A LOCAL PROPERTY OF MEASURABLE SETS

W. EAMES

1. Introduction.[1] Let Ω be a metric space with metric ρ, let $\mathbf{C}$ be a class of closed sets from Ω and let τ be a non-negative real-valued set function on $\mathbf{C}$. We assume that the empty set ϕ is in $\mathbf{C}$ and that $\tau(I) = 0$ if and only if $I = \phi$. For each set A in Ω, we define $\varphi(A)$, $0 \leqslant \varphi(A) \leqslant \infty$ by:

$$\varphi(A) = \lim_{\epsilon \to 0^+} \left[\inf \sum_{n=1}^{\infty} \tau(I(n)) \right]$$

where the infimum is taken for all possible countable collections of sets $I(n)$ from $\mathbf{C}$ such that:

$$A \subset \bigcup_{n=1}^{\infty} I(n)$$

and the diameter of $I(n)$, $d(I(n))$, is less than ϵ for every n. We assume that such a countable collection exists for every set A and every $\epsilon > 0$. φ is an outer measure function (**3**, p. 85), that is:

(i) $\varphi(\phi) = 0$,

(ii) If $A \subset B$, then $\varphi(A) \leqslant \varphi(B)$,

(iii) For any sequence of sets $A(n)$, $n = 1, 2, 3, \ldots$, in Ω,

$$\varphi\left(\bigcup_{n=1}^{\infty} A(n) \right) \leqslant \sum_{n=1}^{\infty} \varphi(A(n)) .$$

A set A is said to be *measurable* if

$$\varphi(B) = \varphi(A \cap B) + \varphi(\tilde{A} \cap B)$$

for every set B in Ω. All Borel sets are measurable (**3**, pp. 102–106). For every set A in Ω, there is a measurable set B, called a *measurable cover* for A, such that $A \subset B$ and $\varphi(B) = \varphi(A)$ (**3**, pp. 107–108). That is, φ is a regular metric outer measure function.

We also define for a set A and a point p in Ω, the number $D(A,p)$, $0 \leqslant D(A,p) \leqslant \infty$, by:

$$D(A, p) = \lim_{\epsilon \to 0^+} \left[\sup \frac{\varphi(A \cap I)}{\tau(I)} \right]$$

where the supremum is taken as I ranges over all sets in $\mathbf{C}$ such that $p \in I$ and $d(I) < \epsilon$.

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We show that, if the sets in **C** satisfy a certain regularity condition and $\varphi(A)$ is finite, then:

(i) $D(A,p) = 1$ for almost all $p \in A$,

(ii) $D(A,p) = 0$ for almost all $p \in \tilde{A}$ if and only if A is measurable,

(iii) By considering the behaviour of $D(A,p)$ it is possible to construct a measurable cover for A in a unique manner.

These results are similar to those previously obtained for the case of Hausdorff outer measure in Euclidean space **(2, 5)**.

2. Separated sets. Let A and B be two sets in Ω. A is said to be *separated* from B if, for every $\epsilon > 0$, there is an open set O such that $\varphi(A \cap \tilde{O}) < \epsilon$ and $\varphi(B \cap O) < \epsilon$ **(2)**. Equivalently, A is separated from B if, for every $\epsilon > 0$, there is a closed set F such that $\varphi(\tilde{F} \cap B) < \epsilon$ and $\varphi(A \cap F) < \epsilon$. If A is separated from B then B is not necessarily separated from A. For example, let φ be linear outer measure in the plane, let A be the interior of a circle and let B be the circumference of this circle. However, we have:

THEOREM 1. *If $\varphi(A)$ is finite, and A is separated from B, then B is separated from A.*

Proof. Let O be an open set such that $\varphi(A \cap \tilde{O}) < \frac{1}{2}\epsilon$ and $\varphi(B \cap O) < \frac{1}{2}\epsilon$. Because Ω is a metric space, O is the union of a countable number of closed sets, $C(n)$, $n = 1, 2 \ldots$. From the measurability of O,

$$\varphi(A) - \varphi(A \cap O) < \tfrac{1}{2}\epsilon.$$

Choose an integer m such that

$$\varphi(A \cap O) < \varphi(A \cap \mathrm{F}) + \tfrac{1}{2}\epsilon$$

where F is the closed set

$$\bigcup_{n=1}^{m} C(n).$$

Then $\varphi(A \cap \tilde{F}) < \epsilon$ and $\varphi(B \cap F) < \epsilon$ so that B is separated from A.

