done when Γ is elementary.

When Γ is non-elementary, Theorem D is covered by the following more precise statement (recall that X is automatically projective in this case [CD23a]).

Proposition 3.12. *Let X be a complex projective surface and Γ be a non-elementary subgroup of $\mathsf{Aut}(X)$. Then there exists a loxodromic element f in Γ such that every f-periodic curve is Γ-periodic.*

If, in addition, Γ contains a parabolic element g, then f can be chosen of the form $f = h^N \circ g^N$ for some h which is conjugate to g in Γ and N is any sufficiently large integer. If, moreover, N is sufficiently divisible, then $(g^N)^*$ is unipotent.

Proof. Consider a subset $S \subset \Gamma_{\mathrm{lox}}$ such that $\{(\mathbb{P}(\theta_f^+), \mathbb{P}(\theta_f^-));\ f \in S\}$ is dense in $\mathrm{Lim}(\Gamma)^2$, as in Lemma 3.3. Let us exhibit an $f \in S$ such that every f-periodic curve is Γ-periodic. By contradiction, we assume that every $f \in S$ admits at least one irreducible periodic curve $C(f)$ which is not Γ-periodic, and we set $c(f) = [C(f)]$. By Lemma 3.8, $c(f)$ does not belong to $\Pi_\Gamma^\perp$, thus $u \mapsto \langle c(f) | u \rangle$ is a non-trivial linear form on Π_Γ. Since the class of any periodic curve is orthogonal to Π_f, $\langle c(f) | \theta_f^+ \rangle = \langle c(f) | \theta_f^- \rangle = 0$.

Let U and U' be open subsets of $\partial \mathbb{H}_X$ intersecting $\mathrm{Lim}(\Gamma)$ non-trivially, and such that $\overline{U} \cap \overline{U}' = \emptyset$; let x be an element of $U \cap \mathrm{Lim}(\Gamma)$. Define

$$A(U, U') = \{f \in \mathsf{Aut}(X);\ f \text{ is loxodromic, } \mathbb{P}(\theta_f^+) \in U \text{ and } \mathbb{P}(\theta_f^-) \in U'\}, \tag{3.14}$$

and

$$D(U, U') = \{c(f);\ f \in A(U, U') \cap S\}. \tag{3.15}$$

By Lemma 3.5, $D(U, U')$ is a finite set. From our assumption on S, there is a sequence (f_n) of elements in $A(U, U') \cap S$ such that $x = \lim_n (\mathbb{P}(\theta_{f_n}^+))$. Extracting a subsequence if necessary we

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may assume that $c(f_n)$ is constant, equal to some $c_\infty \in D(U,U')$, and we infer that x is contained in $c_\infty^\perp$. As a consequence, the limit set $\mathrm{Lim}(\Gamma) \subset \mathbb{P}(\Pi_\Gamma)$ is locally contained in the finite union of hyperplanes $\mathbb{P}(c^\perp \cap \Pi_\Gamma)$, for $c \in D(U,U')$. By compactness, $\mathrm{Lim}(\Gamma)$ is contained in a finite union of hyperplanes, which contradicts the strong irreducibility of the action of Γ on Π_Γ.

Now, to prove the first assertion of the proposition, we simply put $S = \Gamma_{\mathrm{lox}}$; by Lemma 3.3, it satisfies the desired density property.

For the second assertion, we remark that the set of fixed points (u,u') of all pairs (h,h'), where h and h' run independently over the conjugacy class of g, is dense in $\mathrm{Lim}(\Gamma)^2$. If we apply Lemma 3.13 we see that the set of all loxodromic elements $f = h^N \circ (h')^N$ with h and h' conjugate to g satisfies the desired density property. The property '*any irreducible f-periodic curve is also Γ-periodic*' being invariant under conjugacy, we can choose $f = h^N \circ g^N$ for some conjugate h of g and any large enough N.

The last assertion follows from Lemma 3.1. □

Lemma 3.13. *Let h and h' be two parabolic elements of $\mathsf{Aut}(X)$ with distinct fixed points u and u' in $\partial\mathbb{H}_X$. Let U and U' be small, disjoint neighborhoods of u and u', respectively, in $\mathbb{P}(\Pi_\Gamma)$. Then if $N \in \mathbf{Z}$ is large enough, $f_N := h^N \circ (h')^N$ is a loxodromic automorphism such that $\mathbb{P}(\theta^+_{f_N}) \in U$ and $\mathbb{P}(\theta^-_{f_N}) \in U'$.*

Proof. Let us denote by $\mathbb{P}(h^*)$ the linear projective transformation induced by h^* on $\mathbb{P}(\mathrm{NS}(X;\mathbf{R}))$. Since U does not contain u', $\mathbb{P}(h'^*)^N(U) \subset U'$ if $|N|$ is large enough; similarly $\mathbb{P}(h^*)^N(U') \subset U$. So for $f_N = h^N \circ (h')^N$, $\mathbb{P}(f_N^*)$ maps U' strictly inside itself and likewise $\mathbb{P}((f_N^{-1})^*)$ maps U strictly inside itself. This implies that f_N is loxodromic, with its α-limit and ω-limit points in U' and U, respectively. □

4. Complex tori and Kummer examples

This section gathers some facts on automorphism groups of complex tori. We also introduce and study the notion of a Kummer group. Part of this material is well known, we provide the details for completeness.

4.1 Finite orbits on tori

Consider a compact complex torus $A = \mathbf{C}^k/\Lambda$. Each automorphism f of A is an affine transformation $f(z) = L_f(z) + t_f$, where $z \mapsto z + t_f$ is the translation part and L_f is a linear automorphism, induced by a linear transformation of $\mathbf{C}^k$ that preserves Λ. Let Γ be a subgroup of $\mathsf{Aut}(A)$.

Warning. By definition, compact tori and abelian varieties come equipped with their group structure, in particular with their neutral element, or 'origin'. On the other hand, an automorphism f with a non-trivial translation part t_f does not preserve this group structure. If $x \in A$ is fixed by Γ, conjugating Γ by the translation $z \mapsto z + x$ we may assume that Γ fixes the origin of A and acts by linear isomorphisms. Alternatively, we can transport the group structure by this translation and put the origin at x: this changes the group structure but not the underlying complex manifold. We shall frequently do this operation without always specifying the change in the group structure of A.

Suppose the orbit $\Gamma(x) \subset A$ is finite, of cardinality m, and consider the stabilizer $\Gamma_0 \subset \Gamma$ of x; its index divides $m!$. Conjugating Γ by $z \mapsto z + x$, as explained above, all elements $f \in \Gamma_0$ are linear. In that case, every torsion point has a finite Γ_0-orbit, hence also a finite Γ-orbit; in particular, finite orbits of Γ form a dense subset of A for the Euclidean topology. The next proposition summarizes this discussion.

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Proposition 4.1. *Let A be a compact complex torus, and let Γ be a subgroup of $\mathsf{Aut}(A)$. If Γ has a finite orbit, then its finite orbits form a dense subset of A. More precisely, if a periodic point of Γ is chosen as the origin of A for its group law, then all torsion points of A are periodic points of Γ.*

Remark 4.2. If in Proposition 4.1 we moreover assume that $\dim_{\mathbf{C}} A = 2$ and Γ contains a loxodromic element fixing the origin, then conversely *all periodic points of Γ are torsion points.* This follows from Lemma 4.3.

4.2 Dimension 2

Let $A = \mathbf{C}^2/\Lambda$ be a compact complex torus of dimension 2, and let $f(z) = L_f(z) + t_f$ be a loxodromic element of $\mathsf{Aut}(A)$. The loxodromy means exactly that the eigenvalues α and β of L_f satisfy $|\alpha| < 1 < |\beta|$. Pick a basis of Λ, and use it to identify Λ with $\mathbf{Z}^4$ and $\mathbf{C}^2$ with $\mathbf{R}^4$, as real vector spaces. Then, L_f corresponds to an element $M_f \in \mathsf{GL}_4(\mathbf{Z})$.

Lemma 4.3. *Let f be a loxodromic automorphism of a compact complex torus A of dimension 2. Then:*

(1) *f has a fixed point, and after translation $z \mapsto z + x$ by such a fixed point, its periodic points are exactly the torsion points of A;*
(2) *there is no f-invariant curve: the orbit $f^{\mathbf{Z}}(C)$ of any curve is dense in A.*

Proof. With the above notation, the fixed points of f are determined by the equation $(L_f - \mathrm{id})(z) \in \Lambda - t_f$ or, equivalently, $(M_f - \mathrm{id})(z) \in \mathbf{Z}^4 - t_f$. The complex eigenvalues of L_f being $\neq 1$, there is at least one fixed point. Thus, after conjugation by a translation, we may assume that $t_f = 0$. Then, the periodic points of f correspond to the points $x \in \mathbf{R}^4$ such that $M_f^n(x) - x \in \mathbf{Z}^4$; any solution to such an equation being rational, it corresponds to a torsion point in A. This proves property (1). For property (2), we may now assume that $t_f = 0$. There are two linear forms ξ_f^+, $\xi_f^- : \mathbf{C}^2 \to \mathbf{C}$ such that $\xi_f^+ \circ L_f = \alpha\xi_f^+$ and $\xi_f^- \circ L_f = \beta\xi_f^-$. They determine two linear foliations on A: the stable and unstable foliations of f. Both have dense leaves. If y is a torsion point, y is f-periodic, and its stable manifold, being dense, intersects C. Thus, the sequence $(f^n(C))$ accumulates at every torsion point, so it is dense in A, and C is not invariant. $\square$

4.3 Kummer structures

4.3.1 *Kummer groups.* Let X be a compact Kähler surface, and let Γ be a subgroup of $\mathsf{Aut}(X)$. By definition (X, Γ) is a *Kummer group* if there is a (compact) torus A, a finite subgroup G of $\mathsf{Aut}(A)$, a subgroup Γ_A of $\mathsf{Aut}(A)$ containing G, and a bimeromorphic morphism $q_X \colon X \to A/G$ such that the following hold.

(a) Γ_A normalizes G. Thus, if $q_A \colon A \to A/G$ is the quotient map, there is a homomorphism $h \in \Gamma_A \mapsto \overline{h} \in \mathsf{Aut}(A/G)$ such that $q_A \circ h = \overline{h} \circ q_A$ for every $h \in \Gamma_A$; we shall denote by $\overline{\Gamma}_A$ the image of this homomorphism.
(b) The bimeromorphic map q_X is Γ-equivariant: there is a homomorphism $\Gamma \ni f \mapsto \overline{f} \in \mathsf{Aut}(A/G)$, whose image is denoted by $\overline{\Gamma}$, such that $q_X \circ f = \overline{f} \circ q_X$ for every $f \in \Gamma$.
(c) The subgroups $\overline{\Gamma}$ and $\overline{\Gamma}_A$ of $\mathsf{Aut}(A/G)$ coincide.

To each $f \in \Gamma$ corresponds an element $f_A \in \Gamma_A$, unique up to composition with elements of G; the type of f as an automorphism of $\mathsf{Aut}(X)$ coincides with the type of f_A as an automorphism of A, and $\lambda(f) = \lambda(f_A)$. When X is projective, A is an abelian surface and q is a birational morphism. According to the usual terminology, we also say that $f \in \mathsf{Aut}(X)$ is a Kummer example when $(X, f^{\mathbf{Z}})$ is a Kummer group.

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Remark 4.4. Consider a section Ω_A of the canonical bundle K_A such that $\int_A \Omega_A \wedge \overline{\Omega_A} = 1$; it is unique up to multiplication by a complex number of modulus one. In particular, the volume form $\mathsf{vol}_A = \Omega_A \wedge \overline{\Omega_A}$ is invariant under $\mathsf{Aut}(A)$. The quotient of vol_A by the action of G determines a probability measure on A/G, and then on X. This probability measure coincides with the measure of maximal entropy μ_f for every $f \in \Gamma_{\mathrm{lox}}$.

From the definition of a Kummer group, Proposition 4.1 and Remark 4.2 we obtain the following.

Proposition 4.5. *Let (X,Γ) be a Kummer group with at least one finite orbit. Then, its finite orbits are dense in X for the euclidean topology, and there is a dense, Γ-invariant, Zariski-open subset in which all periodic points of loxodromic elements of Γ coincide.*

4.3.2 *Classification of Kummer examples.* Let us consider, first, the case of an infinite cyclic group generated by a loxodromic Kummer example f. From the classification given in [CF03, CF05], we may assume that the finite group G is a cyclic group fixing the origin of A; in other words, G is induced by a cyclic subgroup of $\mathsf{GL}_2(\mathbf{C})$ preserving the lattice Λ such that $A = \mathbf{C}^2/\Lambda$. There are only seven possibilities:

(1) $G = \{\mathrm{id}\}$ and X is a blow-up of a torus A;
(2) $G = \{\mathrm{id}, -\mathrm{id}\}$ and A/G is a Kummer surface, in the classical sense; in particular, X is a blow-up of a K3 surface;
(3) A is the torus $(\mathbf{C}/\mathbf{Z}[\mathsf{i}])^2$ and G is the group of order 4 generated by $\mathsf{i}\mathrm{id}$; in this case X is a rational surface;
(4) A is the torus $(\mathbf{C}/\mathbf{Z}[\exp(2\mathsf{i}\pi/3)])^2$ and G is the group of order 3 generated by $\exp(2\mathsf{i}\pi/3)\mathrm{id}$; in this case X is a rational surface;
(5) A is the torus $(\mathbf{C}/\mathbf{Z}[\exp(2\mathsf{i}\pi/3)])^2$ and G is the group of order 6 generated by $\exp(\mathsf{i}\pi/3)\mathrm{id}$; in this case X is a rational surface;
(6) let ζ_5 be a primitive fifth root of unity; the cyclotomic field $\mathbf{Q}[\zeta_5]$ has two distinct non-conjugate embeddings in $\mathbf{C}$, σ_1 and σ_2, determined by $\sigma_1(\zeta_5) = \zeta_5$ and $\sigma_2(\zeta_5) = \zeta_5^2$; the ring of integers coincides with $\mathbf{Z}[\zeta_5]$ and its image by $\sigma = (\sigma_1, \sigma_2)$ is a lattice $\Lambda_5 \subset \mathbf{C} \oplus \mathbf{C}$; the abelian surface A is the quotient $\mathbf{C}^2/\Lambda_5$; the group G is generated by the diagonal linear map $(x,y) \mapsto (\zeta_5 x, \zeta_5^2 y)$ and has order 5; here, X is rational too;
(7) as in the previous example, $A = \mathbf{C}^2/\Lambda_5$, but now G has order 10 and is generated by $(x,y) \mapsto (-\zeta_5 x, \zeta_5 y)$, and again X is rational.

These constraints on (A,G) apply to non-elementary Kummer groups; in particular, we shall always assume that G is cyclic and fixes the neutral element of A. In cases (1)–(5) of the above list, the abelian surface is $\mathbf{C}^2/(\Lambda_0 \times \Lambda_0)$ for some lattice Λ_0 in $\mathbf{C}$. The natural action of $\mathsf{GL}(2,\mathbf{Z})$ on $\mathbf{C}^2$ preserves $\Lambda_0 \times \Lambda_0$, and induces a non-elementary subgroup of $\mathsf{Aut}(A)$, which commutes to G; hence, it determines also a non-elementary subgroup of $\mathsf{Aut}(A/G)$. On the other hand, cases (6) and (7) do not appear.

Lemma 4.6. *If (X,Γ) is a non-elementary Kummer group, then G is generated by a homothety and the quotient A/G is not of type (6) or (7) in the classification above.*

Proof. The group Γ_A permutes the fixed points of G. Thus, the stabilizer $\Gamma_A^\circ = \mathrm{Stab}_{\Gamma_A}(0)$ of the neutral element is a finite index, non-elementary subgroup of Γ_A. Pick any loxodromic element f in Γ_A°; it acts by conjugacy on G, which is finite, so there is a positive iterate such that $f^n \circ g = g \circ f^n$ for all $g \in G$. Near the origin, f^n and g are two commuting linear transformations, f^n has two eigenvalues, of modulus < 1 and > 1, respectively, and g must preserve the corresponding

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stable and unstable directions of f. Since Γ_A° is non-elementary, these tangent directions form an infinite set as f varies in the set of loxodromic elements of Γ_A°, so g is a homothety, and we are done. $\square$

4.3.3 *Invariant curves.* We keep the notation from the previous paragraphs and consider a non-elementary Kummer group (X, Γ). The singularities of A/G are cyclic quotient singularities and X dominates a minimal resolution of A/G.

Let us examine case (2), when $G = \{\mathrm{id}, -\mathrm{id}\}$. Then G has 16 fixed points, and to resolve the 16 singularities of A/G one can proceed as follows. First, one blows up the fixed points, creating 16 rational curves. Then one lifts the action of G to the blow-up $\hat{A}$. If E is one of the exceptional divisors, then G fixes E pointwise and acts as $w \mapsto -w$ transversally, so locally the quotient map can be written $(w, z) \mapsto (w^2, z)$, with $E = \{w = 0\}$ giving rise to a smooth rational curve of self-intersection -2 on $\hat{A}/G$. This construction provides the minimal resolution $X_{\min} = \hat{A}/G$ of A/G, the singularities being replaced by disjoint (-2)-curves. Cases (3), (4), and (5) can be handled with a similar process because if $x \in A$ is stabilized by a subgroup H of G, then H is locally given around x as a cyclic group of homotheties; so, in the minimal resolution of A/G the singularities are replaced by disjoint rational curves E_i of negative self-intersection $E_i^2 \in \{-2, \dots, -6\}$. Cases (6) and (7) are more delicate; however, by Lemma 4.6, we do not need to deal with them.

Lemma 4.7. *Let (X, Γ) be a non-elementary Kummer group on a smooth projective surface. Then:*

(1) *X is an abelian surface if and only if Γ admits no invariant curve;*
(2) *any connected Γ-periodic curve D is a smooth rational curve, and the induced dynamics of $\mathrm{Stab}_D(\Gamma)$ on D has no periodic orbit.*

Moreover, $D_f = D_\Gamma$ for every $f \in \Gamma_{\mathrm{lox}}$.

Proof. The minimal resolution $X_{\min}$ of A/G is unique, up to isomorphism (see [BHPV04, § III.6], Theorems (6.1) and (6.2), and their proofs). Thus, X dominates $X_{\min}$ and every $f \in \Gamma$ preserves the exceptional divisor of the morphism $X \to X_{\min}$ and induces an automorphism $f_{\min}$ of $X_{\min}$. In particular, Γ admits an invariant curve, unless $G = \{\mathrm{id}\}$ and $X = X_{\min} = A$. Conversely, in that case Γ has no invariant curve, by Lemma 4.3. This proves the first assertion.

Let us prove the second assertion for the induced group $\Gamma_{\min} \subset \mathsf{Aut}(X_{\min})$. Let E be a connected periodic curve for $\Gamma_{\min}$. If $f_{\min} \in \Gamma_{\min}$ is loxodromic, it comes from an Anosov map $f_A \colon A \to A$, as in Lemma 4.3, and f_A does not have any periodic curve. Since E is $f_{\min}$-periodic, it is contained in the exceptional divisor of the resolution $X_{\min} \to A/G$; as explained before the lemma, this divisor is a disjoint union of rational curves, so E is one of these rational curves $E_x = q_{X_{\min}}^{-1}(q_A(x))$, where $x \in A$ has a non-trivial stabilizer $G_x \subset G$. In particular x is fixed by a finite index subgroup $\Gamma_{A,x}$ of Γ_A. Now since Γ is non-elementary, Γ_A and $\Gamma_{A,x}$ are non-elementary as well, and since the action of $\Gamma_{A,x}$ on A is by affine transformations, its action on the exceptional divisor E_x is that of a non-elementary subgroup of $\mathsf{PGL}_2(\mathbf{C}) = \mathsf{Aut}(E_x)$. In particular, $\Gamma_{A,x}$ does not admit any finite orbit in E_x.

The birational morphism $\pi : X \to X_{\min}$ is equivariant with respect to Γ and its image $\Gamma_{\min}$ in $\mathsf{Aut}(X_{\min})$. Thus, π^{-1} blows up periodic orbits of $\Gamma_{\min}$. The last few lines show that, when such a periodic point $y \in X_{\min}$ is blown up, first y does not lie on the exceptional locus of $X_{\min} \to A/G$ and, second, the exceptional divisor E_y does not contain any finite orbit. Thus, X is obtained by simple blow-ups centered on a finite set of distinct periodic points of $\Gamma_{\min}$, every connected component of the exceptional locus of q_X is a smooth rational curve, and there is no Γ-periodic point in these curves. □

5. Unlikely intersections for non-elementary groups

This section is devoted to the proof of Theorem B.

