ON ALGEBRAIC SURFACES TERMWISE INVARIANT UNDER CYCLIC COLLINEATIONS

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1. Introduction. In algebraic geometry it is of interest to examine polynomial surfaces F which transform into themselves under the collineation T defined by:

$$(x'_1, x'_2, x'_3, x'_4) = (x_1, Ex_3, E^2x_3, E^3x_4)$$

where $E^p = 1$, and p is a prime (2). One of the most obvious ways to ensure invariance of a surface is for each term $x_1^a x_2^b x_3^c x_4^d$ of F to go into itself. We present initially, therefore, a theorem which will be useful in the study of such termwise invariance for polynomials of composite degree. Specializations of this result are then developed for the case when the polynomial degree is a prime.

2. Polynomial surfaces of composite degree. Assume that each term $x_1^a x_2^b x_3^c x_4^d$ of the polynomial is invariant and of degree mp. Then since the transform of this term under T is $x_1^a x_2^b x_3^c x_4^d (E^b E^{2c} E^{3d})$ there must hold the simultaneous equations $b + 2c + 3d = kp$ and $a + b + c + d = mp$. Diophantine solutions in terms of two parameters n and r are readily developed as

$$a = n, \quad b = r - 2n + (2m - k)p, \quad c = n - 2r + (k - m)p, \quad d = r.$$

Hence, in terms of congruence classes with respect to the modulus p, $a \in (n)$, $b \in (r - 2n)$, $c \in (n - 2r)$, $d \in (r)$.

On the other hand let a, b, c and d (which are such that $a + b + c + d = mp$) be assumed to have membership in the classes (n), $(r - 2n)$, $(n - 2r)$, (r). Then the term $x_1^a x_2^b x_3^c x_4^d$ transforms under T into $x_1^a x_2^b E^b x_3^c E^{2c} x_4^d E^{3d}$. Since

$$b + 2c + 3d \equiv (r - 2n) - (2n - 4r) + (3r) \equiv 0 \pmod{p},$$

and $E^{kp} = 1$, invariance is established. Thus we have

THEOREM 1. *A polynomial term $x_1^a x_2^b x_3^c x_4^d$, which has degree a multiple of a prime, p, will go into itself under T if and only if a, b, c, d are respectively members of the congruence classes* (mod p)*:* (n), $(r - 2n)$, $(n - 2r)$, (r).

3. Polynomial surfaces of prime degree. Of particular interest in algebraic geometry are surfaces where the degree of the defining polynomial is not a composite integer but is exactly equal to a prime $p > 3$. For this case n and r are subject to certain restrictions which we now study by setting $m = 1$ in the solution to the simultaneous equations used to develop Theorem 1.

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Since a, b, c, d are non-negative exponents where each is at most equal to p, it is clear on inspecting the equation $b + 2c + 3d = kp$, that k can take on only the values 0, 1, 2, and 3.

If $k = 0, b = c = r = 0$, and $n = p$.

If $k = 1$, we have the four inequalities:

$$\begin{aligned} &(1) \quad 0 \leqslant n, \\ &(2) \quad 0 \leqslant r - 2n + p, \\ &(3) \quad 0 \leqslant n - 2r, \\ &(4) \quad 0 \leqslant r. \end{aligned}$$

From inequality (3) we have $2r \leqslant n$, and from (2) it follows that $2n \leqslant r + p$, or $n \leqslant \frac{1}{2}(p + r)$. Thus n must lie within the range defined by $2r \leqslant n \leqslant \frac{1}{2}(p+r)$.

Since $4r \leqslant 2n$ we may replace $2n$ by $4r$ in the inequality (2) obtaining $3r \leqslant p$, or $r \leqslant \frac{1}{3}p$. Hence, to obtain an invariant term of degree p, necessary restrictions on r and n are (for the case $k = 1$) that r first be chosen on the range $0 \leqslant r \leqslant \frac{1}{3}p$, and that then n be chosen on the range $2r \leqslant n \leqslant \frac{1}{2}(p + r)$.

A similar analysis for the case when $k = 2$ demonstrates that r be chosen on the range $0 \leqslant r \leqslant \frac{2}{3}p$, and then n be chosen on the range $2r - p \leqslant n \leqslant \frac{1}{2}r$. An inspection of the latter inequality shows that n is non-negative only if $r \geqslant \frac{1}{2}(p + 1)$. This suggests splitting the range of r obtained into the two ranges

$$0 \leqslant r \leqslant \tfrac{1}{2}p \quad \text{and} \quad \tfrac{1}{2}(p + 1) \leqslant r \leqslant \tfrac{2}{3}p,$$

with corresponding ranges for n of

$$0 \leqslant n \leqslant \tfrac{1}{2}r \quad \text{and} \quad 2r - p \leqslant n \leqslant \tfrac{1}{2}r.$$

Finally, then, if $k = 3$, $b + 2c + 3d =$ p, hence $r = p$ and $n = b = c = 0$.