THEOREM 2. *If A is separated from B, then $\varphi(A \cup B) = \varphi(A) + \varphi(B)$.*

Proof. We can assume that $\varphi(A)$ and $\varphi(B)$ are finite. Since A is separated from B and B is therefore separated from A, there are open sets O and G such that $\varphi(O \cap B)$, $\varphi(\tilde{O} \cap A)$, $\varphi(G \cap A)$, and $\varphi(\tilde{G} \cap B)$ are all less than a preassigned $\epsilon > 0$. From the measurability of O and G, $\varphi(A \cap O) > \varphi(A) - \epsilon$, and $\varphi(B \cap G) > \varphi(B) - \epsilon$. From the subadditivity of φ, $\varphi(O \cap G \cap (A \cup B)) < 2\epsilon$. Thus,

$$\varphi(A \cup B) \geqslant \varphi((A \cup B) \cap (O \cup G)) > \varphi(A) + \varphi(B) - 4\epsilon$$

which, with the subadditivity of φ, proves the theorem.

THEOREM 3. *If $\varphi(A)$ is finite, then A is measurable if and only if $\tilde{A}$ is separated from A.*

Proof. Assume first that A is measurable. From the definition of φ we can, for a given integer n, obtain a sequence of closed sets $I(n,i)$, $i = 1, 2, \ldots,$ from **C** with diameter $< 1/n$, whose union covers A and such that:

$$\sum_{i=1}^{\infty} \tau(I(n, i)) - \frac{1}{n} \leqslant \varphi(A) \leqslant \sum_{i=1}^{\infty} \tau(I(n, i)) + \frac{1}{n}.$$

From the regularity of φ,

$$\varphi(A) = \lim_{m\to\infty} \varphi\left(A \cap \bigcup_{i=1}^{m} I(n, i)\right)$$

so there is an integer $m(n)$ such that

$$\varphi(A) - \varphi\left(A \cap \bigcup_{i=1}^{m(n)} I(n, i)\right) < \frac{\epsilon}{2^n}.$$

Let

$$F = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{m(n)} I(n, i).$$

F is closed, $\varphi(A \cap \tilde{F}) < \epsilon$, and, using the measurability of A and F,

$$\varphi(F \cap \tilde{A}) - \varphi(\tilde{F} \cap A) = \varphi(F) - \varphi(A) \leqslant 0,$$

so that $\varphi(F \cap \tilde{A}) \leqslant \varphi(\tilde{F} \cap A) < \epsilon$, and $\tilde{A}$ is separated from A.

If $\tilde{A}$ is separated from A then $B \cap \tilde{A}$ is separated from $B \cap A$ for any set B so, from Theorem 2, $\varphi(B) = \varphi(B \cap \tilde{A}) + \varphi(B \cap A)$ and A is thus measurable.

THEOREM 4. *If $\varphi(A)$ and $\varphi(B)$ are finite and $\varphi(A \cup B) = \varphi(A) + \varphi(B)$, then A is separated from B.*

Proof. Assume first that B is measurable and assign an $\epsilon > 0$. From the measurability of B and the hypothesis, $\varphi(A \cap B) = 0$. By Theorem 3 there is a closed set F such that $\varphi(B \cap \tilde{F}) < \epsilon$ and $\varphi(\tilde{B} \cap F) < \epsilon$. Because $F \subset B \cup (F \cap \tilde{B})$, we have $\varphi(A \cap F) < \epsilon$ and A is separated from B. If B is not measurable let C be a measurable cover for B. By the preceding, A is separated from C and thus from B.

3. The function $D(A, p)$. This function has been defined in the introduction. Several functions of this type, differing only in the class of sets I over which the supremum is taken, have been studied for the case of Hausdorff outer measure in Euclidean space by various authors (**1**; **2**; **4**; **5**) and examples due to Besicovitch (**1**) and Nikodym (**4**) show that a slight change in this class radically affects the function. If $\varphi(A)$ is finite, then our choice, namely "all I in **C** such that $p \in I$ and $d(I) < \epsilon$," results in a function which is, for

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almost all p, the characteristic function of A if and only if A is measurable and thus it affords a local characterization of the measurability of A.