5.1 Strategy of the proof

If Γ has a Zariski-dense set F of finite orbits, a standard argument shows that there is a sequence $(x_n) \in F^{\mathbf{N}}$ which is generic (given any fixed proper subvariety $Y \subset X$, $x_n \notin Y$ for n large enough). Let m_{x_n} be the probability measure equidistributed over the Galois orbit of x_n. We want to use arithmetic equidistribution to show that the sequence of measures (m_{x_n}) converges when $n \to \infty$. For this, we need a height function h_L (associated to some line bundle L on X with appropriate positivity properties) that vanishes on F. In §§ 5.2–5.4 we construct such height functions: they are associated to the choice of certain finitely supported probability measures ν on Γ. Indeed, to such a measure we associate the linear endomorphism $P_\nu^* = \sum \nu(f) f^*$ of the Néron–Severi group $\mathrm{NS}(X; \mathbf{R})$, and we construct a big and nef line bundle L, that depends on ν, such that $P_\nu^*[L] = \alpha(\nu)[L]$ for some $\alpha(\nu) > 1$; then, h_L will be a Weil height that satisfies the invariance $\sum \nu(f) h_L \circ f = \alpha(\nu) h_L$. The arithmetic equidistribution theorem of Yuan shows that the measures m_{x_n} converge to a measure $\mu_\nu = S \wedge S$, where S is a dynamically defined closed positive current with cohomology class equal to $[L]$. On the other hand, the measures m_{x_n}, hence their limit $\mu := \mu_\nu$, do not depend on ν. As we vary the choice of ν, the construction has enough flexibility to show that for every $f \in \Gamma_{\mathrm{lox}}$, μ_ν can be made arbitrary close to the maximal entropy measure μ_f. It follows that $\mu_f = \mu$ is independent of f and is Γ-invariant. In § 5.5, the dynamics of parabolic elements of Γ is used to deduce that $\mathrm{Supp}(\mu) = X$. Then the classification of Γ-invariant measures from [Can01b, CD23b] implies that μ has a smooth density, and the main result of [CD20] shows that every $f \in \Gamma_{\mathrm{lox}}$ is a Kummer example (Theorem 5.17 in § 5.6). At this point the Kummer structure may *a priori* depend on f, as in Example 5.1 below. This issue is solved in § 5.7 by adding an argument based on Theorem D which finally shows that (X, Γ) is a Kummer group.

Example 5.1. Let X be a Kummer surface possessing both a Kummer automorphism f and a non-Kummer one h, as in [KK01]. Then, f and $h \circ f \circ h^{-1}$ are two Kummer automorphisms which are not associated with the same Kummer structure; the pair $(X, \langle f, h \circ f \circ h^{-1}\rangle)$ is not a Kummer group.

5.2 Kawaguchi's currents

5.2.1 *Action on $H^{1,1}$.* Let X be a compact Kähler surface and let ν be a probability measure on $\mathsf{Aut}(X)$ satisfying the (exponential) moment assumption

$$\int (\|f\|_{C^1} + \|f^{-1}\|_{C^1})^2 \, d\nu(f) < +\infty. \tag{5.1}$$

By [CD23c, Lemma 5.1], this implies

$$\int \|f^*\| \, d\nu(f) < +\infty, \tag{5.2}$$

where f^* is the endomorphism of $H^2(X; \mathbf{R})$ determined by f and $\|\cdot\|$ is any operator norm. (For the proof of Theorem 5.17, we will only consider finitely supported measures so the moment conditions will be trivially satisfied.) Let Γ_ν be the subgroup of $\mathsf{Aut}(X)$ generated by the support

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of ν. We let P_ν be the linear endomorphism of $H^2(X;\mathbf{C})$ defined for every $u \in H^2(X;\mathbf{C})$ by

$$P_\nu(u) = \int f^*(u)\, d\nu(f). \tag{5.3}$$

The following lemma is a strong version of the Perron–Frobenius theorem. Recall that the Kähler form κ_0 was fixed in § 3.

Lemma 5.2. *Assume that Γ_ν is non-elementary. Then, P_ν has a unique eigenvector $w_\nu \in H^{1,1}(X;\mathbf{R})$ such that $w_\nu^2 = 1$ and $\langle w_\nu | [\kappa_0]\rangle > 0$. This eigenvector is big and nef. The eigenvalue $\alpha(\nu)$ such that*

$$P_\nu(w_\nu) = \alpha(\nu) w_\nu$$

is larger than 1 and coincides with the spectral radius of P_ν; the multiplicity of $\alpha(\nu)$ is equal to 1, and all other eigenvalues $\beta \in \mathbf{C}$ of P_ν satisfy $|\beta| < \alpha(\nu)$.

Proof. Recall the definition of the open convex cone $\mathrm{Pos}(X) \subset H^{1,1}(X;\mathbf{R})$ from § 3.1. Since every $f \in \mathsf{Aut}(X)$ preserves $\mathrm{Pos}(X)$, P_ν preserves $\mathrm{Pos}(X)$ too, by convexity; moreover, if $u \in \partial\mathrm{Pos}(X) \setminus \{0\}$ and $u' := P_\nu(u) \in \partial\mathrm{Pos}(X)$, then $f^*(u) \in \mathbf{R}_+ u'$ for all f in $\mathrm{Supp}(\nu)$. Thus, we can apply the results of Vandergraft [Van68]. In his terminology, the *faces* of $\overline{\mathrm{Pos}}(X)$ are the isotropic rays in $\partial\mathrm{Pos}(X)$, so P_ν is *irreducible*, because if a face $\mathbf{R}_+ u$ were fixed by P_ν it would be fixed by every f in $\mathrm{Supp}(\nu)$, and Γ_ν would be elementary. Thus, Theorems 4.2 and 4.3 of [Van68] imply that the spectral radius $\alpha(\nu)$ of P_ν is a simple eigenvalue of P_ν and the corresponding eigenline $\mathbf{R}w_\nu$ intersects $\mathrm{Pos}(X)$; choosing $w_\nu \in \mathbb{H}_X$, w_ν is uniquely determined.

If $w, w' \in \mathbb{H}_X$, then $\langle w | w'\rangle \geq 1$ with equality if and only if $w = w'$. Computing the self-intersection of $P_\nu(w_\nu)$, we deduce that $\alpha(\nu) \geq 1$. In addition, $\alpha(\nu) = 1$ if and only if $f^*(w_\nu) = w_\nu$ for every f in $\mathrm{Supp}(\nu)$, which would imply that Γ_ν is elementary; hence, $\alpha(\nu) > 1$. The Kähler cone satisfies also $P_\nu(\mathrm{Kah}(X)) \subset \mathrm{Kah}(X)$. Hence, w_ν is nef (it is in the closure of $\mathrm{Kah}(X)$). It is big because it is nef and has positive self-intersection (see, e.g., [Laz04, Theorem 2.2.16]).

Let us show that $\alpha(\nu)$ is greater in magnitude than any other eigenvalue of P_ν; for this, we can of course replace P_ν by P_ν^N, for some $N \geq 1$ to be chosen below. According to [Van68, Theorem 4.4] it is sufficient to show that $P_\nu^N(\overline{\mathrm{Pos}(X)}) \subset \mathrm{Pos}(X)$. As said above, if this fails there exists $u, u' \in \partial\mathrm{Pos}(X) \setminus \{0\}$ such that $f^* u \in \mathbf{R}u'$ for every $f \in \mathrm{Supp}(\nu^{\otimes N})$. On the other hand, for N large, there exist three loxodromic elements $f_1, f_2, f_3 \in \mathrm{Supp}(\nu^{\otimes N})$ with disjoint fixed points (apply [BQ16, Theorem 6.36] and [CD23c, Proposition 2.8]). Replacing N by NN' and f_i by $f_i^{N'}$ if necessary, we can further assume that there are six disjoint open sets $B_i^\pm$, $i = 1, 2, 3$, in $\partial\mathbb{H}_X = \mathbb{P}(\partial\mathrm{Pos}(X)) \subset \mathbb{P}(H^{1,1}(X,\mathbf{R}))$ such that $\mathbb{P}(f_i)(\partial\mathbb{H}_X \setminus B_i^-) \subset B_i^+$ for every i. Now, suppose $u, u' \in \partial\mathrm{Pos}(X) \setminus \{0\}$ satisfy $f_i^* u \in \mathbf{R}u'$ for $i = 1, 2, 3$. If $\mathbb{P}(u)$ is in none of the B_i^-, then $\mathbb{P}(u')$ must be in the intersection of the B_i^+, a contradiction. Otherwise, $\mathbb{P}(u)$ is in exactly one of the B_i^-, say B_1^-, and $\mathbb{P}(u')$ should be in $B_2^+ \cap B_3^+$, a contradiction. Thus, we are done. ∎

Example 5.3 (see [Can14, § 2] and [CD20, § 2.2]). Let f be a loxodromic automorphism, and take ν to be the probability measure $p\delta_f + q\delta_{f^{-1}}$ with $p, q \geq 0$ and $p + q = 1$. Note that $\Gamma_\nu = f^{\mathbf{Z}}$ does not satisfy the assumption of Lemma 5.2. Then $P_\nu = pf^* + q(f^{-1})^*$ preserves the f^*-invariant plane $\Pi_f \subset H^{1,1}(X;\mathbf{R})$. If $p > q$, the spectral radius of P_ν is equal to $p\lambda(f) + q/\lambda(f)$, and the corresponding eigenspace is the isotropic line of Π_f corresponding to the eigenvalue $\lambda(f)$ of f; if $p < q$, the spectral radius is equal to $p/\lambda(f) + q\lambda(f)$ and the eigenspace is the other isotropic

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line in Π_f. If $p = q = 1/2$, then $(P_\nu)_{|\Pi_f}$ is the scalar multiplication by

$$\alpha(f) = \tfrac{1}{2}(\lambda(f) + 1/\lambda(f)), \tag{5.4}$$

$\alpha(f) > 1$, and all vectors $u \in \Pi_f$ satisfy $P_\nu(u) = \alpha(f)u$. This example shows that the previous lemma fails for ν, whatever the values of p and q are: the dominant eigenvector is at the boundary of the hyperbolic space, or is not unique.

5.2.2 *Stationary currents.* Let us borrow some notation from [CD23c, § 6.1]: we fix Kähler forms κ_i, whose cohomology classes provide a basis $([\kappa_i])$ of $H^{1,1}(X;\mathbf{R})$. Then, if a is any element of $H^{1,1}(X;\mathbf{R})$, there is a unique $(1,1)$-form $\Theta(a) = \sum_i a_i\kappa_i$ in $\mathrm{Vect}(\kappa_i,\ 1 \le i \le h^{1,1}(X))$ whose class $[\Theta(a)]$ is equal to a. If S is any closed positive current of bidegree $(1,1)$, then $S = \Theta([S]) + dd^c u_S$ for some upper semi-continuous function $u_S \colon X \to \mathbf{R}$: this function is locally the difference of a plurisubharmonic (psh) function and a smooth function, and it is unique up to an additive constant.

The following proposition is essentially due to Kawaguchi, who proved it in [Kaw06] under slightly more restrictive assumptions.

Proposition 5.4. *Let X be a compact Kähler surface, and let vol be a smooth volume form on X. Let ν be a probability measure on $\mathsf{Aut}(X)$ satisfying the moment condition (5.1). Assume that we are in one of the following two situations:*

- (i) *Γ_ν is non-elementary; in this case, set $w = w_\nu$, and $\alpha = \alpha(\nu)$, as in Lemma 5.2;*
- (ii) *$\nu = \frac{1}{2}(\delta_f + \delta_{f^{-1}})$ for some loxodromic automorphism and w is a fixed point of $(1/\alpha(f))P_\nu$ in $\mathbb{H}_X$, as in Example 5.3; in this case, set $\alpha = \alpha(f)$.*

Then, there is a unique closed positive current S such that

$$\int f^*(S)\, d\nu(f) = \alpha S \quad \text{and} \quad [S] = w. \tag{5.5}$$

This current has continuous potentials: $S = \Theta(w) + dd^c(u)$ for a unique continuous function u such that $\int_X u \,\mathsf{vol} = 0$. In particular, the product $S \wedge S$ is a well-defined probability measure on X.

Proof of Proposition 5.4 in case (i). Let β be a smooth form with $[\beta] = w$. For simplicity, we denote by the same letter P_ν the operator $\int f^*(\cdot)\, d\nu(f)$ acting on the cohomology, on differential forms, or on currents. Write $(\alpha^{-1}P_\nu)\beta = \beta + dd^c(h)$ for some smooth function h. Then,

$$\left(\frac{1}{\alpha}P_\nu\right)^n \beta = \beta + dd^c\left(\sum_{j=0}^{n-1} \frac{1}{\alpha^j}(P_\nu)^j(h)\right), \tag{5.6}$$

where $(P_\nu)^j(h)(x) = \int h \circ f(x)\, d\nu^{\star j}(f)$ and $\nu^{\star j}$ denotes the jth convolution power of ν. Using the fact that $\|(P_\nu)^j(h)\|_\infty \le \|h\|_\infty$ for all $j \ge 1$. We deduce that the series on the right-hand side of (5.6) converges geometrically:

$$\sum_{j=k}^{\ell} \left\| \frac{1}{\alpha^j}(P_\nu)^j(h) \right\|_\infty \le \frac{\alpha - \alpha^{k-\ell}}{\alpha - 1} \frac{\|h\|_\infty}{\alpha^k} \le \frac{\alpha}{\alpha - 1} \frac{\|h\|_\infty}{\alpha^k}. \tag{5.7}$$

Thus, if we set $h_\infty = \sum_{j\ge 0}(1/\alpha^j)(P_\nu^*)^j(h)$ and $S = \beta + dd^c(h_\infty)$ we see that S is a closed current which satisfies $P_\nu(S) = \alpha S$.

Let us show that S is positive. By Lemma 5.2, there is a P_ν-invariant hyperplane $W \subset H^{1,1}(X;\mathbf{R})$ such that the spectral radius of $P_{\nu|W}$ is $< \alpha(\nu)$. Since w is nef, there is a smooth

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and closed form β' such that $[\beta'] \in W$ and $\kappa := \beta + \beta'$ is Kähler. Choose $\dim(W)$ smooth, closed $(1,1)$-forms γ_i such that $\gamma_1 = \beta'$ and the classes $[\gamma_i]$ form a basis of W. Then,

$$\frac{1}{\alpha} P_\nu \gamma_i = \sum_j Q_{i,j} \gamma_j + dd^c(u_i), \tag{5.8}$$

where $Q := (Q_{i,j})$ is the matrix of $(1/\alpha) P_{\nu|W}$ in the basis $([\gamma_i])$ and the u_i are smooth functions. Fix some $\eta < 1$ larger than the spectral radius of Q. Iterating (5.8), one sees that $((1/\alpha)P_\nu)^n \gamma_i$ is the sum of a closed form bounded by $C_1 \eta^n$, plus the dd^c of a function which is bounded by $C_2 \sum_{j=0}^n \eta^{n-j} \alpha^{-j}$. Thus, for each i, $((1/\alpha)P)^n \gamma_i$ converges to zero in the space of currents. In particular, $((1/\alpha)P)^n \beta'$ converges to 0, and $((1/\alpha)P)^n \kappa$ converges towards S, so S is positive.

Let us prove that S is unique. Let T be a closed positive current such that $P_\nu(T) = \alpha T$ and $[T] = [S]$. Since $T - S$ is cohomologous to zero and is a difference of closed positive currents, according to [CD23c, Lemma 6.1], we can write $T - S = dd^c v$ where $v = u_1 - u_2$, and each u_i is an $(A\kappa_0)$-psh function (A depends only of the mass of S and T). Changing u_i into $u_i - \int_X u_i \mathsf{vol}$ we may assume that $\int_X u_i \mathsf{vol} = 0$ for $i = 1, 2$. From the invariance of $T - S$ under $(1/\alpha)P_\nu$ we obtain that

$$\frac{1}{\alpha} P_\nu(v) = v + c \tag{5.9}$$

for some constant $c \in \mathbf{R}$; thus, $\alpha^{-n} P_\nu^n(v) = v + c_n$ where c_n converges geometrically towards some $c_\infty \in \mathbf{R}$. From [CD23c, Lemma 6.5], there is a constant $C > 1$ such that

$$\int_X \frac{1}{\alpha^n} |P_\nu^n(v)| \mathsf{vol} \leq \frac{C}{\alpha^n} \int_X \log(C \|\mathrm{Jac}(f^{-1})\|_\infty) \, d\nu^{\star n}(f) \tag{5.10}$$

for all $n \geq 1$. Thanks to the moment condition (5.2) and the subadditivity property

$$\log(\|\mathrm{Jac}((f \circ g)^{-1})\|_\infty) \leq \log(\|\mathrm{Jac}(f^{-1})\|_\infty) + \log(\|\mathrm{Jac}(g^{-1})\|_\infty),$$

we see that

$$\begin{aligned} \int_X \log(\|\mathrm{Jac}(f^{-1})\|_\infty) \, d\nu^{\star n}(f) &= \int_X \log(\|\mathrm{Jac}(f_n^{-1} \circ \cdots \circ f_1^{-1})\|_\infty) \, d\nu(f_1) \cdots d\nu(f_n) \\ &\leq \sum_{j=1}^n \int_X \log(\|\mathrm{Jac}(f_j^{-1})\|_\infty) \, d\nu(f_1) \cdots d\nu(f_n) \\ &= O(n), \end{aligned}$$

so the right-hand side of the inequality (5.10) tends to 0 as n goes to $+\infty$. Hence, $\alpha^{-n} P_\nu^n(v)$ converges towards 0 in $L^1(X; \mathsf{vol})$. Since $\alpha^{-n} P_\nu^n(v) = v + c_n$ also converges towards $v + c_\infty$, we deduce that v is a constant, namely $v = -c_\infty$, and finally $T = S$, as was to be proved. □

Proof of Proposition 5.4 in case (ii). Here, we use the notation of Example 5.3. There are two closed positive currents T_f^+ and T_f^-, with continuous potentials, such that $f^*(T_f^\pm) = \lambda(f)^{\pm 1} T_f^\pm$; they are unique up to a positive scalar factor and their classes generate the isotropic lines $\mathbf{R}\theta_f^\pm$ (see [Can14, § 5]). By convention, we choose them so that $\langle [T_f^+] | [T_f^-] \rangle = 1$ or, equivalently, $T^+ \wedge T^-$ is a probability measure; to determine them uniquely we further require $\langle [T_f^+] | [\kappa_0] \rangle = \langle [T_f^-] | [\kappa_0] \rangle$. Beware that this normalization is different from that of $\theta_f^\pm$ so a priori $[T_f^\pm] \neq \theta_f^\pm$.

Let $w \in \mathbb{H}_X$ be of the form $w = a[T_f^+] + b[T_f^-]$, with $a, b > 0$, and set $S = aT_f^+ + bT_f^-$. Then S satisfies the invariance property $(1/\alpha) P_\nu S = S$, and its uniqueness as a fixed current of P_ν in the class w is obtained as in case (i). □

Remark 5.5. Note that, in case (ii), the measure $S \wedge S$ is equal to $T_f^+ \wedge T_f^-$ which is the measure of maximal entropy μ_f (see § 3.1 and [Can14, §§ 5.2, 8.2]).

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5.2.3 *Continuity properties of stationary currents.* We keep the same setting as above and consider a sequence of probability measures (ν_n) with supports contained in a fixed finite set $\{f_1, \dots, f_m\}$:

$$\nu_n = \sum_i \nu_n(f_i)\delta_{f_i}, \tag{5.11}$$

with coefficients in the simplex of dimension $m-1$ determined by the constraints $\nu_n(f_i) \geq 0$ and $\sum_i \nu_n(f_i) = 1$. Assume that (ν_n) converges towards $\nu_\infty = \sum_i \nu_\infty(f_i)\delta_{f_i}$, and that each Γ_{ν_n} is non-elementary. For $n \in \mathbf{N}$, we denote by $w_{\nu_n} \in \mathbb{H}_X$ the eigenvector of P_{ν_n} given by Lemma 5.2, and by S_n the current given by Proposition 5.4; we write $S_n = \Theta(w_{\nu_n}) + dd^c u_n$, as in Proposition 5.4. For the measure ν_∞, we make one of the following two assumptions:

(a) either Γ_{ν_∞} is non-elementary;
(b) or $\nu_\infty = \frac{1}{2}(\delta_f + \delta_{f^{-1}})$ for some loxodromic automorphism f, and w_{ν_n} converges to $w_{\nu_\infty} := (1/\sqrt{2})([T_f^+] + [T_f^-])$, with notation as in case (ii) of Proposition 5.4.

In both cases, Proposition 5.4 provides a unique closed positive current S_∞ such that $[S_\infty] = w_{\nu_\infty}$ and $P_{\nu_\infty} S_\infty = \alpha(\nu_\infty) S_\infty$; it coincides with $(1/\sqrt{2})(T_f^+ + T_f^-)$ in case (b).

In case (a), by the uniqueness assertion of Lemma 5.2, the classes w_{ν_n} converge towards w_{ν_∞}; in case (b) this convergence holds by assumption. Note that the corresponding constants $\alpha(\nu_n)$ converge as well.