THEOREM 5. *If $\varphi(A)$ is finite, then, for every $\epsilon > 0$ there is a $\delta > 0$ such that*:

$$\varphi\left(A \cap \bigcup_{n=1}^{\infty} I(n)\right) \leqslant \sum_{n=1}^{\infty} \tau(I(n)) + \epsilon$$

for any sequence of sets $I(n)$, $n = 1, 2, \ldots$, from $\mathbf{C}$ with diameters less than δ.

Proof. This has been proved (**1**, p. 427) for the case in which φ is linear outer measure in the plane and A is measurable. The proof here is the same, using the measurability of the sets in $\mathbf{C}$ instead of the measurability of A.

THEOREM 6. *If $\varphi(A)$ is finite, then $D(A, p) \geqslant 1$ for almost all $p \in A$.*

Proof. Let $a > 0$ and $0 < b < 1$. Let $T(a, b)$ be the set:

$$T(a, b) = \{p\colon p \in A \text{ and if } I \in \mathbf{C} \text{ and } p \in I \\ \text{and } d(I) < a, \text{ then } \varphi(A \cap I) \leqslant b \cdot \tau(I)\}.$$

It is sufficient to prove that $T(a, b)$ is null for all such a and b. Assign an $\epsilon > 0$ and obtain a δ by Theorem 5. Choose $\delta < a$. Let $\{I(n)\}$ be a sequence of sets from $\mathbf{C}$ with diameter $< \delta$, whose union covers A, and such that:

$$\varphi(A) - \epsilon \leqslant \sum_{n=1}^{\infty} \tau(I(n)) \leqslant \varphi(A) + \epsilon.$$

Let $\mathbf{A}$ be the class of all sets from this sequence such that $\varphi(A \cap I(n)) > b \cdot \tau(I(n))$ and let $\mathbf{B}$ be the class of all the other sets from the sequence.

Then:

$$\begin{aligned}\varphi(A) &\leqslant \varphi(A \cap \bigcup_{\mathbf{A}} I(n)) + \varphi(A \cap \bigcup_{\mathbf{B}} I(n)) \\ &\leqslant \sum_{\mathbf{A}} \tau(I(n)) + \epsilon + b \cdot \sum_{\mathbf{B}} \tau(I(n)) \\ &\leqslant \varphi(A) + 2\epsilon + (b - 1) \cdot \sum_{\mathbf{B}} \tau(I(n)).\end{aligned}$$

Thus

$$\sum_{\mathbf{B}} \tau(I(n)) \leqslant 2\epsilon/(1 - b)$$

and also,

$$\varphi(T(a, b)) \leqslant \varphi(A \cap \bigcup_{\mathbf{B}} I(n)) \leqslant \sum_{\mathbf{B}} \tau(I(n)) + \epsilon,$$

so that $T(a, b)$ is null, which proves the theorem.

THEOREM 7. *If $\varphi(B)$ is finite and $D(A, p) = 0$ for almost all $p \epsilon B$, then A is separated from B.*

Proof. Assign an $\epsilon > 0$ and consider the set

$$B(n) = \{p: p \in B \text{ and if } I \in \mathbf{C} \text{ and } p \in I \text{ and } d(I) < 1/n, \text{ then } \varphi(A \cap I) < \epsilon\cdot\tau(I)\}.$$

By hypothesis,

$$B - N \subseteq \bigcup_{n=1}^{\infty} B(n) \subseteq B$$

where N is a null subset of B, and thus **(3**, p. 95**)**,

$$\varphi(B) = \lim_{n\to\infty} \varphi(B(n)).$$

Choose an integer m so that $\varphi(B) \leqslant \varphi(B(m)) + \epsilon$ and let δ be determined by applying Theorem 5 to B. Let $\{I(n)\}$ be a sequence of sets from $\mathbf{C}$ with diameter less than the smaller of $1/m$ and δ, whose union covers $B(m)$, and such that:

$$\sum_{n=1}^{\infty} \tau(I(n)) - \epsilon \leqslant \varphi(B(m)) \leqslant \sum_{n=1}^{\infty} \tau(I(n)) + \epsilon.$$

Choose an integer r such that

$$\sum_{n=r+1}^{\infty} \tau(I(n)) < \epsilon.$$

Then, by Theorem 5,

$$\varphi\left(B \cap \bigcup_{n=r+1}^{\infty} I(n)\right) < 2\epsilon.$$

By the definition of $B(m)$,

$$\varphi\left(A \cap \bigcup_{n=1}^{r} I(n)\right) \leqslant \sum_{n=1}^{r} \varphi(A \cap I(n)) < \epsilon\cdot \sum_{n=1}^{\infty} \tau(I(n))$$
$$< \epsilon(\varphi(B(m)) + \epsilon).$$

Let

$$C = \bigcup_{n=1}^{r} I(n).$$

C is closed and $\varphi(B \cap \tilde{C}) \leqslant 3\epsilon$ and $\varphi(A \cap C) < \epsilon\ (\varphi(B) + \epsilon)$ so that A is separated from B.

4. The regularity conditions. For the remainder of the paper we assume that to every set I from $\mathbf{C}$ there corresponds a set I' from $\mathbf{C}$ such that:

(i) $I' \supset \{p: \rho(p, I) \leqslant \alpha\cdot d(I)\}$ where α is a finite number greater than 1 and independent of I and $\rho(p, I)$ is the distance from p to I.

(ii) $\tau(I') \leqslant \beta\cdot\tau(I)$ where β is a finite number independent of I.

(iii) for every $\epsilon > 0$ there is a $\delta > 0$ such that $d(I) < \delta$ implies $d(I') < \epsilon$.

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THEOREM 8. *If $\varphi(A)$ is finite and B is separated from A, then $D(A, p) = 0$ for almost all $p \in B$.*

Proof. Assign $\epsilon > 0$, $\eta > 0$, and $\delta > 0$. Let O be an open set such that $\varphi(A \cap O) < \epsilon^2$ and $\varphi(B \cap \tilde{O}) < \epsilon^2$. Let $A(\epsilon) = \{p: p \in B \text{ and } D(A, p) > \epsilon\}$. For every $p \in A(\epsilon) \cap O$ there is a set $I \in \mathbf{C}$ such that $p \in I$, $d(I) < \eta$, $I \subset O$ and $\varphi(A \cap I) > \epsilon \cdot \tau(I)$. Also, $d(I) \neq 0$, since $d(I) = 0$ implies that I is a set consisting of exactly one point, $I \subset A \cap B$, and $\varphi(I) > 0$, which is impossible if B is separated from A. Let the class of all such sets be denoted by $\mathbf{J}$.

Choose a maximal class $\mathbf{I}(1)$ of disjoint sets from $\mathbf{J}$ with the property that $\delta/\alpha \leqslant d(I) < \delta$ where α is the number referred to in (i) of the regularity conditions. Continue this process in the following way: let $\mathbf{I}(1), \mathbf{I}(2), \mathbf{I}(3), \ldots,$ be a sequence of maximal classes of disjoint sets from $\mathbf{J}$ such that if $I \in \mathbf{I}(n)$ then I is disjoint from all the sets in any preceding class and $\delta/\alpha^n \leqslant d(I) < \delta/\alpha^{n-1}$. Each $\mathbf{I}(n)$ contains at most a countable number of sets since $\tau(I) > 0$ for every $I \in \mathbf{J}$ and

$$\sum_I \tau(I) < \frac{1}{\epsilon} \cdot \sum_I \varphi(A \cap I) = \frac{1}{\epsilon} \cdot \varphi\left(A \cap \bigcup_I I\right) < \infty$$

for any countable selection of sets from $\mathbf{I}(n)$.

Thus the union of all the classes $\mathbf{I}(n)$ is a class containing at most a countable number of sets. With every set $I(n)$, $n = 1, 2, 3, \ldots,$ in this union associate a set $I'(n)$ by the regularity condition. Then,

$$A(\epsilon) \cap O \subset \bigcup_{n=1}^{\infty} I'(n)$$

for, suppose $p \in A(\epsilon) \cap O$. There is a set $I \in \mathbf{J}$ such that $p \in I$ and

$$\frac{\delta}{\alpha^n} \leqslant d(I) < \frac{\delta}{\alpha^{n-1}}$$

for some integer n. If

$$p \notin \bigcup_{n=1}^{\infty} I'(n)$$

then $I \notin \mathbf{I}(n)$ so there is a set $J \in \mathbf{I}(m)$ for an $m \leqslant n$ such that $J \cap I$ is not empty. Then:

$$\rho(p, J) \leqslant d(I) < \frac{\delta}{\alpha^{n-1}} < \alpha \cdot d(J)$$

so that $p \in J'$ which contradicts the assertion. Thus:

$$\begin{aligned}\sum_{n=1}^{\infty} \tau(I'(n)) &\leqslant \beta \cdot \sum_{n=1}^{\infty} \tau(I(n)) \\ &\leqslant \beta/\epsilon \cdot \sum_{n=1}^{\infty} \varphi(A \cap I(n)) \\ &\leqslant \beta/\epsilon \cdot \varphi\left(A \cap \bigcup_{n=1}^{\infty} I(n)\right) < \beta \cdot \epsilon.\end{aligned}$$