Lemma 5.6. *Let X be a smooth compact Kähler surface. Under the above assumptions,*

(1) *the sequence of closed positive currents (S_n) converges towards S_∞;*
(2) *the canonical (continuous) potentials u_n converge uniformly to that of S_∞;*
(3) *the sequence of measures $\mu_n := S_n \wedge S_n$ converges towards $S_\infty \wedge S_\infty$.*

In case (b), *$S_\infty \wedge S_\infty$ is the unique measure of maximal entropy μ_f of f.*

Proof. The first assertion follows from the uniqueness of the current S obtained in Proposition 5.4, and the compactness of the space of currents of mass 1. The speed of convergence obtained in (5.7) shows that the sequence of potentials u_n is equicontinuous, and by the uniqueness of the normalized potentials, it follows that (u_n) converges uniformly to u_∞. Then the convergence of the sequence of measures (μ_n) follows from the continuity properties of wedge products of currents (see [Dem12, III.3.6]). Finally, the characterization of $S_\infty \wedge S_\infty$ in case (b) follows from Remark 5.5. □

5.2.4 *Singular setting.* In this section, we present a variation of Proposition 5.4 in which X is projective but singular; on the other hand, the class w is supposed to be ample. Two main issues have to be considered when X has singularities, the first is the existence of local potentials for positive closed currents, and the second is to control $\mathrm{Jac}(f)$. The notions we need are described in [BM14], [Dem85], and [GR65]: we present them for a (reduced and pure-dimensional) compact complex analytic surface Y. Let Y^{reg} and Y^{sing} denote its regular and singular parts. There is a finite cover of Y by open sets $Y_\alpha \subset Y$ such that each Y_α is isomorphic, via an embedding j_α, to an analytic subset of the unit ball in $\mathbf{C}^N$ (for some N). By definition, a (p,q)-form of class $\mathcal{C}^k$ on Y_α is a (p,q)-form on Y_α^{reg} which is the pull-back by j_α of a (p,q)-form of class $\mathcal{C}^k$ on the unit ball; a form is positive if it is induced by a positive form. By [BM14], Proposition 2.4.4, there are smooth functions $\varphi_\alpha \geq 0$ with compact support $\subset Y_\alpha$ such that $\sum_\alpha \varphi_\alpha = 1$ on Y. Let ω be the standard hermitian $(1,1)$-form of $\mathbf{C}^N$ and set $\kappa_\alpha := j_\alpha^* \omega$. Then $\kappa := \sum_\alpha \varphi_\alpha \kappa_\alpha$ is a

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smooth positive $(1,1)$-form on Y. As a volume form, we shall use $\mathsf{vol} := \kappa \wedge \kappa$. From [GR65, § 5.A, Theorems 14 and 16] and the compactness of Y, we get:

(1) the notion of (p,q)-form of class $\mathcal{C}^k$ does not depend on the local embedding j_α;
(2) if κ' is a positive hermitian form on Y, then $a^{-1}\kappa \leq \kappa' \leq a\kappa$ for some $a > 0$;
(3) if $f \in \mathsf{Aut}(Y)$, the jacobian determinant $\mathrm{Jac}(f)$ of f with respect to vol is a smooth, bounded function on Y^{reg}; in particular, $\log|\mathrm{Jac}(f)| \in L^1(Y, \mathsf{vol})$.

A $(1,1)$-current on Y_α is an element of the dual of the space of smooth $(1,1)$-forms (see [Dem85]). A current T on Y_α induces a current $(j_\alpha)_*T$ on $\mathbf{C}^N$ by the formula $\langle (j_\alpha)_*T|\phi\rangle := \langle T|(j_\alpha)^*\phi\rangle$; it is positive if $(j_\alpha)_*T$ is positive for every α. A function $u\colon Y_\alpha \to [-\infty, +\infty[$ is psh if it is the restriction to Y_α of a psh function on the unit ball of $\mathbf{C}^N$ (see [FN80, Theorem 5.3.1] or [Dem85]). If, furthermore, u is continuous, then u is the restriction of a continuous and psh function $\tilde{u}$ on the unit ball (see [Ric68, Satz 2.4]). If u is psh, the current dd^cu is defined on Y_α by

$$\langle dd^cu, \varphi\rangle := \int_{j_\alpha(Y)} \tilde{u}\, dd^c\tilde{\varphi} \tag{5.12}$$

for every form φ on Y_α induced by a $(1,1)$-test form $\tilde{\varphi}$ on $\mathbf{C}^N$; the integral is computed on the smooth part and is therefore equal to $\int_{Y_\alpha^{\mathrm{reg}}} u dd^c\varphi$ (see [Dem85]). A function defined on an open subset $U \subset Y$ is psh if it is psh on each $U \cap Y_\alpha$. A closed positive $(1,1)$-current T on Y has local (continuous) potentials if every point x has a neighborhood U on which $T = dd^cu$ for some (continuous) psh function on U; note that the existence of a local potential is a non-trivial property when $x \in Y^{\mathrm{sing}}$. If T has a continuous potential u in U, its wedge product with any closed $(1,1)$-current S on U is defined by

$$S \wedge T(\varphi) = \langle (j_\alpha)_*S|\tilde{u}\, dd^c\tilde{\varphi}\rangle \tag{5.13}$$

for every smooth real-valued function φ on $U \cap Y_\alpha$ (and any smooth extension $\tilde{\varphi}$ of φ to the unit ball). This is a positive measure on U, see [Dem12, Chapter III.3].

Example 5.7. Let Y be a (singular) complex projective surface, and $w \in \mathrm{NS}(Y;\mathbf{Q})$ be an ample class on Y. Then, there is an embedding $\iota\colon Y \to \mathbb{P}^N(\mathbf{C})$ such that the restriction of the Fubini–Study metric ω_{FS} to $\iota(Y)$ is a Kähler form representing a positive multiple of w. This form has smooth local potentials, because so does ω_{FS}, and there is a constant $a > 1$ such that $a^{-1}\kappa \leq \iota^*\omega_{FS} \leq a\kappa$.

Proposition 5.8. *Let X be a (singular) complex projective surface, and let vol be a smooth volume form on X. Let ν be a probability measure on $\mathsf{Aut}(X)$ satisfying the moment condition (5.1). Assume that we are in one of the situations (i) or (ii) of Proposition 5.4, and that some real positive multiple of w is an integral ample class. Then, there is a closed positive current S such that:*

(a) *$P_\nu S = \alpha S$ and $[S] = w$;*
(b) *this current has continuous potentials, $S = \Theta(w) + dd^c(u)$ for a unique continuous function u such that $\int_X u\, \mathsf{vol} = 0$;*
(c) *if ω is a Kähler form representing the class w, then $(\alpha^{-1}P_\nu)^n(\omega)$ converges towards S as n goes to $+\infty$.*

In particular, the product $S \wedge S$ is a well-defined probability measure on X. Furthermore, if T is another closed positive current, $P_\nu T = \alpha T$, and $T = S + dd^c(v)$ for some function v, then $T = S$.

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The proof of properties (a), (b), and (c) is the same as that of Proposition 5.4, but here we start with $\beta = s^{-1}\iota^*\omega_{FS}$ instead of an arbitrary smooth form, where $\iota^*\omega_{FS}$ is as in Example 5.7 and $s \in \mathbf{R}_+^*$ is chosen so that sw belongs to $\mathrm{NS}(X;\mathbf{Z})$ and is very ample.

Remark 5.9. Inspecting the proof reveals that *our construction coincides with Kawaguchi's construction in* [Kaw06, Theorem 3.2.1]. In other words, the current S coincides with the current T of [Kaw06, Theorem 3.2.1], except that Kawaguchi assumed X to be smooth to avoid working with Kähler forms on singular surfaces.

The only point that remains to be explained is the validity of the volume argument in the uniqueness assertion of the proposition. First, as observed above, the Jacobian of an automorphism is uniformly bounded. Second, we have to extend [CD23c, Lemma 6.5] to the singular case. The key step of this lemma is the fact that for a quasi-psh function u, there is a constant $c > 0$ such that $\mathsf{vol}\{|u| \geq t\} \leq c\exp(-t/c)$ (see [Kis00]). To extend this result to the singular case it is enough to deal with a local psh function u on a singular surface Y_α in the unit ball of $\mathbf{C}^N$. Consider a linear projection $\pi\colon Y_\alpha \to \mathbf{C}^2$ onto a 2-dimensional plane which is a finite ramified cover. Then, one can define $\pi_*(u)$ outside the ramification locus R_π of π, by $\pi_*(u)(x) = \sum_{y\in Y_\alpha, \pi(y)=x} u(y)$. This function is psh, and it is bounded, because u is locally bounded; so, it extends through R_π as a psh function. In particular, $\pi_*(u)$ satisfies $\mathsf{vol}_{\mathbf{C}^2}\{|\pi_*(u)| \geq t\} \leq c_\pi \exp(-t/c_\pi)$ for some $c_\pi > 0$. Choosing finitely many such projections π_j to ensure that $\sum_j \pi_j^*\mathsf{vol}_{\mathbf{C}^2}$ is a volume form on Y_α, we obtain $\mathsf{vol}\{|u| \geq t\} \leq c\exp(-t/c)$ for some $c > 0$, as desired.

5.3 Rational invariant classes

We now construct sequences of probability measures for which the fixed classes w_{ν_n} have good positivity and integrality properties; the last assertion makes use of the contraction $\pi_0\colon X \to X_0$ constructed in Proposition 3.9.

Proposition 5.10. *Let X be a smooth complex projective surface and Γ be a non-elementary subgroup of $\mathsf{Aut}(X)$ such that Π_Γ is defined over $\mathbf{Q}$. Let f be a loxodromic element of Γ. There is a sequence (ν_n) of probability measures on $\mathsf{Aut}(X)$ such that:*

(1) *the support $\mathrm{Supp}(\nu_n)$ is a finite subset F of Γ that does not depend on n and generates a non-elementary subgroup of Γ containing f;*
(2) *$\nu_n(g)$ is a positive rational number for all $g \in F$;*
(3) *the unique eigenvector w_{ν_n} of P_{ν_n} in $\mathbb{H}_X$ is an element of $\mathbf{R}_+\mathrm{NS}(X;\mathbf{Z})$;*
(4) *the corresponding eigenvalue $\alpha(\nu_n)$ belongs to $\mathbf{Q}_+\cap]1,+\infty[$;*
(5) *ν_n converges to the measure $\frac{1}{2}(\delta_f + \delta_{f^{-1}})$ and w_{ν_n} converges to $(1/\sqrt{2})([T_f^+] + [T_f^-])$.*

If Γ contains a parabolic element g, one can furthermore assume that g belongs to F and that $w_{\nu_n} \in \mathbf{R}_+\pi_0^*[A_n]$ for some ample line bundle A_n on X_0.

Proof. For the proof we use the conventions of § 3.1.2, in particular the classes $\theta_f^\pm$, which can be defined by $\theta_f^\pm = \langle T_f^\pm | [\kappa_0]\rangle^{-1}[T_f^\pm]$.

Step 1. Since the representation of Γ on Π_Γ is irreducible, it is also irreducible over $\mathbf{C}$. Indeed, if W is a proper, Γ-invariant, complex subspace of $\Pi_\Gamma \otimes_{\mathbf{R}} \mathbf{C}$, then W does not contain any non-zero real vector $u \in H^{1,1}(X;\mathbf{R})$; in particular, it does not contain any isotropic eigenvector of any loxodromic element of Γ. This implies that W is contained in the orthogonal complement $(\theta_h^+)^\perp$ for all $h \in \Gamma_{\mathrm{lox}}$. But in Π_Γ the intersection $\bigcap_{h\in\Gamma_{\mathrm{lox}}} (\theta_h^+)^\perp$ is defined over $\mathbf{R}$ and is Γ-invariant, so it is trivial.

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Thus, according to Burnside's theorem (see [HR80]), Γ contains a basis of the real vector space $\mathsf{End}(\Pi_\Gamma)$. More precisely, one can find a basis $(f_1^*, f_2^*, \ldots, f_N^*)$ with $N = (\dim \Pi_\Gamma)^2$ such that $f_i \in \Gamma$ for all i, $f_1 = f$ and $f_2 = f^{-1}$ (indeed, f and f^{-1} are linearly independent endomorphisms). In particular, the set of linear combinations $\sum_i \alpha_i f_i^*$ with $\alpha_i \geq 0$ contains a non-empty, open, and convex cone of $\mathsf{End}(\Pi_\Gamma)$.

If Γ contains a parabolic element g we can require that g belongs to the basis, because f^*, g^*, and $(f^{-1})^*$ are linearly independent: indeed, g^* preserves a unique isotropic line, while any linear combination of f^* and $(f^{-1})^*$ preserves at least two isotropic lines.

Step 2. Set $F = \{f_1, f_2, \ldots, f_N\}$ and $\Delta_N = \{(\nu_i) \in \mathbf{R}_+^N;\ \sum_i \nu_i = 1\}$. Let Δ_N° be the interior of this simplex. Points in Δ_N° correspond to probability measures $\nu = \sum_i \nu_i \delta_{f_i}$ whose support is equal to F. When $\nu \in \Delta_N^\circ$, Γ_ν is non-elementary; so, by Lemma 5.2, P_ν has a unique fixed point w_ν in $\mathbb{H}_X$. As a consequence, the map $\nu \in \Delta^\circ \mapsto w_\nu$ is continuous. If U is an open subset of Δ_N°, and ν_0 is in U, then the image of U by $\nu \mapsto P_\nu$ is an open subset U' of the hyperplane

$$\left\{ P \in \mathsf{End}(\Pi_\Gamma);\ P = \sum \alpha_i f_i,\ \sum \alpha_i = 1 \right\} \subset \mathsf{End}(\Pi_\Gamma). \tag{5.14}$$

Now, take a vector w' in $\mathbb{H}_X \cap \Pi_\Gamma$ near w_{ν_0}. Since $P_{\nu_0}(\mathbb{P}(w'))$ is near $\mathbb{P}(w')$, there is a $B \in \mathsf{GL}(\Pi_\Gamma)$ close to id such that $P_{\nu_0}(\mathbb{P}(w')) = B^{-1}(\mathbb{P}(w'))$. Thus, $B \circ P_{\nu_0}$ is close to P_{ν_0} and fixes $\mathbb{P}(w')$. A positive multiple of $B \circ P_{\nu_0}$ belongs to U', hence is of the form $P_{\nu'}$ for some $\nu' \in U$. Then, w' is the leading eigenvector of $P_{\nu'}$ and w' is in the image of $\nu \mapsto w_\nu$. This shows that $\nu \mapsto w_\nu$ contains an open set around w_{ν_0}.

Step 3. Now, consider a sequence (ν_n) of elements of Δ_N° converging to $a\delta_f + (1-a)\delta_{f^{-1}}$, with $0 < a < 1$. Normalize the fixed point w_{ν_n} by setting $\overline{w}_n := \langle w_{\nu_n} | [\kappa_0] \rangle^{-1} w_{\nu_n}$, so that $\langle \overline{w}_n | [\kappa_0] \rangle = 1$ and $\overline{w}_n$ stays in a compact subset of $H^{1,1}(X; \mathbf{R})$. If $\overline{w}_{n_j}$ converges to $\overline{w}$ along a subsequence (n_j) the limit is a nef eigenvector of the operator $af^* + (1-a)(f^{-1})^*$ associated to an eigenvalue ≥ 1. Thus, if a is small the limit must be equal to θ_f^- and the sequence $(\overline{w}_n)$ converges towards this eigenvector (see Example 5.3). Conversely, if $1-a$ is small, then the limit is θ_f^+. The subset

$$\Delta_N^\circ(\varepsilon) = \{(\nu_i) \in \Delta_N^\circ;\ \nu_i \leq \varepsilon,\ \forall i \geq 3\} \tag{5.15}$$

is connected. Thus, the closure of its image by the continuous map $\nu \mapsto \langle w_\nu | [\kappa_0] \rangle^{-1} w_\nu$ is a compact and connected subset of Π_Γ, and the intersection of these compact sets is also connected. This set is contained in the segment $[\theta_f^-, \theta_f^+]$ because it is contained in $\overline{\mathrm{Pos}(X)}$ and in the union of eigenvectors of $af^* + (1-a)(f^{-1})^*$, for $a \in [0,1]$. Since it contains the endpoints of this segment, it actually coincides with it. From this we deduce that there exists a sequence of probability measures $\nu_n \in \Delta_N^\circ$ such that $\langle w_{\nu_n} | [\kappa_0] \rangle^{-1} w_{\nu_n}$ converges to the class $\frac{1}{2}(\theta_f^+ + \theta_f^-)$, hence:

– w_{ν_n} converges to the class $(1/\sqrt{2})([T_f^+] + [T_f^-])$.

Then Example 5.3, the proof of case (ii) of Proposition 5.4, and Lemma 5.6 show that:

– P_{ν_n} converges towards $\frac{1}{2}(f^* + (f^{-1})^*)$;
– $\alpha(\nu_n)$ converges towards $\alpha(f) = \frac{1}{2}(\lambda(f) + 1/\lambda(f))$.

Step 4. At this stage the coefficients $\nu_n(f_i)$ and the eigenvalues $\alpha(\nu_n)$ are positive real numbers. Let U_n be a small open neighborhood of $\nu_n = (\nu_n(f_i))$ in Δ_N°. By Step 2, the image of $\nu' \in U_n \mapsto w_{\nu'}$ contains a neighborhood of w_{ν_n} in $\Pi_\Gamma \subset \mathrm{NS}(X; \mathbf{R})$. Thus, after a small perturbation of ν_n we may assume that $w_{\nu_n} \in \mathbf{R}_+ \mathrm{NS}(X; \mathbf{Z})$. According to Proposition 3.9 and Remark 3.11, when Γ contains parabolic elements, we may further choose w_{ν_n} to be proportional to the pullback $[\pi_0^* A_n]$ of an ample class.

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The equation satisfied by w_{ν_n} is $\alpha(\nu_n)w_{\nu_n} = \sum_i \nu_n(f_i) f_i^*(w_{\nu_n})$. Write $w_{\nu_n} = \eta_n \tilde{w}_n$ for some $\tilde{w}_n$ in $\mathrm{NS}(X;\mathbf{Q})$ and η_n in $\mathbf{R}_+$; the equation becomes

$$\alpha(\nu_n)\tilde{w}_n = \sum_{i=1}^{N} \nu_n(f_i) f_i^*(\tilde{w}_n). \tag{5.16}$$

This is a linear relation of the form $\beta_0 \tilde{w}_n = \sum_i \beta_i f_i^*(\tilde{w}_n)$, where $\tilde{w}_n$ and the $f_i^*(\tilde{w}_n)$ belong to $\mathrm{NS}(X;\mathbf{Q})$ and the β_i are positive real numbers (with $\beta_0 > 1$). Thus, given any $\varepsilon > 0$, there is a relation of the form $\tilde{\beta}_0 \tilde{w}_n = \sum_i \tilde{\beta}_i f_i^*(\tilde{w}_n)$ where the coefficients $\tilde{\beta}_i$ are rational numbers which are ε-close to the original β_i. This proves that we can perturb ν_n one more time to insure that the $\nu_n(f_i)$ and $\alpha(\nu_n)$ are positive rational numbers. □

5.4 Arithmetic equidistribution

Here, X is a normal projective surface (possibly singular) and both X and the subgroup Γ of $\mathsf{Aut}(X)$ are defined over some number field $\mathbf{k} \subset \overline{\mathbf{Q}}$. For y in $X(\overline{\mathbf{Q}})$, let m_y denote the uniform probability measure supported on the Galois orbit of y,

$$m_y = \frac{1}{\deg(y)} \sum_{y' \in \mathrm{Gal}(\overline{\mathbf{Q}}:\mathbf{k})(y)} \delta_{y'}; \tag{5.17}$$

here, $\deg(y)$ is the degree of the closed point defined by y or, equivalently, the cardinality of the orbit of y under the action of the Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}:\mathbf{k})$, and the sum ranges over all points y' in this orbit. A sequence (x_j) of points of $X(\overline{\mathbf{Q}})$ is *generic* if the only Zariski-closed subset of X containing infinitely many of the x_j is X. Equivalently, (x_j) converges to the generic point of X for the Zariski topology.

Theorem 5.11. *Let X be a normal projective surface defined over a number field $\mathbf{k}$. Let ν be a probability measure on $\mathsf{Aut}(X_{\mathbf{k}})$ with finite support F and rational weights $\nu(f) \in \mathbf{Q}_+$, for f in F, and such that Γ_ν is non-elementary. Assume that:*

(i) *the class w_ν such that $P_\nu^* w_\nu = \alpha(\nu) w_\nu$ is ample and contained in $\mathrm{NS}(X;\mathbf{Q})$;*
(ii) *$(x_j) \in X(\overline{\mathbf{Q}})^{\mathbf{N}}$ is a generic sequence such that each x_j is a periodic point of Γ_ν.*

Then, the sequence of probability measures (m_{x_j}) converges towards the measure $S_\nu \wedge S_\nu$, and this measure is Γ_ν-invariant.

Here w_ν is given by Lemma 5.2 and S_ν is the current associated to w_ν by Proposition 5.8. It is important that w_ν be a rational class, that is $w_\nu \in \mathrm{NS}(X;\mathbf{Q})$ instead of just $\mathrm{NS}(X;\mathbf{R})$, since we rely on results of Kawaguchi, Yuan, and Zhang that require this assumption. It is also crucial that X is not supposed to be smooth because this result will be applied to the model X_0 constructed in § 3.3.

Remark 5.12. By [Kaw08, Lee12], Theorem 5.11 also holds when Γ_ν is elementary (with essentially the same proof).

Example 5.13. Under the assumption of Theorem 5.11, assume furthermore that X is an abelian surface. Since Γ_ν has a periodic point x_1, the stabilizer $\Gamma_{x_1} = \mathrm{Stab}_{\Gamma_\nu}(x_1)$ has finite index in Γ_ν; conjugating by a translation we can take x_1 as the neutral element for the group law of $X \simeq \mathbf{C}^2/\Lambda$. Then, the periodic points of Γ_{x_1} (and of Γ_ν) are exactly the torsion points of X (see § 4.1). By the equidistribution theorem of Szpiro, Ullmo, and Zhang, the measures m_{x_j} converge towards the Haar measure of X (see [SUZ97]). In addition, Γ_{x_1} is induced by a subgroup $\tilde{\Gamma}_{x_1}$ of $\mathsf{GL}_2(\mathbf{C})$ preserving the lattice Λ, and Γ_ν is a group of affine transformations with linear part

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given by $\tilde{\Gamma}_{x_1}$ and translation part given by the finite subset $\Gamma_\nu(0) \subset X$. Every cohomology class u in $H^{1,1}(X;\mathbf{R})$ has a distinguished representative, given by the unique translation invariant $(1,1)$-form ω_u on X such that $[\omega_u] = u$. Since Γ_ν acts by affine automorphisms, the operator $(1/\alpha)P_\nu$ preserves ω_{w_ν}, and $S_\nu = \omega_{w_\nu}$. Thus, for abelian surfaces, Theorem 5.11 corresponds to the theorem of Szpiro, Ullmo, and Zhang together with the fact that $\omega_{w_\nu} \wedge \omega_{w_\nu}$ is equal to the volume form inducing the Haar measure on X.

Proof of Theorem 5.11. For notational ease, set $\Gamma = \Gamma_\nu$, $w = w_\nu$, $\alpha = \alpha(\nu)$. Let $\pi\colon Y \to X$ be a minimal resolution; by [BHPV04, § III.6, Theorems (6.1) and (6.2)], it is unique up to isomorphism and Γ lifts to a subgroup Γ_Y of $\mathsf{Aut}(Y)$. We shall also consider ν as a measure on Γ_Y. The pull-back $\pi^*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is an isometric embedding for the intersection form; moreover, $\pi^*\mathrm{NS}(X;\mathbf{R})$ is Γ_Y-invariant and contains classes with positive self-intersection. Thus, Γ_Y and Γ are both non-elementary. Moreover, $P_\nu(\pi^* w) = \alpha\pi^* w$ in $\mathrm{NS}(Y;\mathbf{R})$, so $\pi^* w$ coincides with the unique eigenclass provided by Lemma 5.2 in $\mathrm{NS}(Y;\mathbf{R})$.

Case 1. Assume that $\mathrm{Pic}^0(X)$ is non-trivial; equivalently, $\mathrm{Pic}^0(Y)$ is non-trivial. Since Γ_Y contains a loxodromic element, we deduce from [Can14, Theorem 10.1] that Y is a blow-up of an abelian surface (for $\mathrm{Pic}^0(Y)$ is trivial when Y is birationally equivalent to a rational, K3, or Enriques surfaces). But then X is smooth and is a blow-up of an abelian surface. If X itself is not an abelian surface, the exceptional divisor E of the blow-up is Γ-invariant. From this invariance we get $\alpha\langle w|[E]\rangle = \langle P_\nu w|[E]\rangle = \langle w|[E]\rangle$, but since w is ample and $\alpha > 1$, this is a contradiction. Therefore, X is abelian, and Theorem 5.11 follows from the discussion in Example 5.13.

Case 2. Assume now that $\mathrm{Pic}^0(X) = 0$. The proof is based on standard ideas from arithmetic equidistribution theory. For the reader's convenience we provide background and details (see also [Lee12] for the applicability of arithmetic equidistribution in this context). Changing w into a multiple, we assume $w \in \mathrm{NS}(X;\mathbf{Z})$. Multiplying the equation $\sum_f \nu(f) f^*(w) = \alpha w$ by the least common multiple b of the denominators, we obtain the linear relation

$$\sum_{f\in F} n(f) f^*(w) = dw; \tag{5.18}$$

in which $d = b\alpha$ and the coefficients $n(f) = b\nu(f)$ are positive integers such that

$$\sum_{f\in F} n(f) = b < d = b\alpha \tag{5.19}$$

because $\alpha > 1$. Denote by D a divisor with class w, and by L the line bundle $\mathcal{O}_X(D)$. Since $\mathrm{Pic}^0(X)$ is trivial, the Equality (5.18) implies

$$\bigotimes_{f\in F} (f^* L)^{\otimes n(f)} = L^{\otimes d} \tag{5.20}$$

up to an isomorphism of line bundles that we do not specify. From this identity, Kawaguchi constructs in [Kaw06, § 1] a function $\hat{h}_L\colon X(\overline{\mathbf{Q}}) \to \mathbf{R}_+$ which satisfies the relation $\sum_f n(f)\hat{h}_L \circ f = d\hat{h}_L$ and differs from the naive Weil height function associated to L only by a bounded error. It will be referred to as the *canonical stationary height* associated to ν and L. This height function can be decomposed as a sum of continuous local height functions, see [Kaw06, § 4]. Arakelov theory also provides a canonical adelic metric on (X, L); for each place v of $\mathbf{k}$, there is a metric $|\cdot|_v$ on the line bundle $(X_{\mathbf{k}_v}, L_{\mathbf{k}_v})$, where $\mathbf{k}_v$ is an algebraic closure of the v-adic

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completion of $\mathbf{k}$, such that

$$\prod_{f \in F} |s(f(x))|_v^{n(f)} = |s(x)|_v^d \tag{5.21}$$

for every local section s of L defined over $\mathbf{k}$. In our setting, an embedding $\mathbf{k} \subset \mathbf{C}$ is fixed; it corresponds to one of the places of $\mathbf{k}$. The adelic metric corresponding to that place gives a continuous metric on L. By construction, the curvature current of the metric is precisely the current S, here denoted by S_ν, constructed in Proposition 5.8 (see Remark 5.9).

Lemma 5.14. *A point $x \in X(\overline{\mathbf{Q}})$ satisfies $\hat{h}_L(x) = 0$ if and only if its Γ_ν-orbit is finite.*

Proof (see [Kaw06, Proposition 1.3.1.]). Let $\mathbf{k}'$ be any number field containing $\mathbf{k}$. The set $\{x \in X(\mathbf{k}');\ \hat{h}_L(x) = 0\}$ is Γ_ν-invariant and by Northcott's theorem it is finite, so every element of that set has a finite orbit. Let us prove the other implication. Iterating the relation $\sum_f n(f)\hat{h}_L \circ f = d\hat{h}_L$ and evaluating it on a periodic point x yields $\alpha^n \hat{h}_L(x) = \sum_{g \in \Gamma} \nu^{\star n}(g)\hat{h}_L(g(x))$ where $\nu^{\star n}$ is the nth convolution of ν. The right-hand side is bounded because $\hat{h}_L(g(x))$ takes only finitely many values, and on the left-hand side the term α^n goes to $+\infty$; thus, $\hat{h}_L(x) = 0$, as asserted. ☐

Let $\mathbf{A}_\mathbf{k}$ denote the ring of adèles of the number field $\mathbf{k}$. The sections of L defined over $\mathbf{k}$ determine a lattice $H^0(X, L)$ in $H^0(X, L) \otimes \mathbf{A}_\mathbf{k}$, and the quotient $(H^0(X, L) \otimes \mathbf{A}_\mathbf{k})/H^0(X, L)$ is therefore compact. Denote by $\overline{L}$ the line bundle L endowed with its canonical adelic metric. For each place v, denote by $B_v \subset H^0(X, L) \otimes \mathbf{k}_v$ the unit ball with respect to the v-adic component $|\cdot|_v$ of the adelic metric of $\overline{L}$. Let λ_L be a Haar measure on $H^0(X, L) \otimes \mathbf{A}_\mathbf{k}$. The quantity

$$\chi(X, \overline{L}) = \log \frac{\lambda_L(\prod_{v \in M_\mathbf{k}} B_v)}{\lambda_L(H^0(X, L) \otimes \mathbf{A}_\mathbf{k}/H^0(X, L))} \tag{5.22}$$

does not depend on the choice of Haar measure. Taking tensor products, we get a sequence of adelic metrized line bundles $(\overline{L}^{\otimes n})_{n \geq 1}$, and by definition the arithmetic volume of $\overline{L}$ is

$$\widehat{\mathsf{vol}}_\chi(X, \overline{L}) = \limsup_{n \to +\infty} \frac{\chi(X, \overline{L}^{\otimes n})}{n^3/6}. \tag{5.23}$$

This is to be compared with the usual volume $\mathsf{vol}(X, L)$ of L, which by definition is the limsup of $(2/n^2)h^0(X, L^{\otimes n})$, as n tends to $+\infty$. A fundamental inequality of Zhang asserts that if (x_j) is a generic sequence in $X(\overline{\mathbf{Q}})$,

$$\liminf_j\ \hat{h}_L(x_j) \geq \frac{\widehat{\mathsf{vol}}_\chi(X, \overline{L})}{3\mathsf{vol}(X, L)}. \tag{5.24}$$

This follows from an adelic version of the Minkowski theorem on the existence of integer points in lattices (see, e.g., [CT09, Lemma 5.1] or [Zha95, Theorem 1.10]).

As for the usual volume, the arithmetic volume can be interpreted in terms of arithmetic intersection. Indeed, to $\overline{L}$ is associated an arithmetic degree $\widehat{\deg}(c_1(\overline{L})^3)$, and it is shown in [Zha95] that $\widehat{\mathsf{vol}}_\chi(X, \overline{L}) = \widehat{\deg}(c_1(\overline{L})^3) \geq 0$ (see also [Kaw06, Theorem 2.3.1]). Thus, the existence of a generic sequence of periodic points (x_j) shows that $\widehat{\mathsf{vol}}_\chi(X, \overline{L}) = 0$ and $\hat{h}_L(x_j) = \widehat{\mathsf{vol}}_\chi(X, \overline{L})$ for all j.

We are now in a position to apply Yuan's equidistribution theorem (see [Yua08, BB10]): the sequence of measures (m_{x_j}) converges towards the probability measure $S_\nu \wedge S_\nu$ as j goes to ∞. If f is any element of Γ_ν, the points $f(x_j)$ also form a generic sequence of Γ-periodic points. Since the actions of Γ and $\mathrm{Gal}(\overline{\mathbf{Q}} : \mathbf{k})$ commute, we infer that $f_*(m_{x_j}) = m_{f(x_j)}$, so taking the limit as $j \to \infty$ yields $f_*(S_\nu \wedge S_\nu) = S_\nu \wedge S_\nu$, and finally $S_\nu \wedge S_\nu$ is Γ_ν-invariant. ☐

5.5 Density of active saddle periodic points

Let f be a loxodromic automorphism of X. We say that a periodic point of f is *active* if it is contained in the support of the measure of maximal entropy μ_f. From [Duj06, Can14] we know that a saddle periodic point that is not contained in any f-periodic curve is active (see [Can14, Theorem 8.2]).

Theorem 5.15. *Let X be a compact Kähler surface and let Γ be a non-elementary subgroup of $\mathsf{Aut}(X)$ that contains a parabolic automorphism. Then, given any non-empty open subset $\mathcal{V} \subset X$ (for the Euclidean topology), there exists a point $x \in \mathcal{V}$ and a loxodromic element $f \in \Gamma$ such that x is an active saddle periodic point of f. In particular, the union of the supports of the measures μ_f, for $f \in \Gamma_{\rm lox}$, is dense in X.*

Before proceeding to the proof, let us point out the following lemma, which readily follows from Lemma 3.5, together with the fact that an irreducible curve with negative self-intersection is determined by its class in $\mathrm{NS}(X;\mathbf{Z}) \subset \mathrm{NS}(X;\mathbf{R})$.

Lemma 5.16. *Let U and U' be two disjoint open subsets of $\mathbb{P}(\mathrm{NS}(X;\mathbf{R}))$ containing nef classes. Set*

$$A(U,U') = \{f \in \mathsf{Aut}(X);\ f \text{ is loxodromic, } \mathbb{P}([T_f^+]) \in U \text{ and } \mathbb{P}([T_f^-]) \in U'\}.$$

Then, the union of all periodic curves of all elements of $A(U,U')$ is a finite set of curves.

Proof of Theorem 5.15. Pick g in $\Gamma_{\rm par}$. Since Γ is non-elementary we can conjugate g by an element of $\Gamma_{\rm lox}$ to produce a pair $g, h \in \Gamma_{\rm par}$ with distinct fixed points in $\partial\mathbb{H}_X$.

Step 1. Assume that X is a blow-up of an abelian surface A, and pick f in $\Gamma_{\rm lox}$. By Lemma 4.3, its periodic points are dense, and all of them are active because μ_f is the pull-back to X of the Haar measure on A. Thus, any open subset of X contains active saddle periodic points.

From now on, assume that X is not a blow-up of an abelian surface.

Step 2. From § 3.1, g preserves a unique fibration $\pi_g\colon X \to B_g$ and the automorphism induced by g on B_g is periodic. Replacing g by some iterate, we assume that $\pi_g \circ g = \pi_g$. Let $\mathcal{U} \subset B_g$ be a small disk containing no critical value of π_g. There is a real analytic diffeomorphism $\Phi\colon \pi_g^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathbf{R}^2/\mathbf{Z}^2$ and a real analytic map $\varphi\colon \mathcal{U} \to \mathbf{R}^2$ such that $\pi_{\mathcal{U}} \circ \Phi = \pi_g$ and $g_\Phi := \Phi \circ g \circ \Phi^{-1}$ satisfies

$$g_\Phi(b,z) = (b, z + \varphi(b)) \tag{5.25}$$

for all points $(b,z) \in \mathcal{U} \times \mathbf{R}^2/\mathbf{Z}^2$. According to [Can01b, CD23b], φ is generically of maximal rank: there is a finite set $Z \subset \mathcal{U}$ such that $(D\varphi)_b\colon T_b\mathcal{U} \to \mathbf{R}^2$ has rank 2 for every $b \in \mathcal{U} \setminus Z$; hence, $\{b \in \mathcal{U};\ \varphi(b) \in \mathbf{Q}^2/\mathbf{Z}^2\}$ is dense in $\mathcal{U}$. If $\varphi(b) = (a_0/N, b_0/N)$ for some integers a_0, b_0, and N, then every point $q = (b,z)$ in the fiber is fixed by g_Φ^N and

$$(Dg_\Phi^N)_x = \begin{pmatrix} \mathrm{id}_2 & 0 \\ N(D\varphi)_b & \mathrm{id}_2 \end{pmatrix}. \tag{5.26}$$

Thus, in any holomorphic coordinate system (x,y) in which π_g expresses as $\pi_g(x,y) = x$, the differential of g^N at the fixed point $\Phi^{-1}(q)$ is of the form $\left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right)$ with $a \neq 0$.

Step 3. The invariant fibrations π_g and π_h are transversal in the complement of a proper Zariski-closed set $\mathrm{Tang}(\pi_g, \pi_h)$. According to Lemmas 5.16 and 3.13, we can find an integer $N > 0$, and a divisor $F \subset X$ such that all elements $g^{\ell N} \circ h^{\ell N}$ with $\ell \geq 1$ are loxodromic and do not have any periodic curve outside F.

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Step 4. Let D be the union of the singular and multiple fibers of π_g and of π_h, of $\mathrm{Tang}(\pi_g, \pi_h)$, and of the divisor F; D is a divisor of X. Let $\mathcal{V}$ be an open subset of X. Then $\mathcal{V}$ contains a small ball $\mathcal{V}'$ such that:

- $\mathcal{V}'$ does not intersect D;
- $\pi_g(\mathcal{V}')$ and $\pi_h(\mathcal{V}')$ are topological disks $\mathcal{U}_g$ and $\mathcal{U}_h$ in B_g and B_h, respectively;
- there are local coordinates (x, y) in $\mathcal{V}'$ (respectively, x in $\mathcal{U}_g$ and y in $\mathcal{U}_h$) such that $(\pi_g)_{|\mathcal{V}'}(x, y) = x$ and $(\pi_h)_{|\mathcal{V}'}(x, y) = y$.

Step 2 provides a point $(x_0, y_0) \in \mathcal{V}'$ and an integer $N > 0$ such that g^N fixes the fiber of π_g through (x_0, y_0) pointwise, h^N fixes the fiber of π_h through (x_0, y_0) pointwise, and

$$(Dg^N)_{(x_0,y_0)} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \quad \text{and} \quad (Dh^N)_{(x_0,y_0)} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \tag{5.27}$$

for some non-zero complex numbers a and b. If $\ell \in \mathbf{Z}$ is sufficiently large, $f_{\ell N} = (Dg^{\ell N})_{(x,y)} \circ (Dh^{\ell N})_{(x,y)}$ is a loxodromic automorphism, (x_0, y_0) is a fixed point of $f_{\ell N}$ which is not contained in a periodic curve of $f_{\ell N}$ (because (x_0, y_0) is not in F), which is shown to be a saddle by an explicit computation. Thus, as explained before the proof, (x_0, y_0) is active, and we are done. □

5.6 Measure rigidity and Kummer examples

THEOREM 5.17. *Let X be a smooth complex projective surface and let Γ be a subgroup of* $\mathsf{Aut}(X)$. *Assume that:*

(i) *X and Γ are defined over a number field $\mathbf{k} \subset \overline{\mathbf{Q}}$;*
(ii) *Γ is non-elementary and contains a parabolic automorphism.*

If Γ has a Zariski-dense set of finite orbits, then every loxodromic automorphism in Γ is a Kummer example.

Proof. Step 1: from the Zariski-dense set of finite orbits we can extract a generic sequence of Γ-periodic points $(x_j) \in X(\overline{\mathbf{Q}})^{\mathbf{N}}$. Since Γ is non-elementary, it contains a loxodromic element f. The isolated periodic points of f are defined over $\overline{\mathbf{Q}}$, because X and f are defined over $\overline{\mathbf{Q}}$, and the non-isolated periodic points of f form a finite number of f-periodic curves (see § 3.2). Thus, we can find a Zariski-dense set of Γ-periodic points x_i' in $X(\overline{\mathbf{Q}})$. If $Z \subset X$ is an irreducible curve that contains infinitely many of the x_i', then Z is defined over $\overline{\mathbf{Q}}$ too. There are only countably many curves defined over $\overline{\mathbf{Q}}$. Thus, by a diagonal argument, we find an infinite sequence of periodic points $x_j \in X(\overline{\mathbf{Q}})$ such that (x_j) is generic.

In what follows, (x_j) denotes such a generic sequence of periodic points. Consider the contraction $\pi_0 \colon X \to X_0$ of the union D_Γ of all Γ-periodic curves (see Proposition 3.9); the group Γ also acts on the normal projective surface X_0. Note that the projection $(\pi_0(x_j)) \in X_0(\overline{\mathbf{Q}})^{\mathbf{N}}$ is also generic.

Step 2: there exists a Γ-invariant measure μ such that $\mu_f = \mu$ for all loxodromic f. Fix an arbitrary element f in Γ_{lox}. By [CD23c, Lemma 2.9], Π_Γ is defined over $\mathbf{Q}$ so applying Proposition 5.10 we obtain a sequence of probability measures (ν_n). Denote by S_{ν_n} and $S_{\nu_n,0}$ the currents, on X and X_0, respectively, given by Propositions 5.4 and 5.8; by construction $\pi_0^* S_{\nu_n,0} = S_{\nu_n}$, where the pull-back is obtained by locally pulling back the continuous potentials.

Fix an integer $n \geq 1$. Theorem 5.11 shows that the sequence of probability measures $m_{\pi_0(x_j)}$ converges towards $S_{\nu_n,0} \wedge S_{\nu_n,0}$ as j goes to $+\infty$. Therefore, $S_{\nu_n,0} \wedge S_{\nu_n,0}$ coincides with the Γ-invariant probability measure $\mu_0 := \lim_j m_{\pi_0(x_j)}$ and does not depend on n. Since $S_{\nu_n,0}$ has continuous potentials, this measure gives no mass to proper analytic subsets of X_0. Let μ be

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the probability measure which is equal to $\pi_0^*(\mu_0)$ on $X \setminus D_\Gamma$ and gives no mass to D_Γ. Since S_{ν_n} has continuous potentials, $\mu = S_{\nu_n} \wedge S_{\nu_n}$. In X, the sequence (m_{x_j}) converges to μ. Indeed, if a subsequence of (m_{x_j}) converges towards some measure λ. Then $(\pi_0)_*\lambda = \mu_0$, and since μ_0 does not charge $X_0^{\rm sing}$, we infer that λ is equal to $\pi_0^*(\mu_0)$ on $X \setminus D_\Gamma$ and does not charge D_Γ, so $\lambda = \mu$. Thus, by compactness of the set of probability measures, m_{x_j} converges towards μ.

Now, we let $n \to \infty$. By Proposition 5.10(5), Proposition 5.4, and Lemma 5.6, $S_{\nu_n} \wedge S_{\nu_n} = \mu$ converges towards μ_f as n goes to $+\infty$. Thus $\mu = \mu_f$ for all loxodromic elements f in Γ. In particular, μ is f-ergodic, hence Γ-ergodic.

Step 3: conclusion. As already explained, μ gives no mass to proper algebraic subsets of X. Furthermore, Theorem 5.15 implies that the support of μ is equal to X. Thus, Theorem A of [CD23b] shows that μ is absolutely continuous with a smooth density. Since $\mu = \mu_f$, the Main Theorem of [CD20] implies that (X, f) is a Kummer example, as was to be shown. □

5.7 From Kummer examples to Kummer groups

Proposition 5.18. *Let X be a compact Kähler surface, and Γ be a non-elementary subgroup of $\mathsf{Aut}(X)$. Let $f \in \Gamma$ be a loxodromic element whose maximal invariant curve D_f coincides with D_Γ. If f is a Kummer example, then (X, Γ) is a Kummer group.*

Proof. Consider the birational morphism $\pi_0 \colon X \to X_0$ contracting D_f (see [Can14, § 4.1] or [CD20, Proposition 6.1]. By assumption, f is a Kummer example, which entails that X_0 is a quotient A/G, with $A = \mathbf{C}^2/\Lambda$ a compact torus and G a finite subgroup of $\mathsf{Aut}(A)$ generated by a diagonal map $g_0 \in \mathsf{GL}_2(\mathbf{C})$ of order 2, 3, 4, 5, 6, or 10 (see § 4.3.2).