By regularity condition (iii), $d(I'(n))$ may be made as small as desired for all n by choosing η sufficiently small. Thus, $\varphi(A(\epsilon) \cap O) < \beta\cdot\epsilon$ so that

$$\varphi(A(\epsilon)) < \beta\cdot\epsilon + \varphi(A(\epsilon) \cap \tilde{O}) \leqslant \epsilon(\beta + \epsilon).$$

Since $A(\epsilon) \subset A(\epsilon')$ if $\epsilon < \epsilon'$, we have:

$$\begin{aligned}\varphi(\{p: p \in B \quad \text{and} \quad D(A, p) > 0\}) &= \varphi\ (\lim_{\epsilon\to 0^+} A(\epsilon)) \\ &= \lim_{\epsilon\to 0^+} \varphi(A(\epsilon)) \\ &= 0\end{aligned}$$

which proves the theorem.

THEOREM 9. *If $\varphi(A)$ is finite, then $D(A, p) = 1$ for almost all $p \in A$.*

Proof. In view of Theorem 6 we need only show that the set $A(b) = \{p: p \in A \text{ and } D(A, p) > 1 + b\}$ is null for all $b > 0$. Fix $b > 0$, assign $\epsilon > 0$ and obtain $\delta > 0$ by applying Theorem 5 to the set A. As in Theorem 8, obtain a sequence of disjoint sets $I(n)$, $n = 1, 2, 3, \ldots$, such that

$$A(b) \subset \bigcup_{n=1}^{\infty} I'(n)$$

and $d(I'(n)) < \delta$, $\tau(I'(n)) \leqslant \beta\cdot\tau(I(n))$ and $\varphi(A \cap I(n)) > (1 + b)\cdot\tau(I(n))$ for all n. Then:

$$\varphi\left(A \cap \bigcup_{n=1}^{\infty} I(n)\right) = \sum_{n=1}^{\infty} \varphi(A \cap I(n)) \geqslant (1 + b)\cdot \sum_{n=1}^{\infty} \tau(I(n))$$

and, by Theorem 5,

$$\varphi(A(b)) \leqslant \varphi\left(A \cap \bigcup_{n=1}^{\infty} I'(n)\right) \leqslant \sum_{n=1}^{\infty} \tau(I'(n)) + \epsilon$$

and

$$\varphi\left(A \cap \bigcup_{n=1}^{\infty} I(n)\right) \leqslant \sum_{n=1}^{\infty} \tau(I(n)) + \epsilon.$$

From these relations, $\varphi(A(b)) \leqslant \epsilon\cdot(1 + \beta/b)$ so that $\varphi(A(b)) = 0$.

THEOREM 10. *If $\varphi(A)$ is finite and B is a measurable cover for A, then $\varphi(A \cap I) = \varphi(B \cap I)$ for every measurable set I. Thus, $D(A, p) = D(B, p)$ for every point p.*

Proof. Since I is measurable,

$$\varphi(A) = \varphi(A \cap I) + \varphi(A \cap \tilde{I})$$

and

$$\varphi(B) = \varphi(B \cap I) + \varphi(B \cap \tilde{I}).$$

Thus, $[\varphi(A \cap I) - \varphi(B \cap I)] + [\varphi(A \cap \tilde{I}) - \varphi(B \cap \tilde{I})] = 0$. Since $A \subset B$, both numbers in the square brackets are non-positive, so both are 0, which proves the theorem.

The following four theorems contain results similar to those obtained by Jeffery **(2)** and Randolph **(5)** for the case of Hausdorff outer measure in Euclidean space. It is interesting to note that they hold for any function $D(A, p)$ such that, if $\varphi(A)$ is finite, then:

(i) $D(A, p) > 0$ for almost all $p \in A$.
(ii) $D(A, p) = 0$ for almost all $p \in \tilde{A}$ if A is measurable.
(iii) $D(A, p) = D(B, p)$ for every p if B is a measurable cover for A.