The group Γ induces a group of automorphisms of X_0. View X_0 as an orbifold: its fundamental group is $\Lambda \rtimes G$ and its universal cover $\widetilde{X_0}$ is $\mathbf{C}^2$. Concretely, this means that X_0 is the quotient of $\mathbf{C}^2$ by the group of affine transformations with linear part in G and translation part in Λ. The canonical hermitian metric on $\mathbf{C}^2$ is invariant under the affine action of $\Lambda \rtimes G$. If $\tilde{h} \colon \mathbf{C}^2 \to \mathbf{C}^2$ is a lift of some $h \in \Gamma$ to $\widetilde{X_0}$,[5] the norm of $D\tilde{h}_{(x,y)}$ with respect to this hermitian metric is constant along the orbits of $\Lambda \rtimes G$, hence it is bounded since the action is co-compact. This implies that the holomorphic map $(x,y) \in \mathbf{C}^2 \mapsto D\tilde{h}_{(x,y)}$ is constant. Thus, if we denote by $\widetilde{\Gamma}$ the group of all possible lifts of all elements of Γ to $\mathbf{C}^2 = \widetilde{X_0}$, then $\widetilde{\Gamma}$ is a group of affine transformations that contains $\Lambda \rtimes G$ as a normal subgroup and satisfies $\tilde{\Gamma}/(\Lambda \rtimes G) = \Gamma$. The action by conjugation of $\tilde{\Gamma}$ on $\Lambda \rtimes G$ preserves the subgroup Λ of translations. Therefore, Λ is also normal in $\widetilde{\Gamma}$: this shows that $\widetilde{\Gamma}$ induces a group of automorphisms $\Gamma_A = \widetilde{\Gamma}/\Lambda$ of $A = \mathbf{C}^2/\Lambda$ that covers X_0, and the proof is complete. □

Conclusion of the proof of Theorem B. By Proposition 3.12, there exists $f \in \Gamma_{\rm lox}$ such that $D_f = D_\Gamma$. Theorem 5.17 implies that f is a Kummer example, so Proposition 5.18 concludes the proof. □

6. Around Theorem B: consequences and comments

6.1 Corollaries

The following corollary of Theorem B applies, for instance, to general Wehler examples defined over $\overline{\mathbf{Q}}$.

[5] To prove the existence of such a lift, note that h maps the regular part of X_0 to itself, so first lift $h|_{{\rm Reg}(X_0)}$ to $\mathbf{C}^2 \setminus \pi^{-1}({\rm Sing}(X_0))$, which is simply connected, and then use Hartogs extension to extend $\tilde{h}$ across the discrete set $\pi^{-1}({\rm Sing}(X_0))$.

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Corollary 6.1. *Let X be a smooth projective surface and let Γ be a subgroup of* $\mathsf{Aut}(X)$*. Assume that:*

(i) *X and Γ are defined over a number field;*
(ii) *X is not an abelian surface;*
(iii) *Γ contains a parabolic automorphism, and has no invariant curve.*

Then Γ admits only finitely many finite orbits.

Proof. Suppose Γ has infinitely many finite orbits; since Γ does not preserve any curve, these orbits form a Zariski dense subset. Let g be a parabolic automorphism of Γ. If the fibration π_g were Γ-invariant, then Γ would preserve the curve $\bigcup_{y\in\Gamma(x)} \pi_g^{-1}(\pi_g(y))$ for every Γ-periodic point x. Thus, there is an element h in Γ that does not preserve π_g, and $h^{-1}\circ g\circ h \in \Gamma$ is a parabolic map associated to a different fibration. Hence, Γ is non-elementary (see § 3.1.3) and Theorem B shows that Γ is a Kummer group. But, since X is not abelian, a Kummer subgroup of $\mathsf{Aut}(X)$ admits an invariant curve (see Lemma 4.7): this contradiction concludes the proof. □

The next result is in the spirit of the 'dynamical Manin–Mumford problem'.

Corollary 6.2. *Let X be a smooth projective surface and Γ be a subgroup of* $\mathsf{Aut}(X)$*, both defined over a number field. Suppose that Γ is non-elementary and contains parabolic elements. Let $C \subset X$ be an irreducible curve containing infinitely many periodic points of Γ. Then:*

(1) *either C is Γ-periodic and is fixed pointwise by a finite index subgroup of Γ;*
(2) *or (X, Γ) is a Kummer group and C comes from a translate of an abelian subvariety (of dimension 1).*

In both cases the genus of C is 0 or 1. Thus, a curve of genus ≥ 2 contains at most finitely many periodic points of Γ.

To be specific, with the notation of § 4.3, the second assertion means the following: there is a translate $E + t$ of a 1-dimensional abelian subvariety $E \subset A$ such that $q_X(C) = q_A(E + t)$. Moreover, if we choose the origin of A at a periodic point of Γ_A, we can choose t to be a torsion point of A. We keep this notation in the proof.

Proof. Let $\mathrm{Per}(C)$ be the set of periodic points of Γ in C; it is Zariski dense in C, for C is irreducible. The Zariski closure of $\Gamma(\mathrm{Per}(C))$ is either a Γ-invariant curve or X.

In the first case C is contained in D_Γ, a finite index subgroup $\Gamma' \subset \Gamma$ preserves C, and the restriction $\Gamma'|_C$ has infinitely many periodic points in C. In this case C has (arithmetic) genus 0 or 1 by [DJS07, Theorem 1.1]. A group of automorphisms of a curve with at least three periodic orbits is finite, because it admits a finite index subgroup fixing 3 points; thus, a finite index subgroup of Γ fixes C pointwise.

In the second case, Theorem B shows that (X, Γ) is a Kummer group. Since C cannot be periodic, its image $q_X(C) \subset A/G$ is a non-trivial curve whose lift to A contains a Zariski-dense subset of Γ_A-periodic points. Choose one of these periodic points as the origin of A. By Proposition 4.1 and Remark 4.2, the Γ_A-periodic points are exactly the torsion points of A, and conclusion (2) follows from Raynaud's theorem (formerly known as the Manin–Mumford conjecture) [Ray83]. □

6.2 Finitely generated groups

It turns out that Γ is often defined over a number field when X is.

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Proposition 6.3. *Let X be a projective surface defined over a number field $\mathbf{k}$. Assume that $\mathsf{Aut}(X)$ contains a loxodromic element, and that X is not an abelian surface. Then any finitely generated subgroup of $\mathsf{Aut}(X)$ is defined over a finite extension of $\mathbf{k}$.*

Corollary 6.4. *If X is a K3 or Enriques surface defined over a number field $\mathbf{k}$, $\mathsf{Aut}(X)$ is finitely generated in this case (see [Ste85]) and is defined over a finite extension of $\mathbf{k}$.*

Proof of the Proposition. It is enough to show that any automorphism $f \in \mathsf{Aut}(X)$ is defined over a finite extension of $\mathbf{k}$. Under our assumption, $\mathsf{Aut}(X)^* \subset \mathsf{GL}(H^*(X, \mathbf{Z}))$ is infinite, $\mathsf{Aut}(X)^0$ is trivial, and the homomorphism $f \in \mathsf{Aut}(X) \mapsto f^* \in \mathsf{GL}(H^*(X, \mathbf{Z}))$ has finite kernel (see [Can14, Theorem 10.1]); more precisely, if H is any ample divisor, the stabilizer of $[H]$ is a finite subgroup $\mathsf{Aut}(X; [H])$ of $\mathsf{Aut}(X)$.

Fix a finite extension $\mathbf{k}'$ of $\mathbf{k}$ and a basis of $\mathrm{NS}(X; \mathbf{Z})$ given by classes of divisors D_i which are defined over $\mathbf{k}'$. Fix an ample divisor H defined over $\mathbf{k}'$. By assumption X and the D_i are defined by polynomial equations over $\mathbf{k}'$, in some $\mathbb{P}^N$. Now, consider an automorphism f of X, defined by polynomial formulas with coefficients in some extension $\mathbf{K}$ of $\mathbf{k}'$. Any field automorphism $\varphi \in \mathrm{Gal}(\mathbf{K} : \mathbf{k}')$ conjugates f to an automorphism f^φ of X: this defines a map $\varphi \in \mathrm{Gal}(\mathbf{K} : \mathbf{k}') \mapsto f^\varphi \in \mathsf{Aut}(X)$. On the other hand, $\langle (f^\varphi)^*[D_i] | [D_j] \rangle = \langle f^*[D_i] | [D_j] \rangle$ for any pair (i, j) because the divisors D_i are defined over $\mathbf{k}'$; thus, $(f^\varphi)^* = f^*$ on $\mathrm{NS}(X; \mathbf{Z})$, and $f^\varphi \circ f^{-1}$ belongs to the finite group $\mathsf{Aut}(X; [H])$, so the set $\{f^\varphi;\ \varphi \in \mathrm{Gal}(\mathbf{K} : \mathbf{k})\}$ is finite, and we are done. $\square$

6.3 Open problems

In the case of the affine plane $\mathbb{A}^2$, it follows from [DF17] that any non-elementary subgroup of $\mathsf{Aut}(\mathbb{A}^2_{\mathbf{k}})$, for any number field $\mathbf{k}$, has at most finitely many finite orbits (see [DF17] for the definition of 'non-elementary' in this case). This motivates the following question.

Question 6.5. Is Theorem B true without assuming the existence of a parabolic element in Γ?

To understand the difficulties behind Question 6.5, let us comment on three arguments that required the hypothesis $\Gamma_{\mathrm{par}} \neq \emptyset$. First, it was used to show that $\Pi_\Gamma \subset \mathrm{NS}(X; \mathbf{R})$ is defined over $\mathbf{Q}$ and to construct the projective surface X_0 (which is then used in the construction of the canonical stationary height). The point is that, in general, the contraction of the divisor D_Γ is a well-defined complex analytic surface, but it is not projective (see [CD20, § 11]). We expect that this issue could be circumvented by applying more advanced techniques from Arakelov geometry. Second, Theorem 5.15 also relies on the existence of parabolic elements; the point was to show that all active periodic points of all loxodromic elements of Γ cannot be simultaneously contained in some real surface. For instance, it is unclear to us whether there can exist a real projective surface $X_{\mathbf{R}}$, with a non-elementary subgroup $\Gamma \subset \mathsf{Aut}(X_{\mathbf{R}})$, such that all periodic points of all elements $f \in \Gamma \setminus \{\mathrm{id}\}$ are contained in the real part $X(\mathbf{R})$ of X. Third, parabolic automorphisms are crucially used in the classification of Γ-invariant probability measures given in [Can01b, CD23b]. We expect that the techniques from [BH17, CD23c] will soon lead to a complete classification of Γ-invariant probability measures, for any non-elementary group $\Gamma \subset \mathsf{Aut}(X)$. Such a classification would then open the way to an extension of Theorem B to all non-elementary groups (defined over a number field).

Remark 6.6. In [Kaw13, Question 3.3], Kawaguchi formulates two interesting questions which are closely related to our main results as well as to Question 6.5.

(1) First, he asks whether two loxodromic automorphisms f and g of a complex projective surface X with a Zariski-dense set of common periodic points automatically have the same

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periodic orbits. As it is formulated, the answer is no, because of Kummer groups: if we start with two loxodromic automorphisms of an abelian surface A fixing the origin and generating a non-elementary subgroup, then one can blow-up the origin, and the automorphisms lift to automorphisms with the same periodic orbits (coming from torsion points of A), *except for their fixed points on the exceptional divisors, which do differ* (see Lemma 4.7, assertion (2)). Thus, his question needs to be modified by asking whether f and g have the same periodic points, *except for finitely many of them*.

(2) The second part of [Kaw13, Question 3.3] asks whether two loxodromic automorphisms of a Wehler surface having a Zariski-dense set of common periodic points automatically generate an elementary group. There are (singular) Kummer examples in the Wehler family (see [Can10, § 8.2]), and they provide counter-examples to this question. Taking these comments into consideration, Kawaguchi's second question can now be reformulated as: *if two loxodromic automorphisms f and g of a complex projective surface X have a Zariski-dense set of common periodic points, then is it true that either $f^m = g^n$ for some $m, n \geq 1$, or f and g generate a Kummer group?* This seems harder than Question 6.5, because common periodic points do not directly provide common periodic orbits. A natural companion to the last question is: *when do two loxodromic automorphisms have the same measure of maximal entropy?*

One may also ask for effective bounds on the cardinality of a maximal finite Γ-invariant subset of $X(\mathbf{C})$ in terms of the data (compare [DKY22]). Proposition 2.8 says that such a bound should at least depend on the degrees of the generators of Γ.

Lastly, a natural question is whether the number field assumption in Theorem B is necessary at all: this is what the next section is about.

7. From number fields to C

In this section we show how a specialization argument makes it possible to extend Corollary 6.1 beyond the number field case. A full generalization of Theorem B to complex coefficients would require further ideas (see § 7.4 for a short discussion). For concreteness we first treat the case of Wehler surfaces and then explain the extra ingredients required to address the general case.

7.1 Wehler surfaces

We resume the notation from § 2. The complete linear system $|L|$ parameterizing Wehler surfaces is a projective space of dimension 26, which yields a moduli space of dimension 17 modulo the action of $\mathsf{Aut}(\mathbb{P}^1)^3$. There is a dense, Zariski-open subset $W_0 \subset |L|$ such that if $X \in W_0$, then X is a smooth Wehler surface and for every $1 \leq j \neq k \leq 3$, $\pi_{j,k} : X \to \mathbb{P}^1 \times \mathbb{P}^1$ is a finite morphism. Let Γ_X be the group generated by the three involutions σ_i.

Theorem 7.1. *If X is a smooth Wehler surface for which the projections $\pi_{j,k} \colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ are finite maps, then Γ_X admits only finitely many finite orbits.*

For the proof we follow the approach of [DF17, § 5 and Theorem D] closely.

Proof. Let $G = (\mathbf{Z}/2\mathbf{Z}) \star (\mathbf{Z}/2\mathbf{Z}) \star (\mathbf{Z}/2\mathbf{Z})$ with generators a_1, a_2, a_3 and let $\chi : G \to \mathsf{Aut}(X)$ be the unique homomorphism such that $\chi(a_i) = \sigma_i$. By definition, $\Gamma_X = \chi(G)$. Let c_i denote the class of the curve $X \cap \{z_i = \mathrm{C^{st}}\}$. The subspace $\mathbf{Z}c_1 \oplus \mathbf{Z}c_2 \oplus \mathbf{Z}c_3$ of $\mathrm{NS}(X, \mathbf{Z})$ is invariant by $\chi(G)^* \subset \mathsf{GL}(\mathrm{NS}(X, \mathbf{Z}))$ and this representation does not depend on $X \in W_0$: the matrices of the involutions $\sigma_i^* = \chi(a_i)^*$ in the basis (c_1, c_2, c_3) have constant integer coefficients (see, e.g., [CD23c, § 3]). Thus, we can define G_{lox} (respectively, G_{par}) to be the set of elements $h \in G$ such that for any $X \in W_0$, $\chi(h)$ acts as a loxodromic (respectively, parabolic) map on $\mathrm{NS}(X, \mathbf{Z})$.

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Here, we implicitly use the fact that the type of $h \in \mathsf{Aut}(X)$ is the same as the type of h^* in restriction to any h^*-invariant subspace of $H^{1,1}(X;\mathbf{R})$ on which the intersection form is not negative definite. In particular, the type of h coincides with the type of h^* as an isometry of $\mathrm{Vect}(c_1, c_2, c_3)$.

Fix a system of affine coordinates (x, y, z) and write the equations of Wehler surfaces as in (2.1); this gives a system of homogeneous coordinates on $|L|$, and $|L|$ can be considered as a projective space defined over $\mathbf{Q}$. Then, endow $|L| \simeq \mathbb{P}^{26}(\mathbf{C})$ with the $\overline{\mathbf{Q}}$-Zariski topology. Fix $X \in W_0$, let $b \in \mathbb{P}^{26}$ (for 'base point') denote the parameter corresponding to X, and S be the closure of $\{b\}$ for this topology: this is a subvariety of $\mathbb{P}^{26}$ defined over $\overline{\mathbf{Q}}$ in which b is, by construction, a generic point. We put $S_0 = S \cap W_0$, and we restrict the universal family $\mathcal{X} \to \mathbb{P}^{26}$ of Wehler surfaces to a family $\mathcal{X}_{S_0} \to S_0$, with a fiber-preserving action of the group G. The fiber over s is denoted by $\mathcal{X}_s$ and the natural homomorphism $G \to \mathsf{Aut}(\mathcal{X}_s)$ by χ_s; thus, X coincides with $\mathcal{X}_b$.

Lemma 7.2. *For every $s \in S_0(\mathbf{C})$:*

(1) *$\mathcal{X}_s$ is a smooth $K3$ surface which does not contain any fiber of $\pi_{i,j}$, $i \neq j \in \{1, 2, 3\}$;*
(2) *$h \in G$ belongs to G_{lox} (respectively, G_{par}) if and only if $\chi_s(h)$ is a loxodromic (respectively, parabolic) element of $\mathsf{Aut}(\mathcal{X}_s)$;*
(3) *$\chi_s(G)$ is a non-elementary subgroup of $\mathsf{Aut}(\mathcal{X}_s)$ without invariant curve.*

Proof of Lemma 7.2. The first assertion follows from the results of § 2.1 and the inclusion $S_0 \subset W_0$. Likewise $\chi_s(G)$ has no invariant curve by § 2.2. The second assertion follows from our preliminary remarks on the definition of G_{lox} and G_{par}, and it also implies that $\chi_s(G)$ is non-elementary. □

Assume now by contradiction that Γ_X admits infinitely many finite orbits. Then the following holds.

Lemma 7.3. *For every $s \in S_0(\mathbf{C})$, $\chi_s(G)$ has infinitely many finite orbits.*

This lemma concludes the proof of the theorem. Indeed pick $s \in S_0(\overline{\mathbf{Q}})$. By Lemma 7.2, $\chi_s(G)$ is non-elementary, contains parabolic elements; and the Zariski closure of the set of finite orbits of $\chi_s(G)$ coincides with $\mathcal{X}_s$, as otherwise it would be an invariant curve. Then, by Theorem B, $(\mathcal{X}_s, \chi_s(G))$ must be a Kummer group. But $\mathcal{X}_s$ is a K3 surface and a Kummer group on a non-abelian surface admits an invariant curve, so that we get a contradiction with Lemma 7.2.(3). □

Proof of Lemma 7.3. We first describe the set of finite Γ_X-orbits as a countable union of subvarieties by arguing as in § 2.4.1. Let G_d be the intersection of the kernels of all homomorphisms from G to groups of order $\leq d!$; it is a finite index subgroup of G. For any action of G, if the orbit of a point x has cardinality $\leq d$, then x is fixed by G_d. Conversely, if x is fixed by G_d, then its G-orbit is finite. Define a subvariety Z_d of X by

$$Z_d = \{x \in X;\ \forall g \in G_d,\ \chi(g)(x) = x\}. \tag{7.1}$$

Finally, put $Z = \bigcup_{d \geq 1} Z_d$. Then the Γ_X-orbit of $x \in X$ is finite if and only if $x \in Z$. We can now define a subvariety $\mathcal{Z}_d$ of $\mathcal{X}_{S_0}$ which is the fibered analogue of Z_d, namely

$$\mathcal{Z}_d = \{(s, x);\ x \in \mathcal{X}_s,\ \forall g \in G_d,\ \chi_s(g)(x) = x\}, \tag{7.2}$$

and put $\mathcal{Z} = \bigcup \mathcal{Z}_d$. We let $\mathcal{Z}_s$ (respectively, $\mathcal{Z}_{d,s}$) be the intersection of $\mathcal{Z}$ (respectively, $\mathcal{Z}_d$) with $\mathcal{X}_s$.

Set $f = a_3 a_2 a_1 \in G$. An explicit computation shows that $f \in G_{\mathrm{lox}}$ and the eigenvalues of $\chi_s(f)^*$ on $\mathrm{Vect}(c_1, c_2, c_3)$ are -1, $\lambda(f) = 9 + 4\sqrt{5}$, and $1/\lambda(f)$. The eigenline corresponding to -1 is $\mathbf{R} \cdot (c_1 - 3c_2 + c_3)$, its orthogonal complement in $\mathrm{Vect}(c_1, c_2, c_3)$ is the plane $\Pi_{\chi_s(f)}$, and this plane contains the class $c_1 + 2c_2 + c_3$. This class is ample, because it is a convex combination, with positive coefficients, of the Chern classes c_i of the line bundles $\pi_i^*(\mathcal{O}_{\mathbb{P}^1}(1))$, $i = 1, 2, 3$. Since any invariant curve must be orthogonal to $\Pi_{\chi_s(f)}$, we deduce that $\chi_s(f)$ has no invariant curve (for all $s \in S_0$).

Now assume by contradiction that there is a parameter $t \in S_0(\mathbf{C})$ such that $\mathcal{Z}_t$ is finite. Let P_n be the set of fixed points of $\chi(f^n)$, so that $P = \bigcup_n P_n$ is the set of all periodic points of $\chi(f)$; likewise, let $\mathcal{P}_n$ and $\mathcal{P}$ be their respective fibered versions. Note that $\mathcal{Z} \subset \mathcal{P}$. For fixed n, let $\mathcal{Y}_n$ be the (reduced) subvariety of $\mathcal{X}_{S_0}$ whose underlying set is $\mathcal{Z} \cap \mathcal{P}_n$. More precisely, the sequence of subvarieties $\mathcal{Y}_n^m := \bigcup_{d=1}^m \mathcal{Z}_d \cap \mathcal{P}_n$ is non-decreasing with m, so it stabilizes, and we define $\mathcal{Y}_n = \mathcal{Y}_n^m$ for m sufficiently large; its fibers will be denoted by $\mathcal{Y}_{n,s}$ ($\mathcal{Y}_{n,s}$ is the intersection of $\mathcal{Y}_n$ with $\mathcal{X}_s$, it may be non-reduced). For the generic point b the cardinality of $\mathcal{Y}_{n,b}$ tends to infinity with n.

From this point, the argument is identical to that of Lemma 5.3 and Theorem D in [DF17].[6] For $x \in \mathcal{Y}_{n,s}$, its multiplicity $\mathrm{mult}(x, \mathcal{Y}_{n,s})$ as a point in $\mathcal{Y}_{n,s}$ is equal to its multiplicity as a fixed point of $\chi_s(f)^n$. Nakayama's lemma implies that the function

$$s \mapsto \sum_{x \in \mathcal{Y}_{n,s}} \mathrm{mult}(x, \mathcal{Y}_{n,s})$$

is upper semicontinuous for the Zariski topology, hence

$$\sum_{x \in \mathcal{Y}_{n,t}} \mathrm{mult}(x, \mathcal{Y}_{n,t}) \geq \sum_{x \in \mathcal{Y}_{n,b}} \mathrm{mult}(x, \mathcal{Y}_{n,b}) \underset{n \to \infty}{\longrightarrow} +\infty. \tag{7.3}$$

On the other hand, $\mathcal{Z}_t$ is a finite set, so there exists n_0 such that for all $n \geq 1$, $\mathcal{Y}_{n,t} \subset \mathcal{P}_{n_0,t}$, and the theorem of Shub and Sullivan [SS74] asserts that for every $x \in \mathcal{P}_{n_0,t}$, the multiplicity of x as a fixed point of $\chi_t(f)^n$ is bounded as $n \to \infty$. This contradicts (7.3) and concludes the proof. ☐

7.2 Groups without invariant curve

Let us recall Theorem C.

Theorem 7.4. *Let X be a compact Kähler surface and let Γ be a subgroup of $\mathsf{Aut}(X)$. Assume that (i) X is not an abelian surface, and (ii) Γ contains a parabolic element and has no invariant curve. Then Γ admits only finitely many finite orbits.*

Proof. The idea is the same as for Theorem 7.1, however new technicalities arise. As in Corollary 6.1, Γ is automatically non-elementary, so X is projective. Arguing by contradiction, we suppose that Γ admits infinitely many finite orbits. Applying Theorem D from § 3, we fix $f \in \Gamma_{\mathrm{lox}}$ without invariant curve. We also fix a parabolic element $g \in \Gamma$.

Step 1: geometry of X. Since Γ is non-elementary, X is a blow-up of an abelian surface, a K3 surface, an Enriques surface, or the projective plane (see [Can14, Theorem 10.1]). In the first three cases, there is a unique minimal model $\varphi\colon X \to \overline{X}$, and the exceptional divisor of φ is $\mathsf{Aut}(X)$-invariant; since Γ has no invariant curve, X is already equal to its minimal model $\overline{X}$, and since X is not abelian, X is a K3 or an Enriques surface.

[6] Our setting is actually simpler since we are dealing with automorphisms on a projective surface rather than birational mappings, so the properness issue analyzed in [DF17] is not relevant here.

Step 2: reduction to a finitely generated subgroup. The group generated by f and g satisfies assumption (ii) and since it is contained in Γ it also admits infinitely many finite orbits. From now on, we replace Γ by $\langle f, g\rangle$ and assume Γ to be finitely generated.

Step 3: specialization formalism. Embed X into a projective space $\mathbb{P}^N_{\mathbf{C}}$. Fix a finite set of reduced and effective, irreducible divisors E_j in X whose classes form a basis of $\mathrm{NS}(X;\mathbf{Z})$, let $H \subset X$ be a hyperplane section, and let Ω be a non-trivial rational section of $K_X^{\otimes 2}$, where K_X is the canonical bundle. If X is a K3 or an Enriques surface, we assume that Ω is regular, hence does not vanish. Let $R \subset \mathbf{C}$ be the $\overline{\mathbf{Q}}$-subalgebra generated by the coefficients of a system of homogeneous equations for X, the E_j, and H, and by the coefficients of the formulas defining Ω and a finite symmetric set of generators of Γ. (We shall actually further enlarge R in § 7.3.2.)

Let $K = \mathrm{Frac}(R)$. It is isomorphic to the function field of some (irreducible) algebraic variety V, defined over $\overline{\mathbf{Q}}$. There is a dense, Zariski-open subset S of V, which may be assumed to be an affine subset, such that all elements of R correspond to regular functions on S. Note that in what follows, by Zariski topology we mean the $\overline{\mathbf{Q}}$-Zariski topology; nevertheless, since we will use transcendental arguments, $S(\mathbf{C})$ will also be considered as a complex analytic space endowed with its Euclidean topology.

By specialization, i.e. evaluation of the elements of R at $s \in S$, we can view $X \subset \mathbb{P}^N$, Γ, the E_j, H, and Ω as families over S; that is, there is a scheme $\mathcal{X}$ of finite type over $\overline{\mathbf{Q}}$ and a proper morphism $\pi : \mathcal{X} \to S$ endowed with a group of fiber-preserving automorphisms $\widetilde{\Gamma}$, together with a (complex) base point $b \in S$ so that the fiber $\mathcal{X}_b$ may be identified with X and, furthermore, $\widetilde{\Gamma}_b = \Gamma$, $\mathcal{E}_{j,b} = E_j$, $\mathcal{H}_b = H$, $\widetilde{\Omega}_b = \Omega$, etc. The point $b \in S$ may be thought of as the generic point of S (i.e. its closure for the Zariski topology is S) so, in particular, b is a regular point of S and S is smooth in a complex neighborhood of b. If X is a K3 or an Enriques surface, changing S into some Zariski-dense affine open subset, we may assume that $\widetilde{\Omega}_s$ does not vanish on any $\mathcal{X}_s$.

Step 4: types of automorphisms and invariant curves.

Lemma 7.5. *There is a Zariski-open subset $S_1 \subset S$ such that:*

(1) *above $S_1(\mathbf{C})$, the projection $\mathcal{X}(\mathbf{C}) \to S(\mathbf{C})$ is a submersion; for $s \in S_1(\mathbf{C})$, $\mathcal{X}_s$ is smooth and it is not an abelian surface;*
(2) *for $s \in S_1(\mathbf{C})$, f_s is loxodromic and there exists a Euclidean neighborhood B of b such that for $s \in B$, f_s admits no invariant curve;*
(3) *for $s \in S_1(\mathbf{C})$, g_s is parabolic.*

Proof of part (1). The surface $\mathcal{X}_b$ is smooth and, by construction, there is a Zariski-dense open subset S_1 of S containing b above which $\mathcal{X} \to S$ is a submersion; in particular, for $s \in S_1$, $\mathcal{X}_s$ is smooth (we will further reduce S_1 finitely many times in the proof, keeping the same notation). For the second conclusion, observe that S is connected (for the euclidean topology), hence so is S_1; therefore, by Ehresmann's lemma the fibers above S_1 are diffeomorphic to X. On the other hand, a surface which is diffeomorphic to a complex torus and possesses a non-elementary group of automorphisms is automatically an abelian surface. Since X is not abelian, we conclude that the same is true for any fiber $\mathcal{X}_s$, $s \in S_1$. □

Proof of part (2). The loxodromic nature of f_s follows from the lower semi-continuity, in Zariski topology, of the dynamical degree for birational transformations of surfaces (see [Xie15, Theorem 4.3]). Indeed, the set $\{s \in S_1; \lambda(f_s) \leq 1\}$ is Zariski closed but does not contain the generic point, so it is a proper subset, which can be removed from S_1.

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Let us show that f_s has no invariant curve for $s \in S_1$ close to b. Indeed, recall from Proposition 3.7 that there is a uniform bound on the degree of an invariant curve. Here we compute the degree of a curve on $\mathcal{X}_s$ (respectively, of f_s) with respect to the normalized ample class $\langle \mathcal{H}_s | \mathcal{H}_s \rangle^{-1/2} [\mathcal{H}_s]$ induced by the hyperplane section $\mathcal{H}_s$. If c_X is as in Proposition 3.7, the inequality $\frac{1}{2}[\mathrm{Div}(\Omega_s)] \leq c_X \langle \mathcal{H}_s | \mathcal{H}_s \rangle^{-1/2} [\mathcal{H}_s]$ is satisfied on a Zariski-open set. Then by Bishop's theorem, if (s_i) is a sequence of points converging to b such that f_{s_i} preserves a curve C_i, we can extract a subsequence along which (C_i) converges towards a curve C in $\mathcal{X}_b$ (see [Chi89, § 16] for the relevant notions). This curve is f_b-invariant, which contradicts our assumption on f. ☐

The proof of the third assertion of Lemma 7.5 is a little tedious and will be postponed to § 7.3.

Step 5: conclusion. We pick a point t in $S_1(\overline{\mathbf{Q}}) \cap B$, and argue exactly as in the case of Wehler surfaces. Indeed observe first that the assumptions of Corollary 6.1 are satisfied at the parameter t. Next, since all periodic points of f_t of a given period are isolated, we can apply to f_t the strategy of the proof of Theorem 7.1, based on Nakayama's lemma and the theorem of Shub and Sullivan; it implies that Γ_t has infinitely many periodic orbits on $\mathcal{X}_t$, thereby reaching the desired contradiction. ☐

7.3 Proof of Lemma 7.5(3)

By Step 1, X is a K3 surface, an Enriques surface, or a blow-up of the projective plane. By semi-continuity of dynamical degrees and $\lambda(g) = 1$, g_s is parabolic or elliptic for every s in S_1. Thus, we need to show that the set of parameters for which g_s is elliptic is Zariski closed.

7.3.1 *K3 and Enriques surfaces.* Assume that X is a K3 (respectively, an Enriques) surface. Above $S_1(\mathbf{C})$, every fiber $\mathcal{X}_s$ has the diffeomorphism type of $\mathcal{X}_b$, in particular it is simply connected, and $K_{\mathcal{X}_s}$ is trivial (respectively, its fundamental group is $\mathbf{Z}/2\mathbf{Z}$ and $K_{\mathcal{X}_S}^{\otimes 2}$ is trivial), so it is also a K3 (respectively, an Enriques) surface, for K3 (respectively, Enriques) are characterized by these properties (see [BHPV04, Chapter VI]). For such a surface, the group $\{h \in \mathsf{Aut}(\mathcal{X}_s);\ h^* = \mathrm{id} \text{ on } H^2(\mathcal{X}_s; \mathbf{Z})\}$ has at most 4 elements (see [MN84]). The second Betti number is fixed, equal to 22 (respectively, 10), and if $h^* \in \mathsf{GL}(H^2(\mathcal{X}_s; \mathbf{Z}))$ has finite order, then its order divides some fixed integer k, because $\mathsf{GL}_{22}(\mathbf{Z})$ (respectively, $\mathsf{GL}_{10}(\mathbf{Z})$) contains a finite index, torsion free subgroup. Thus, g_s is elliptic if and only if $g_s^{4k} = \mathrm{id}$. This implies that the set of parameters s for which g_s is elliptic is Zariski closed and does not contain b, and we are done in this case.

7.3.2 *Rational surfaces.* Now, we assume that X is rational. This case is slightly more delicate because there exists automorphisms of $\mathbb{P}^2$ of arbitrary large finite order.

Let $\pi_g \colon X \to B$ be the invariant fibration of g, with $B = \mathbb{P}^1$ since X is rational. Replacing g by some positive iterate, we assume that its action on the base B is the identity. As explained in [CD12, CGL21], π_g comes from a Halphen pencil; in particular, there is a pencil of curves in $\mathbb{P}^2$, defined by some rational function $\varphi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, and a birational morphism $\eta \colon X \to \mathbb{P}^2$ that blows up the base points of this pencil (and possibly other points too), such that π_g coincides with $\varphi \circ \eta$. The last blow-up necessary to resolve the indeterminacies of φ provides an irreducible curve E which is generically transverse to the fibration and has negative self-intersection. Let us add to our $\overline{\mathbf{Q}}$-algebra R the coefficients of the formulas defining π_g, φ, η, E, etc. Reducing S_1 if necessary, we get a family of automorphisms g_s preserving each fiber of a genus 1 fibration $\pi_{g,s} \colon \mathcal{X}_s \to \mathbb{P}^1$, with an irreducible multisection E_s of negative self-intersection. As for K3 and Enriques surfaces, the following lemma finishes the proof.

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Lemma 7.6. *There is an integer $\ell > 0$ such that if $s \in S_1$ and g_s is elliptic, then $g_s^\ell = \mathrm{id}$.*

Proof. Set $m = \langle [E] | [F] \rangle$ where F is any fiber of π_g. Above S_1 the surfaces $\mathcal{X}_s$ are pairwise diffeomorphic, so they have the same second Betti number and there is an integer $k > 0$ such that $(h^*)^k = \mathrm{id}$ for every elliptic automorphism of $\mathcal{X}_s$, for every $s \in S_1$. Now, if g_s is elliptic, then $(g_s^k)^*[E_s] = [E_s]$ and this implies $g_s^k(E_s) = E_s$. Since g_s preserves every fiber, and E_s intersects every fiber in at most m points, we deduce that $g_s^{k\cdot m!}$ fixes a point in each fiber. But an automorphism of a curve of genus 1 which fixes a point has order at most 12, so $g_s^{12k\cdot m!} = \mathrm{id}_{\mathcal{X}_s}$, and we are done. □

7.4 Discussion

It would be interesting to extend Theorem B in its general form beyond number fields, that is, without assuming that $D_\Gamma = \emptyset$. Fix $(f, g) \in \Gamma_{\mathrm{lox}} \times \Gamma_{\mathrm{par}}$, as above. The main difficulty appears in the following situation: Γ fixes D_Γ pointwise, and for every parameter $s \in S(\overline{\mathbf{Q}})$, the alleged Zariski-dense set of finite orbits of Γ specializes as a finite subset of $\mathcal{X}_s$ which intersects $(D_\Gamma)_s$. In that case, the theorem of Shub and Sullivan does not apply because it concerns isolated fixed points; so, a finer understanding of the Lefschetz fixed point formula is required. The tools introduced in [IU10] and in a chapter of Xie's thesis [Xie14] may lead to a solution of this problem.

8. Canonical vector heights

Let $\mathbf{k}$ be a number field and $\overline{\mathbf{k}} \simeq \overline{\mathbf{Q}}$ be an algebraic closure of $\mathbf{k}$. Let X be a projective surface defined over $\mathbf{k}$ and Γ be a subgroup of $\mathsf{Aut}(X_{\mathbf{k}})$. We consider the vector space

$$\mathrm{Pic}(X; \mathbf{R}) = \mathrm{Pic}(X_{\overline{\mathbf{k}}}) \otimes_{\mathbf{Z}} \mathbf{R} \tag{8.1}$$

of $\mathbf{R}$-divisors of $X_{\overline{\mathbf{k}}}$ modulo linear equivalence; doing so, we annihilate the torsion part of $\mathrm{Pic}^0(X)$. Keep in mind that when X is birational to an abelian variety, the vector space $\mathrm{Pic}^0(X; \mathbf{R}) := \mathrm{Pic}^0(X) \otimes_{\mathbf{Z}} \mathbf{R}$ is infinite-dimensional. The Weil height machine extends to $\mathrm{Pic}(X; \mathbf{R})$ by $\mathbf{R}$-linearity (see [HS00, § B.3.2]). Recall from § 1.3 that a *canonical vector height* on $X(\overline{\mathbf{k}})$ for the group Γ is, by definition, a function $h : \mathrm{Pic}(X; \mathbf{R}) \times X(\overline{\mathbf{k}}) \to \mathbf{R}_+$ such that:

(a) h is *linear* with respect to the first factor $L \in \mathrm{Pic}(X; \mathbf{R})$;
(b) for every $L \in \mathrm{Pic}(X; \mathbf{R})$, $h(L, \cdot)$ is a *Weil height* associated to L;
(c) h is Γ-*equivariant*; for every $f \in \Gamma$, $h(L, f(x)) = h(f^*L, x)$.

Note that if $\mathrm{Pic}(X_{\overline{\mathbf{k}}})$ is tensorized by $\mathbf{Q}$ instead of $\mathbf{R}$ and property (a) is stated over $\mathbf{Q}$ we get an equivalent notion. Given any Γ-invariant subspace $V \subset \mathrm{Pic}(X; \mathbf{R})$, one may also study the notion of *restricted* canonical vector height $h \colon V \times X(\overline{\mathbf{k}}) \to \mathbf{R}$. This is most significant when V contains classes with positive self-intersection, in which case it surjects onto Π_Γ under the natural map $D \in \mathrm{Pic}(X; \mathbf{R}) \to [D] \in \mathrm{NS}(X; \mathbf{R})$ (see Lemma 8.3(1)). Note that we use brackets to distinguish a class in $\mathrm{NS}(X; \mathbf{R})$ from a class in $\mathrm{Pic}(X; \mathbf{R})$.

If $A_{\mathbf{k}}$ is an abelian variety and Γ is a subgroup of $\mathsf{Aut}(A_{\mathbf{k}})$ fixing its neutral element, the Néron–Tate height on A is a canonical vector height for Γ (see [HS00, Theorem B.5.6]). The same holds if the neutral element is Γ-periodic, because in this case $\Gamma(0)$ is made of torsion points (see Remark 4.2). In this section, we describe automorphism groups of surfaces which are non-elementary, contain parabolic elements, and possess a (restricted) canonical vector height h_{can}: Theorems E, E′, and E″ show that (X, Γ) is a Kummer group and h_{can} is derived from the Néron–Tate height.

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8.1 Invariant classes and canonical vector heights

In the following lemmas, h_{can} is a restricted canonical vector height for (X, Γ), defined on some Γ-invariant subspace $V_{\mathrm{can}} \subset \mathrm{Pic}(X; \mathbf{R})$. We say that a class $[E]$ in $\mathrm{NS}(X; \mathbf{R})$ is *almost Γ-invariant* if $f^*[E] = \pm[E]$ for all f in Γ.

LEMMA 8.1. *Let $[E] \in \mathrm{NS}(X; \mathbf{R})$ be almost Γ-invariant. Let $\varphi \colon X(\overline{\mathbf{k}}) \to \mathbf{R}$ be a function. The function*

$$h_{[E],\varphi}(D, x) = h_{\mathrm{can}}(D, x) + \langle [E] | D \rangle \varphi(x)$$

is a restricted canonical vector height on $V_{\mathrm{can}} \times X(\overline{\mathbf{k}})$ if and only if either $[E]$ is orthogonal to V_{can}, or $\varphi \colon X(\overline{\mathbf{k}}) \to \mathbf{R}$ is bounded and satisfies $\varphi(x)[E] = \varphi(f(x)) f^*[E]$ for all $f \in \Gamma$.

In this situation, we shall say that $h_{[E],\varphi}$ is *derived from* the height h_{can}.

Proof. If $[E]$ is orthogonal to V_{can}, then $h_{[E],\varphi} = h_{\mathrm{can}}$ on $V_{\mathrm{can}} \times X(\overline{\mathbf{k}})$ and there is nothing to prove. Otherwise, we can fix a class $D \in V_{\mathrm{can}}$ such that $\langle [E] | D \rangle \neq 0$. If $h_{[E],\varphi}$ is a canonical vector height, then $\varphi = \langle [E] | D \rangle^{-1} (h_{[E],\varphi} - h_{\mathrm{can}})(D, \cdot)$ is bounded, because $h_{[E],\varphi}(D, \cdot)$ and $h(D, \cdot)$ are Weil heights associated to the same divisor. Furthermore, φ satisfies $\varphi(x)[E] = \varphi(f(x)) f^*[E]$ because $h_{[E],\varphi}$ and h_{can} are Γ-equivariant and $\langle [E] | f^*(D) \rangle = \langle f^*[E] | D \rangle$ for all $f \in \Gamma$. The reverse implication is straightforward. □

LEMMA 8.2. *Assume that Γ contains a loxodromic element. Let $C \subset X$ be an irreducible Γ-periodic curve. If the class of C belongs to V_{can}, or if V_{can} contains a Γ-periodic class D such that $\mathcal{O}(D)_{|C}$ is ample, then the restriction homomorphism $\mathrm{Stab}_\Gamma(C) \ni f \mapsto f_{|C}$ has finite image.*

Proof. If C is Γ-periodic, then its self-intersection is negative, the restriction of $\mathcal{O}_X(-C)$ to C has positive degree, and $(\mathcal{O}_X(-C))_{|C}$ is therefore ample. Thus, it is enough to consider the case where V_{can} contains a periodic class D such that $\mathcal{O}(D)_{|C}$ is ample.

After replacing Γ by a finite index subgroup, we may assume $\Gamma(C) = C$. If σ is an automorphism of $\mathbf{C}$ over $\mathbf{k}$, it maps C to a Γ-invariant curve C^σ. Since Γ contains a loxodromic element, there are only finitely many Γ-invariant irreducible curves (the components of D_Γ, see § 3.2). Thus, the orbit of C under the group of automorphisms of $\mathbf{C}$ over $\mathbf{k}$ is finite, C is defined over a number field, and $C(\overline{\mathbf{k}})$ is dense in $C(\mathbf{C})$.