THEOREM 11. *If $\varphi(A)$ is finite and*

$$G = \{p : p \in \tilde{A} \text{ and } D(A, p) > 0\}$$

then:

(i) *$A \cup G$ is a measurable cover for A.*
(ii) *A is measurable if and only if G is null.*

Proof. Let B be a measurable cover for A and let

$$\begin{aligned} C &= \{p : p \in B \text{ and } D(A, p) = 0\} \\ D &= \{p : p \in \tilde{B} \text{ and } D(A, p) > 0\} \\ E &= \{p : p \in A \text{ and } D(A, p) = 0\}. \end{aligned}$$

By Theorems 9 and 10, C and E are null. By Theorems 3, 10, and 8, D is null. Since $A \cup G = (B - C) \cup D \cup E$, $A \cup G$ is a measurable cover for A. Since $A = (A \cup G) - G$, if G is null then A is measurable. If A is measurable then:

$$\begin{aligned} \varphi(A) = \varphi(A \cup G) &= \varphi((A \cup G) \cap A) + \varphi((A \cup G) \cap \tilde{A}) \\ &= \varphi(A) + \varphi(G) \end{aligned}$$

so that G is null.

THEOREM 12. *If $\varphi(A)$ is finite and A is measurable and $A = B \cup C$ where B is separated from C, then B and C are measurable.*

Proof. The measurability of B follows from Theorems 8, 10, and the relation:

$$\{p : p \in \tilde{B} \text{ and } D(B, p) > 0\} \subset \{p : p \in \tilde{A} \text{ and } D(A, p) > 0\} \cup \{p : p \in C \text{ and } D(B, p) > 0\}.$$

Similarly, C is measurable.

THEOREM 13. *If $\varphi(A)$ is finite, G is defined as in Theorem 11, and:*

$$F = \{p : p \in \tilde{G} \text{ and } D(G, p) > 0\}$$

then:

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(i) *$F \subset A$ and $A - F$ is a measurable kernel for A.*

(ii) *$G \cup F$ is measurable and $\varphi(G \cup F) = \varphi(A) - \phi(A)$, where $\phi(A)$ is the inner measure of A.*

(iii) *If A is not measurable then A is not separated from G.*

(iv) *$\tilde{A} - G$ is separated from $A \cup G$.*

(v) *A is measurable if and only if F is empty.*

Proof. Since $A \cup G$ is a measurable cover for A, $D(A, p) = D(A \cup G, p) \geqslant D(G, p)$, so that $F \subset A$. By Theorem 11, $G \cup F$ is a measurable cover for $(A \cup G) - A$, so a measurable kernel for A is $(A \cup G) - (G \cup F) = A - F$. $G \cup F$ and $A - F$ are measurable and disjoint, so that:

$$\varphi(G \cup F) + \phi(A) = \varphi(G \cup F) + \varphi(A - F) = \varphi(A).$$

If A is not measurable then $\varphi(G) > 0$. Thus:

$$\varphi(A) + \varphi(G) = \varphi(A \cup G) + \varphi(G) > \varphi(A \cup G),$$

so that, by Theorem 2, A is not separated from G. Since $A \cup G$ is measurable, (iv) follows from Theorem 3. If A is measurable then G is null so F is empty. If F is empty then $A = A - F$ is measurable by (i).

THEOREM 14. *If $\varphi(A)$ and $\varphi(B)$ are finite and:*

$$A_b = \{p\colon p \in A \text{ and } D(B, p) > 0\}$$
$$B_a = \{p\colon p \in B \text{ and } D(A, p) > 0\}$$

then $\varphi(A_b) = \varphi(B_a)$.

Proof. If A' and B' are measurable covers for A and B respectively, then:

$$A_b = \{p\colon p \in A \cap B' \text{ and } D(B', p) > 0\} \cup \{p\colon p \in A \cap \tilde{B}' \text{ and } D(B', p) > 0\}.$$

The first set on the right has outer measure $\varphi(A \cap B')$ by Theorem 6, and the second set is null by Theorem 11. By Theorem 10, $\varphi(A \cap B') = \varphi(A' \cap B')$, so that $\varphi(A_b) = \varphi(A' \cap B')$ which, by symmetry, proves the theorem.

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Queen's University
and
Sir John Cass College