Set $\Gamma' = \mathrm{Stab}_\Gamma(D)$, and pick $x_0 \in C(\overline{\mathbf{k}})$. Then $h_{\mathrm{can}}(D, y) = h_{\mathrm{can}}(D, x_0)$ for every y in $\Gamma'(x_0)$; since $h_{\mathrm{can}}(D, \cdot)$ is a Weil height for D, and $\mathcal{O}(D)_{|C}$ is ample, Northcott's theorem implies that $\{x \in C(\mathbf{k}'); h_{\mathrm{can}}(D, x) = h_{\mathrm{can}}(D, x_0)\}$ is finite for every number field $\mathbf{k}'$; thus, $\Gamma'(x_0)$ is a finite set. Since $C(\overline{\mathbf{k}})$ is infinite, we can argue as in the proof of Corollary 6.2 to deduce that $\Gamma'_{|C}$ is finite, as asserted. □

LEMMA 8.3. *Assume $\mathrm{Pic}^0(X) = 0$ and identify $\mathrm{Pic}(X; \mathbf{R})$ with $\mathrm{NS}(X; \mathbf{R})$.*

(1) *If V_{can} contains a class with positive self-intersection, then it contains Π_Γ.*
(2) *If V_{can} contains Π_Γ, and if C is an irreducible rational Γ-periodic curve, then $h_{\mathrm{can}}(D, x) = 0$ for every $D \in \Pi_\Gamma$ and $x \in C(\overline{\mathbf{k}})$.*

Proof. If V_{can} contains a class in the positive cone it contains the limit set $\mathrm{Lim}(\Gamma)$, hence also Π_Γ (see [CD23c, § 2.3]); this proves the first assertion. For the second, pick a probability measure ν on Γ with finite support, and assume that $P_\nu^*(D) = \alpha(\nu) D$ for some D in Π_Γ and some $\alpha(\nu) > 1$. Then, $\sum_f \nu(f) h_{\mathrm{can}}(D, f(x)) = \alpha(\nu) h_{\mathrm{can}}(D, x)$ by equivariance and linearity. On the other hand, $\mathcal{O}(D)_{|C}$ has degree 0, because $\langle D | C \rangle = 0$, and is therefore trivial because C is rational. Thus, $h_{\mathrm{can}}(D, \cdot)$ is bounded on $C(\overline{\mathbf{k}})$. Since $\alpha(\nu) > 1$, this implies that $h_{\mathrm{can}}(D, x) = 0$

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for every $x \in C(\overline{\mathbf{k}})$. To conclude, note that such eigenvectors D generate Π_Γ when we vary ν (see § 5.3). □

8.2 From canonical vector heights to Kummer groups

Theorem E. *Let X be a smooth projective surface and Γ be a subgroup of $\mathsf{Aut}(X)$, both defined over a number field $\mathbf{k}$. Suppose that:*

(i) *Γ is non-elementary and contains parabolic elements;*
(ii) *there exists a canonical vector height h_{can} for $(X(\overline{\mathbf{k}}), \Gamma)$ on a Γ-invariant subspace of $\mathrm{Pic}(X; \mathbf{R})$ which contains a divisor with positive self-intersection.*

Then (X, Γ) is a Kummer group. If, in addition, h_{can} is defined on $\mathrm{Pic}(X; \mathbf{R}) \times X(\overline{\mathbf{k}})$, then X is an abelian surface.

The smoothness of X is essential for the last conclusion to hold; for instance, if (X_0, Γ) is a singular Kummer group with no Γ-invariant curve, we shall see that the Néron–Tate height induces a canonical vector height on $\mathrm{Pic}(X_0; \mathbf{R}) \times X_0(\overline{\mathbf{k}})$.

The remainder of this subsection is devoted to the proof of Theorem E. Let us already observe that once (X, Γ) is known to be a Kummer group, the last conclusion follows from Lemmas 4.7 and 8.2. Thus, all we have to show is that (X, Γ) is a Kummer group.

8.2.1 *Reduction to* $\mathrm{Pic}^0(X) = 0$. Suppose $\mathrm{Pic}^0(X) \neq \{0\}$. Then, Γ being non-elementary, [Can14, Theorem 10.1] shows that X is either an abelian surface or a blow-up of such a surface along a finite orbit of Γ, and by definition (X, Γ) is a Kummer group.

Thus, from now on, we assume $\mathrm{Pic}^0(X) = \{0\}$ and identify $\mathrm{Pic}(X; \mathbf{R})$ with $\mathrm{NS}(X; \mathbf{R})$.

8.2.2 *A key lemma.* Assumption (ii) provides a canonical vector height h_{can} for (X, Γ) defined on Π_Γ (see Lemma 8.3). Recall from [Sil91, Kaw08] that for every $f \in \Gamma_{\mathrm{lox}}$ there exist canonical heights $h_f^{\pm}$, respectively associated to the classes $\theta_f^{\pm}$, such that $h_f^+(f(x)) = \lambda(f) h_f^+(x)$ and $h_f^-(f^{-1}(x)) = \lambda(f) h_f^-(x)$. They satisfy:

- $h_f^{\pm} \geq 0$ on $X(\overline{\mathbf{k}})$;
- if D_f denotes the maximal invariant curve of f, then, for $x \in X(\overline{\mathbf{k}})$, $h_f^+(x) + h_f^-(x) = 0$ if and only if x is a periodic point or $x \in D_f$ (see [Kaw08, § 5]).

Furthermore, any Weil height h associated to θ_f^+ such that $h(f(x)) = \lambda(f) h(x)$ coincides with h_f^+: indeed $k := h - h_f^+$ is bounded because h and h_f^+ are Weil heights associated to the same class, so the relation $k(f(x)) = \lambda(f) k(x)$ forces it to be identically zero. Thus, the next lemma follows immediately from the defining properties (a), (b), and (c) (see [Kaw13, Proposition 3.4] or [Bar04, § 1]).

Lemma 8.4. *If $\mathrm{Pic}^0(X) = \{0\}$ and if h_{can} is a canonical vector height for (X, Γ) defined on Π_Γ, then $h_{\mathrm{can}}(\theta_f^{\pm}, \cdot) = h_f^{\pm}(\cdot)$ for every loxodromic $f \in \Gamma$.*

Remark 8.5. If x belongs to the maximal invariant curve D_Γ and c belongs to Π_Γ then $h_{\mathrm{can}}(c, x) = 0$. Indeed for every $f \in \Gamma_{\mathrm{lox}}$, $D_\Gamma \subset D_f$ so $h_{\mathrm{can}}(\theta_f^{\pm}, x) = h_f^+(x) = 0$, and since the classes $(\theta_f^{\pm})_{f \in \Gamma_{\mathrm{lox}}}$ span Π_Γ, the result follows by linearity. This extends the second assertion of Lemma 8.3 to invariant curves which are not rational.

Now, recall that the classes $\theta_f^{\pm}$ are normalized by $\langle \theta_f^{\pm} | [\kappa_0] \rangle = 1$. Let us view $\mathbb{H}_X$ and $\mathbb{P}(\mathbb{H}_X)$ as subsets of $\{u \in H^{1,1}(X;\mathbf{R});\ \langle u|u\rangle > 0, \langle u|[\kappa_0]\rangle = 1\}$. Setting

$$\widetilde{\Pi}_\Gamma = \Pi_\Gamma \cap \{\langle \cdot | [\kappa_0] \rangle = 1\}, \tag{8.2}$$

$\mathrm{Lim}(\Gamma)$ can now be viewed as a subset of $\widetilde{\Pi}_\Gamma$ which generates Π_Γ as a vector space. The starting point of the proof of Theorem E is the following key lemma, inspired by the approach of Kawaguchi in [Kaw13].

Lemma 8.6. *In addition to the assumptions of Theorem E, suppose that:*

(iii) *there exists $f \in \Gamma_{\mathrm{lox}}$ such that $[\theta_f^+, \theta_f^-] \cap \mathrm{Int}(\mathrm{Conv}(\mathrm{Lim}(\Gamma))) \neq \emptyset$, where $\mathrm{Conv}(\cdot)$ is the convex hull and $\mathrm{Int}(\cdot)$ stands for the interior relative to $\widetilde{\Pi}_\Gamma$.*

Then (X, Γ) is a Kummer group.

Proof. Set $d = \dim \widetilde{\Pi}_\Gamma$. Replacing $\mathbf{k}$ by a finite extension, we may assume that the birational morphism $\pi_0 \colon X \to X_0$ constructed in Proposition 3.9 is defined over $\mathbf{k}$; this morphism contracts the maximal Γ-invariant curve D_Γ.

Let ν be a probability measure on Γ, whose support is finite and contains f as well as elements of Γ_{par}. Let w_ν be the eigenvector of the operator P_ν for the eigenvalue $\alpha(\nu)$ given by Lemma 5.2. As in Proposition 5.10, we may assume that w_ν is a rational class and is the pull-back of an ample class $[A_0]$ on X_0; by multiplying w_ν by a positive integer, we also assume that w_ν is an integral class.

Let L be the line bundle given by the class w_ν, and $\hat{h}_L$ be the associated canonical stationary height, as in the proof of Theorem 5.11. This is the unique Weil height such that $\sum_h \nu(h)\hat{h}_L \circ h = \alpha(\nu)\hat{h}_L$. By the linearity of the canonical vector height and the uniqueness of $\hat{h}_L$, we get $\hat{h}_L(\cdot) = h_{\mathrm{can}}(w_\nu, \cdot)$.

Pick $w = a\theta_f^+ + b\theta_f^-$ in the interior of $\mathrm{Conv}(\mathrm{Lim}(\Gamma))$, with a, b in $\mathbf{R}_+$ and $a + b = 1$. Then by linearity and Lemma 8.4, $h_{\mathrm{can}}(w, \cdot) = ah_f^+ + bh_f^-$. Caratheodory's theorem provides a subset Λ of $\mathrm{Lim}(\Gamma)$ such that $|\Lambda| = d + 1$ and w belongs to the interior of the simplex $\mathrm{Conv}(\Lambda)$. By the density of fixed points of loxodromic elements in $\mathrm{Lim}(\Gamma)$, we may assume that Λ is made of classes θ_g^+ for g in a finite subset Λ_Γ of Γ_{lox}. If $\varepsilon > 0$ is small enough, $w - \varepsilon w_\nu$ stays in $\mathrm{Int}(\mathrm{Conv}(\Lambda))$; so, there are positive coefficients β_g, for $g \in \Lambda_\Gamma$, such that $w - \varepsilon w_\nu = \sum_{g \in \Lambda_\Gamma} \beta_g \theta_g^+$. By the linearity of h_{can}, we infer that

$$ah_f^+(\cdot) + bh_f^-(\cdot) = \sum_{g \in \Lambda_\Gamma} \beta_g h_g^+(\cdot) + \varepsilon \hat{h}_L(\cdot). \tag{8.3}$$

Now, if $x \in X(\overline{\mathbf{k}})$ is f-periodic, then $h_f^+(x) = h_f^-(x) = 0$, and since $\hat{h}_L$ and the $h_g^+(x)$ are non-negative, we deduce that $\hat{h}_L(x) = 0$. The line bundle L is the pull-back of an ample line bundle A_0 on X_0; thus, by Lemma 5.14, the Γ_ν-orbit of $\pi_0(x)$ in X_0 is a finite set. Since f has a Zariski-dense set of periodic points, Theorem B implies that (X, Γ_ν) is a Kummer group.

Since we can further choose Γ_ν to contain any *a priori* given finite subset of Γ, Propositions 3.12 and 5.18 show that (X, Γ) is a Kummer group. □

From this point, the proof of Theorem E is completed in two steps. We first deal with the case $\dim \Pi_\Gamma \leq 4$ by directly checking assumption (iii) of Lemma 8.6. This covers general Wehler surfaces (which is the setting of [Kaw13]) since $\dim \Pi_\Gamma = 3$ in this case. The general case is treated in a second stage by a dimension reduction argument.

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8.2.3 *Conclusion when* $\dim \Pi_\Gamma \leq 4$. Since Γ is non-elementary, $\dim \Pi_\Gamma \geq 3$, so we need to consider the cases $\dim \Pi_\Gamma = 3$ and $\dim \Pi_\Gamma = 4$.

For $\dim \Pi_\Gamma = 3$, i.e. $d = \dim(\widetilde{\Pi}_\Gamma) = 2$, the intersection of $\widetilde{\Pi}_\Gamma$ with the positive cone is the Klein model of the hyperbolic disk $\mathbb{H}^2$. If assumption (iii) is not satisfied, then for every $f \in \Gamma_{\text{lox}}$, $\text{Lim}(\Gamma)$ is entirely contained on one side of the geodesic $[\theta_f^+, \theta_f^-]$. Fix 4 points in $\text{Lim}(\Gamma) \subset \partial\mathbb{H}^2 \simeq \mathbb{S}^1$, labelled in circular order (p_1, p_2, p_3, p_4). Lemma 3.3 provides elements f and g in Γ_{lox} such that $(\theta_f^+, \theta_g^+, \theta_f^-, \theta_g^-)$ is arbitrary close to (p_1, p_2, p_3, p_4). Then $[\theta_f^+, \theta_f^-]$ intersects $[\theta_g^+, \theta_g^-]$ transversally in the disk $\mathbb{H}^2$, so $\text{Lim}(\Gamma)$ intersects both sides of $[\theta_f^+, \theta_f^-]$, a contradiction.

Now assume $\dim \Pi_\Gamma = 4$, i.e. $d = 3$. Then $\text{Conv}(\text{Lim}(\Gamma))$ is a convex body in dimension 3. The conclusion relies on the following lemma (see below for a proof).

Lemma 8.7. *Let $p_1, \ldots, p_5$ be five points in general position in $\mathbf{R}^3$. Then, there is a pair of indices $i \neq j$ such that the line segment between p_i and p_j intersects the interior of the convex hull of p_1, $\ldots$, p_5.*

Indeed, fix such a 5-tuple of points in $\text{Lim}(\Gamma)$ and approximate the given pair (p_i, p_j) by (θ_f^+, θ_f^-), for some $f \in \Gamma_{\text{lox}}$. Then, $[\theta_f^+, \theta_f^-]$ intersects the interior of $\text{Conv}(\text{Lim}(\Gamma))$, and Lemma 8.6 finishes the proof of Theorem E (when $\dim(\Pi_\Gamma) \leq 4$).

Proof of Lemma 8.7. The general position assumption implies that there exists $\beta_i \neq 0$ such that $\sum_i \beta_i p_i = 0$ and $\sum_i \beta_i = 0$. After a permutation of the p_i and possibly replacing the β_i by their opposite, we may assume that

$$\beta_1 < 0 < \beta_2 \leq \beta_3 \leq \beta_4 \leq \beta_5 \quad \text{or} \quad \beta_1 \leq \beta_2 < 0 < \beta_3 \leq \beta_4 \leq \beta_5. \tag{8.4}$$

In the first case, p_1 lies in the interior of the simplex spanned by $p_2, \ldots, p_5$. In the second case, the point

$$\frac{-\beta_1 p_1 - \beta_2 p_2}{-\beta_1 - \beta_2} = \frac{\beta_3 p_3 + \beta_4 p_4 + \beta_5 p_5}{\beta_3 + \beta_4 + \beta_5} \tag{8.5}$$

lies in the interior of the convex hull of $p_1, \ldots, p_5$ and in the interior of the line segment between p_1 and p_2. □

Remark 8.8. The theorem of Steinitz (see [Gru03, § 13.1]) is a far-reaching generalization of Lemma 8.7. There is no analogue of this lemma in higher dimension (see [Gru03, § 4.7]), hence the need for a different argument when $\dim \Pi_\Gamma \geq 5$.

8.2.4 *Conclusion of the proof of Theorem E.* Recall from Lemma 3.1 that g^* is virtually unipotent for every $g \in \Gamma_{\text{par}}$. Thus, if we pick a pair of parabolic elements g_1, g_2 in Γ with distinct invariant fibrations, some positive iterates g_1^N and g_2^N satisfy the assumption of the following lemma.

Lemma 8.9. *Let g_1 and g_2 be parabolic elements in $\mathsf{Aut}(X)$, with distinct invariant fibrations and such that g_1^* and g_2^* are unipotent. Then, $\Gamma_0 := \langle g_1, g_2 \rangle$ is non-elementary and $\dim(\Pi_{\Gamma_0}) \leq 4$.*

Proof. Since $\pi_{g_1} \neq \pi_{g_2}$, Γ_0 is non-elementary (see § 3.1.3). The subspace $W := \text{Fix}(g_1^*) \cap \text{Fix}(g_2^*)$ of $\text{NS}(X; \mathbf{R})$ is fixed pointwise by Γ_0. Thus, $W^\perp$ is Γ_0-invariant, it contains Π_{Γ_0} (see [CD23c, Proposition 2.8]), and all we need to show is that $\dim(W^\perp) \leq 4$. To see this, note that a unipotent Euclidean isometry is the identity, thus if $g \in \mathsf{O}^+(1, d)$ is parabolic and unipotent, the structure of parabolic isometries of $\mathbb{H}_d$ (see [FL12, § I.5]) implies that $\text{Fix}(g^*) \subset \mathbf{R}^{d+1}$ is a subspace of codimension 2, and we are done. □

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Lemma 8.9 and § 8.2.3 imply that $(X, \langle g_1, g_2\rangle)$ is a Kummer group for every pair of unipotent elements $g_1, g_2 \in \Gamma_{\rm par}$ generating a non-elementary subgroup. By Proposition 3.12 we can choose $f \in \langle g_1, g_2\rangle$ such that $D_f = D_\Gamma$, hence from Proposition 5.18 we conclude that (X, Γ) is a Kummer group, and Theorem E is established.

8.3 Canonical vector heights on abelian surfaces

In this section, A is an abelian surface, defined over some number field $\mathbf{k}$ and $\Gamma \subset \mathsf{Aut}(A_{\mathbf{k}})$ is non-elementary. Denote by $h_{\rm NT}\colon \mathrm{Pic}(A) \times A(\overline{\mathbf{k}}) \to \mathbf{R}$ the Néron–Tate height on A; it vanishes identically on the torsion part of $\mathrm{Pic}(A)$, so we may also consider it as a function on $\mathrm{Pic}(A;\mathbf{R}) \times A(\overline{\mathbf{k}})$. When $0 \in A$ has a finite Γ-orbit, $h_{\rm NT}$ is a canonical vector height (see [HS00, Theorem B.5.6]).

Let $h_{\rm can}$ be a restricted canonical vector height for $(A_{\mathbf{k}}, \Gamma)$, defined on some Γ-invariant subspace $V_{\rm can}$ of $\mathrm{Pic}(A;\mathbf{R})$. Our goal is to compare it with $h_{\rm NT}$.

By definition, a divisor D on A is symmetric if $[-1]^*D$ is linearly equivalent to D, where $[m]$ denotes multiplication by m; likewise it is antisymmetric if $[-1]^*D \simeq -D$ or equivalently if $D \in \mathrm{Pic}^0(A)$ (see [HS00, Proposition A.7.3.2]). If $f \in \mathsf{Aut}(A)$ fixes the origin, it commutes to $[-1]$, so that f^* preserves symmetry and antisymmetry.

Remark 8.10. Any class $[D] \in \mathrm{NS}(A)$ can be lifted to a symmetric divisor class $D \in \mathrm{Pic}(A)$, which is unique up to a 2-torsion element in $\mathrm{Pic}^0(A)$. Thus, D admits a unique symmetric lift in $\mathrm{Pic}(A;\mathbf{R})$. By using such a lift it makes sense to consider also $h_{\rm NT}(\cdot,\cdot)$ (respectively, $h_{\rm can}(\cdot,\cdot)$) as a function on $\mathrm{NS}(A;\mathbf{R}) \times A(\overline{\mathbf{k}})$ (respectively, on the projection of $V_{\rm can}$ in $\mathrm{NS}(A;\mathbf{R})$). This observation will be used repeatedly in the following.

Remark 8.11. The Picard number of any complex abelian surface satisfies $\rho(A) \in \{1,2,3,4\}$. When $\mathsf{Aut}(A)$ contains a non-elementary group Γ, we obtain $3 \le \dim \Pi_\Gamma \le \rho(A) \le 4$. Moreover, $\rho(A) = 4$ if and only if A is isogenous to $B \times B$, for some elliptic curve B with complex multiplication (see [BL04, Ex. 10 p. 142]).

Proposition 8.12. *If $V_{\rm can}$ contains $\mathrm{Pic}^0(A) \otimes_{\mathbf{Z}} \mathbf{R}$, then Γ has a finite orbit in $A(\overline{\mathbf{k}})$.*

Proof. In this proof it is enough to consider $h_{\rm can}$ as a function on $\mathrm{Pic}^0(A) \times X(\overline{\mathbf{k}})$, by composing with the natural homomorphism $\mathrm{Pic}^0(A) \to \mathrm{Pic}^0(A;\mathbf{R})$.

Step 1: if D is an element of $\mathrm{Pic}^0(A)$, then for every $f \in \Gamma_{\rm lox}$, every periodic point x of f satisfies $h_{\rm can}(D,x) = 0$. Assume $f^q(x) = x$ for some $q \ge 1$. The endomorphism $f^q - \mathrm{id}$ is an isogeny of A because f is loxodromic (see § 4.2). Thus, its dual $(f^q)^* - \mathrm{id}$ is an isogeny of $\mathrm{Pic}^0(A)$ and we can find $E \in \mathrm{Pic}^0(A)$ such that $(f^q)^*E - E = D$. By equivariance $h_{\rm can}((f^q)^*E, x) = h_{\rm can}(E,x)$, and then by linearity $h_{\rm can}(D,x) = 0$.

Step 2: let $\mathbf{k}'$ be a finite extension of $\mathbf{k}$ and let P be a subset of $A(\mathbf{k}')$. If, for every $D \in \mathrm{Pic}^0(A)$, the set $\{h_{\rm NT}(D,x);\ x \in P\} \subset \mathbf{R}$ is bounded, then P is finite. To see this, consider the abelian group $A(\mathbf{k}')$; by the Mordell–Weil theorem, its rank is finite, so modulo torsion it is isomorphic to $\mathbf{Z}^r$ for some $r \ge 0$. Set $W_{\mathbf{k}'} = A(\mathbf{k}') \otimes_{\mathbf{Z}} \mathbf{R}$, a real vector space of dimension r. Let H be an ample symmetric divisor on $A_{\overline{\mathbf{k}}}$, then $h_{\rm NT}(H,\cdot)$ determines a positive-definite quadratic form on V; let $\langle\cdot|\cdot\rangle_H$ be the bilinear pairing associated to $h_{\rm NT}(H,\cdot)$. If s is an element of $A(\overline{\mathbf{k}})$, and $t_s \in \mathsf{Aut}(A_{\overline{\mathbf{k}}})$ is the translation by s, then $D_s := H - t_s^*H$ is an element of $\mathrm{Pic}^0(A)$ and $h_{\rm NT}(D_s,\cdot)$ induces an affine linear form $A(\mathbf{k}') \to \mathbf{R}$; namely, $h_{\rm NT}(D_s,\cdot) = -2\langle s|\cdot\rangle_H$. Since $\langle\cdot|\cdot\rangle_H$ is positive definite on $W_{\mathbf{k}'}$ (see [HS00, Proposition B.5.3]), one can find r elements $s_i \in A(\mathbf{k}')$ such that the linear forms $\ell_i := \langle s_i|\cdot\rangle_H$ constitute a basis of the dual of $W_{\mathbf{k}'}$. Our assumption says that each

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$\ell_i(P)$ is a relatively compact subset of $\mathbf{R}$; this implies that P is contained in a compact, hence finite, subset of the lattice $A(\mathbf{k}') \subset V$.

Step 3: Γ has a finite orbit. Let f be a loxodromic element of Γ, and x be a fixed point of f. Its Γ-orbit is made of fixed points of conjugates of f. Note that $\Gamma(x)$ is contained in $A(\mathbf{k}')$ for some finite extension of $\mathbf{k}$. By the first step, h_{can} vanishes on $\mathrm{Pic}^0(A) \times \Gamma(x)$. Since h_{can} and h_{NT} are Weil heights, $|h_{\mathrm{can}}(D, \cdot) - h_{\mathrm{NT}}(D, \cdot)| \leq B(D)$ for each divisor class $D \in \mathrm{Pic}^0(A)$, where $B(D) \geq 0$ depends on D. Thus, $|h_{\mathrm{NT}}(D, \Gamma(x))| \leq B(D)$ for every $D \in \mathrm{Pic}^0(A)$, and the second step implies that $\Gamma(x)$ is finite. □

Proposition 8.13. *Assume that the neutral element has a finite Γ-orbit. Then h_{can} coincides with the Néron–Tate height on:*

- *the set of symmetric divisors whose numerical class belongs to Π_Γ;*
- *the set of antisymmetric divisors;*

whenever one of these sets is contained in V_{can}.

In the following proofs, we denote by Π_Γ^{s} the subspace $\mathrm{Pic}(A; \mathbf{R})$ made of symmetric elements $E \in \mathrm{Pic}(A; \mathbf{R})$ such that $[E] \in \Pi_\Gamma$.

Proof. Let $\Gamma_0 \leq \Gamma$ be the finite index subgroup fixing the origin. Let us show that $h_{\mathrm{can}} = h_{\mathrm{NT}}$ on $\Pi_\Gamma^{\mathrm{s}} \times A(\overline{\mathbf{k}})$. For this, we use Remark 8.10, identify Π_Γ^{s} with Π_Γ, and consider h_{can} and h_{NT} as functions on $\Pi_\Gamma \times A(\overline{\mathbf{k}})$. Now, if $f \in \Gamma_{0,\mathrm{lox}}$, we get $h_{\mathrm{NT}}(\theta_f^+, \cdot) = h_{\mathrm{can}}(\theta_f^+, \cdot)$ because the difference is bounded on $A(\overline{\mathbf{k}})$, and is multiplied by $\lambda(f) > 1$ under the action of f (as in Lemma 8.4). Since the classes θ_f^+, for $f \in \Gamma_{\mathrm{lox}}$, generate Π_Γ, our claim is established.

Let us now deal with antisymmetric divisors. Identifying $\mathrm{Pic}^0(A_{\overline{\mathbf{k}}})$ with the dual abelian variety $A^\vee_{\overline{\mathbf{k}}}$ of A, we have to show that h_{can} coincides with h_{NT} on $A^\vee(\mathbf{k}')$ for every finite extension $\mathbf{k}'$ of $\mathbf{k}$. By the Mordell–Weil theorem $A^\vee(\mathbf{k}')$ is a finitely generated abelian group so

$$W^\vee_{\mathbf{k}'} := A^\vee(\mathbf{k}') \otimes_{\mathbf{Z}} \mathbf{R} \tag{8.6}$$

is a real vector space of dimension r, for some $r < +\infty$. Consider the function $\Phi : (D, x) \mapsto h_{\mathrm{can}}(D, x) - h_{\mathrm{NT}}(D, x)$. When D is fixed, $\Phi(D, \cdot)$ is bounded: $|\Phi(D, x)| \leq B(D)$ for all $x \in A(\overline{\mathbf{k}})$. On the other hand, when x is fixed, $\Phi_x(D) := \Phi(D, x)$ defines a linear form $\Phi_x \colon W^\vee_{\mathbf{k}'} \to \mathbf{R}$. Applying the previous boundedness property to $f(x)$, for f ranging in Γ_0, and using the equivariance $\Phi(D, f(x)) = \Phi(f^* D, x)$ we obtain that for every $x \in A(\overline{\mathbf{k}})$, Φ_x is bounded on every Γ_0^*-orbit $\Gamma_0^*(D) \subset W^\vee_{\mathbf{k}'}$.

We claim that this forces Φ_x to vanish, which is the desired result. For this we analyze the dual action of Γ_0. Let f be a loxodromic element of Γ_0, and $f^\vee_{\mathbf{k}'}$ be the induced linear map on $W^\vee_{\mathbf{k}'} = A^\vee(\mathbf{k}') \otimes_{\mathbf{Z}} \mathbf{R}$. Let L_f be the linear lift of f to $\mathbf{C}^2$, as in §§ 4.1 and 4.2.

Lemma 8.14. *The endomorphism $f^\vee_{\mathbf{k}'}$ is semi-simple and its complex eigenvalues are complex conjugate to those of L_f; none of them has modulus* 1.

Let us take this for granted and conclude the proof. Since $f^\vee_{\mathbf{k}'}$ is semi-simple, $W^\vee_{\mathbf{k}'}$ is a direct sum of $f^\vee_{\mathbf{k}'}$-invariant irreducible factors $\bigoplus W_i^\vee$, each of dimension 1 or 2. For each $W_i^\vee$, denote by λ_i the corresponding eigenvalue of $f^\vee_{\mathbf{k}'}$, and pick some $D_i \in W_i^\vee \setminus \{0\}$. If $W_i^\vee$ is a line, then $\lambda_i \in \mathbf{R}^*$ and $|\lambda_i| \neq 1$. Since Φ_x is bounded on $\{(f^\vee_{\mathbf{k}'})^n(D_i);\ n \in \mathbf{Z}\}$, the line $W_i^\vee$ is contained in $\ker \Phi_x$. If $W_i^\vee$ is a plane, then $f^\vee_{\mathbf{k}'}|_{W_i^\vee}$ is a similitude with $|\lambda_i| \neq 1$ and $\mathrm{Arg}(\lambda_i) \neq 0$ mod $(2\pi\mathbf{Z})$. If $\Phi_x|_{W_i^\vee} \neq 0$, $\{|\Phi_x| \leq B(D_i)\} \cap W_i^\vee$ is a strip, which furthermore contains the orbit

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$\{(f_{\mathbf{k}'}^\vee)^n(D_i), n \in \mathbf{Z}\}$. This is not compatible with the properties of λ_i, and this contradiction shows that $W_i^\vee \subset \ker \Phi_x$, so finally $\Phi_x = 0$, as claimed. $\square$

Proof of Lemma 8.14. The complex torus underlying $A_{\mathbf{C}}^\vee$ is isomorphic to a quotient of the space of $\mathbf{C}$-antilinear forms on $\mathbf{C}^2$. Thus, if $f \in \mathsf{Aut}(A)$ is induced by a linear map $L_f \in \mathsf{GL}_2(\mathbf{C})$, the automorphism of $A^\vee$ determined by f^* is induced by the conjugate transpose $\overline{L}_f^t$ (see [BL04, § 2.4]). When f is loxodromic, the eigenvalues of L_f satisfy $|\alpha| < 1 < |\beta|$; we deduce that the automorphism of $\mathsf{Aut}(A_{\mathbf{C}}^\vee)$ determined by f^* is also loxodromic, with eigenvalues $\overline{\alpha}$ and $\overline{\beta}$, and the minimal polynomial of $\overline{L}_f^t$ is $(X-\overline{\alpha})(X-\overline{\beta})$. Let P be the minimal, real, unitary polynomial such that $(X-\overline{\alpha})(X-\overline{\beta})$ divides P (by construction $\deg(P) \in \{2,3,4\}$ and P has no repeated factors). Since $P(\overline{L}_f^t) = 0$, we infer that $P(f_{\mathbf{k}'}^\vee) = 0$ and the result follows. $\square$

Proposition 8.15. *Let $A_{\mathbf{k}}$ be an abelian surface defined over a number field $\mathbf{k}$. Let Γ be a non-elementary subgroup of $\mathsf{Aut}(A_{\mathbf{k}})$, for which the neutral element $0 \in A(\mathbf{k})$ is periodic. Then one of the following situation occurs:*

(1) $\mathrm{NS}(A, \mathbf{R}) = \Pi_\Gamma$ *and the Néron–Tate height is the unique canonical vector height on* $\mathrm{Pic}(A; \mathbf{R})$*;*

(2) $\mathrm{NS}(A, \mathbf{R}) = \Pi_\Gamma \overset{\perp}{\oplus} \mathbf{R}[E]$ *for some* $[E] \in \mathrm{NS}(A; \mathbf{R}) \setminus \{0\}$*, and the canonical vector heights on* $\mathrm{Pic}(A; \mathbf{R})$ *are exactly the functions of the form* $h_{\mathrm{can}}(D, x) = h_{\mathrm{NT}}(D, x) + \langle [E] | D\rangle \varphi(x)$*, where* $\varphi\colon A(\overline{\mathbf{k}}) \to \mathbf{R}$ *is any bounded function such that* $\varphi(f(x)) f^*[E] = \varphi(x)[E]$ *for all* f *in* Γ.

Proof. When $\mathrm{NS}(A, \mathbf{R}) = \Pi_\Gamma$, Proposition 8.13 and the decomposition of any divisor class as a sum $D = D^{\mathrm{s}} + D^{\mathrm{a}}$ with D^{s} symmetric and D^{a} antisymmetric imply that $h_{\mathrm{can}} = h_{\mathrm{NT}}$. Thus, by Remark 8.11 we may assume that $\rho(A) = 4$ and $\dim(\Pi_\Gamma) = 3$. Pick $[E] \in \Pi_\Gamma^\perp \setminus \{0\}$. The line $\mathbf{R}[E]$ is Γ-invariant, and the intersection form is negative on $\mathbf{R}[E]$; as a consequence, there is a homomorphism $\alpha_{[E]}\colon \Gamma \to \{+1, -1\}$ such that $f^*[E] = \alpha(f)[E]$ for all $f \in \Gamma$. Then for fixed x,

$$\Delta_x(D) = h_{\mathrm{can}}(D, x) - h_{\mathrm{NT}}(D, x), \tag{8.7}$$

defines a linear form on $\mathrm{Pic}(A; \mathbf{R})$, which by Proposition 8.13 vanishes identically on Π_Γ^{s}. Thus, $\Delta(D, x) = \langle [E] | D\rangle \varphi(x)$ for some real valued function φ, and the conclusion follows from Lemma 8.1. $\square$

8.4 Synthesis

8.4.1 *Canonical vector heights.* Putting together Theorem E and Proposition 8.15 gives the following.

Theorem E′. *Let X be a smooth projective surface, defined over a number field $\mathbf{k}$. Let Γ be a non-elementary subgroup of $\mathsf{Aut}(X_{\mathbf{k}})$ that contains parabolic elements. Let h_{can} be a canonical vector height on $\mathrm{Pic}(X; \mathbf{R}) \times X(\overline{\mathbf{k}})$ for the group Γ. Then, X is an abelian surface and h_{can} is derived from a translate of the Néron–Tate height by a periodic point y of Γ:*

$$h_{\mathrm{can}}(D, x) = h_{\mathrm{NT}}(D, x + y) + \langle [E] | D\rangle\, \varphi(x)$$

for some almost-invariant class $[E] \in \mathrm{NS}(X; \mathbf{R})$ and some bounded function $\varphi\colon X(\overline{\mathbf{k}}) \to \mathbf{R}$ such that $\varphi(f(x)) f^*[E] = \varphi(x)[E]$ for $f \in \Gamma$.

Note that h_{can} is just a translate of h_{NT} when E is numerically trivial.

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8.4.2 *Restricted canonical vector heights.* Let us add the assumption

$$\mathrm{Pic}^0(X) = 0, \tag{8.8}$$

to the hypotheses of Theorem E. Our goal is to describe all possibilities for $(V_{\mathrm{can}}, h_{\mathrm{can}})$.

Since $\mathrm{Pic}^0(X) = 0$, X is not a blow-up of an abelian surface and Theorem E implies that (X, Γ) is a Kummer group of type (2), (3), (4), or (5) in the nomenclature of § 4.3.2. We make use of the notation of §§ 4.3.1 and 4.3.2. The origin $0 \in A$ is a fixed point of the cyclic group G, and the orbit $\Gamma_A(0)$ is finite. Since G is generated by a finite-order homothety $(x, y) \mapsto (\alpha x, \alpha y)$ on A, G acts trivially on $\mathrm{NS}(A; \mathbf{R})$ and on symmetric divisors. Thus, $\mathrm{NS}(A/G; \mathbf{R})$ can be identified to $\mathrm{NS}(A; \mathbf{R})$ and to the subspace of $\mathrm{Pic}(A; \mathbf{R})$ generated by symmetric divisors; let

$$\iota \colon \mathrm{NS}(A; \mathbf{R}) \to \mathrm{NS}(X; \mathbf{R}) \tag{8.9}$$

denote the corresponding embedding, given by $\iota = q_X^*(q_A)_*$. On the space of symmetric divisors, the Néron–Tate height is G-invariant and Γ-equivariant, so it induces a canonical vector height $h_{\mathrm{NT}}^{A/G}(\cdot, \cdot)$ on A/G for $\overline{\Gamma}_A$. Then, it induces a restricted canonical vector height on $\iota(\mathrm{NS}(A; \mathbf{R})) \times X(\overline{\mathbf{k}})$, namely

$$h_{\mathrm{NT}}^X : (D, x) \longmapsto h_{\mathrm{NT}}^{A/G}((q_X)_* D, q_X(x)). \tag{8.10}$$

In what follows, we denote by E_i the disjoint irreducible rational curves contracted by q_X (see Lemma 4.7); their classes generate $\iota(\mathrm{NS}(A; \mathbf{R}))^\perp \subset \mathrm{NS}(X; \mathbf{R})$. The height h_{NT}^X vanishes on $\bigcup_i E_i(\overline{\mathbf{k}})$, because the E_i are mapped to torsion points of A.

Lemma 8.16. *We have $\Pi_\Gamma = \iota(\Pi_{\Gamma_A}) \subset V_{\mathrm{can}} \subset \iota(\mathrm{NS}(A; \mathbf{R}))$.*

Proof. The first equality comes from the equivariance of q_X and q_A. The first inclusion follows from Lemma 8.3 and the assumption (ii) of Theorem E. It remains to prove the last inclusion. If V_{can} is not contained in $q_X^*(\mathrm{NS}(A/G; \mathbf{R}))$, there is an index i, and a class D in $\Pi_\Gamma^\perp \cap V_{\mathrm{can}}$ such that $\langle D | E_i \rangle > 0$, i.e. $\mathcal{O}(D)|_{E_i}$ is ample. The action of Γ on $\Pi_\Gamma^\perp$ factorizes through a finite group (see [CD23c, Lemma 2.9]), so D is Γ-periodic and by Lemma 8.2, $\Gamma|_{E_i}$ is finite; this contradicts Lemma 4.7, and the conclusion follows. □

Let D be an element of Π_Γ. By Lemma 8.3, $h_{\mathrm{can}}(D, x) = 0$ for all $x \in \bigcup_i E_i(\overline{\mathbf{k}})$. Thus, $(D, x) \mapsto h_{\mathrm{can}}(\iota(D), q_X^{-1}(q_A(x)))$ is a well-defined restricted canonical vector height on $\Pi_{\Gamma_A} \times A(\overline{\mathbf{k}})$ (see Remark 8.10), which gives height 0 to the fixed points of elements of $G \setminus \{\mathrm{id}\}$. By Proposition 8.13, this height coincides with the Néron–Tate height on $\Pi_{\Gamma_A} \times A(\overline{\mathbf{k}})$.

This yields a complete description of h_{can} when $V_{\mathrm{can}} = \Pi_\Gamma$.

By Lemma 8.16 and Remark 8.11, the remaining possibility is that $\dim(V_{\mathrm{can}}) = 4$ and $\dim(\Pi_\Gamma) = 3$. Choose an almost Γ_A-invariant class $[E]$ in $\mathrm{NS}(A; \mathbf{R})$, as in Proposition 8.15, and a divisor F in X such that $[F] = \iota([E])$. Each element $D \in V_{\mathrm{can}}$ decomposes as a sum

$$D = D' + \frac{\langle [F] | D \rangle}{\langle [F] | [F] \rangle} [F] \tag{8.11}$$

with D' in Π_Γ. Then, for x in $\bigcup_i E_i(\overline{\mathbf{k}})$, we get

$$h_{\mathrm{can}}(D, x) = \frac{\langle [F] | D \rangle}{\langle [F] | [F] \rangle} h_{\mathrm{can}}([F], x). \tag{8.12}$$

Define a function by setting $\psi(x) = \langle [F] | [F] \rangle^{-1} h_{\mathrm{can}}([F], x)$ on $\bigcup_i E_i(\overline{\mathbf{k}})$ and $\psi(x) = 0$ otherwise. It satisfies the equivariance $\psi(f(x)) f^*[F] = \psi(x)[F]$ because h_{can} is equivariant and $[F]$ is almost

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invariant, and it is bounded because $\mathcal{O}(F)_{|E_i}$ is trivial for each E_i. Now, if we set

$$h'_{\mathrm{can}}(D,x) = h_{\mathrm{can}}(D,x) - \langle [F] | D\rangle\, \psi(x) \tag{8.13}$$

we get a new restricted canonical vector height on $V_{\mathrm{can}} \times X(\overline{\mathbf{k}})$ that vanishes on $\bigcup_i E_i(\overline{\mathbf{k}})$. This height comes from a canonical vector height on A/G, and since as seen before $\mathrm{NS}(A/G;\mathbf{R})$ can be identified to $\mathrm{NS}(A;\mathbf{R})$, it yields a canonical vector height for (A,Γ_A) restricted to the space of symmetric divisors. The second assertion of Proposition 8.15 entails that this last height is derived from the Néron–Tate height for some function φ; since Γ_A contains G, and G fixes $[E]$, φ is G-invariant. Coming back to X, we get that h_{can} is derived from the Néron–Tate height too. In formulas,

$$h_{\mathrm{can}}(D,x) = h^X_{\mathrm{NT}}(D,x) + \langle [F] | D\rangle\, \Phi(x), \tag{8.14}$$

where $\Phi\colon X(\overline{\mathbf{k}}) \to \mathbf{R}$ is a bounded function which satisfies $\Phi(f(x))f^*[F] = \Phi(x)[F]$ for $f \in \Gamma$. This function is equal to ψ on $\bigcup_i E_i(\overline{\mathbf{k}})$ and to $\varphi \circ q_X$ on its complement.

To conclude, using the above notation, let us summarize these results in a (somewhat imprecise) statement.

Theorem E$''$. *Let X be a smooth projective surface, defined over a number field $\mathbf{k}$, and such that $\mathrm{Pic}^0(X_{\overline{\mathbf{k}}}) = 0$. Let Γ be a non-elementary subgroup of $\mathsf{Aut}(X_{\mathbf{k}})$ containing parabolic elements. Let h_{can} be a restricted canonical vector height on $V_{\mathrm{can}} \times X(\overline{\mathbf{k}})$ for the group Γ, where $V_{\mathrm{can}} \subset \mathrm{Pic}(X;\mathbf{R})$ is Γ-invariant and contains classes with positive self-intersection. Then (X,Γ) is a Kummer group associated to an abelian surface A, V_{can} is contained in $\iota(\mathrm{NS}(A;\mathbf{R}))$ and:*

– *either $V_{\mathrm{can}} = \Pi_\Gamma$ and h_{can} coincides with the Néron–Tate height h^X_{NT};*
– *or Π_Γ is a codimension-1 subspace of V_{can} and h_{can} is derived from h^X_{NT}.*

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Conflicts of Interest

None.

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