To summarize, restrictions on n and r necessary for the degree of the polynomial to be precisely p are:

$$\begin{array}{lll} (k = 0) & \text{(i)} & r = 0, \; n = p; \\ (k = 1) & \text{(ii)} & 0 \leqslant r \leqslant \frac{1}{3}p; \; 2r \leqslant n \leqslant \frac{1}{2}(p + r); \\ (k = 2) & \text{(iiia)} & 0 \leqslant r \leqslant \frac{1}{2}p; \; 0 \leqslant n \leqslant \frac{1}{2}r; \\ & \text{(iiib)} & \frac{1}{2}(p + 1) \leqslant r \leqslant \frac{2}{3}p; \; 2r - p \leqslant n \leqslant \frac{1}{2}r; \\ (k = 3) & \text{(iv)} & r = p, \; n = 0. \end{array}$$

We must now demonstrate the sufficiency of the above inequalities, i.e., that any n and r so chosen will yield a polynomial term of degree p. We illustrate the method of proof for (ii), for the case where p is of the form $6\alpha + 1$ and K is any even member of the range $0 \leqslant r \leqslant \frac{1}{3}p$. L is any corresponding member of the range $2r \leqslant n \leqslant \frac{1}{2}(p + r)$. Then

$$r = \tfrac{1}{3}(p - 3K - 1), \quad 0 \leqslant K \leqslant \tfrac{1}{3}(p - 1)$$

and

$$n = \tfrac{1}{2}(p + r - 2L - 1), \quad 0 \leqslant L \leqslant \tfrac{3}{2}K.$$

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The exponent $b = r - 2n + p$ is equal to $2L + 1$ upon substitution and simplification, and in the same way the exponent $c = n - 2r$ can be shown to equal $(3/2)K - L$. Then the polynomial term corresponding to the particular choice of K and L will be

$$x_1^{\frac{1}{2}(p+r-2L-1)} x_2^{2L+1} x_3^{(3/2)K-L} x_4^{\frac{1}{3}(p-3K-1)}.$$

The degree of this term is p, as can be readily shown by adding the exponents.

In a like manner the sufficiency of (ii) for $p = 6\alpha + 1$ and K odd can be readily established. A similar treatment will establish the inequalities (iiia) and (iiib) for all possible cases where p is either of the form $6\alpha + 1$ or $6\alpha - 1$ and K is either even or odd. Since conditions (i) and (iv) are obvious, sufficiency is completely demonstrated. We summarize these results in the following:

THEOREM 2. *Necessary and sufficient conditions that a polynomial term $x_1^a x_2^b x_3^c x_4^d$ of prime degree $p > 3$ shall be invariant under T are that:*

$$a \in (n), \quad b \in (r - 2n), \quad c \in (n - 2r), \quad d \in (r),$$

and also n and r must satisfy the conditional inequalities:

(a) $r = 0,\ n = p$;
(b) $0 \leqslant r \leqslant \frac{1}{3}p,\ 2r \leqslant n \leqslant \frac{1}{2}(p + r)$;
(c_1) $0 \leqslant r \leqslant \frac{1}{2}p,\ 0 \leqslant n \leqslant \frac{1}{2}r$;
(c_2) $\frac{1}{2}(p + 1) \leqslant r \leqslant \frac{2}{3}p,\ 2r - p \leqslant n \leqslant \frac{1}{2}r$;
(d) $r = p,\ n = 0$.

4. A numerical example. Theorem 2 makes possible the selection with certainty of the largest possible number of terms of a polynomial of degree p which is termwise invariant under the collineation T. For example, by the laborious process of writing out all terms of a given degree and testing them individually for invariance it is a known result **(1)** that the largest such polynomial for degree 5 is

$$F_5 = x_1^5 + x_2^5 + x_1x_2^3x_3 + x_1^2x_2x_3^2 + x_1^2x_2^2x_4 + x_1^3x_3x_4 + x_3^5 + x_2x_3^3x_4 + x_2^2x_3x_4^2 + x_1x_3^2x_4^2 + x_1x_2x_4^3 + x_4^5.$$

Application of Theorem 2 leads to 12 pairs of values for r and n as follows:

(a) $r = 0,\quad n = 5$;
(b) $r = 0,\quad n = 0, 1, 2$;
$r = 1,\quad n = 2, 3$;
(c) $r = 0,\quad n = 0$;
$r = 1,\quad n = 0$;
$r = 2,\quad n = 0, 1$;
$r = 3,\quad n = 1$;
(d) $r = 5,\quad n = 0$.

The application of these pairs leads precisely to the polynomial F_5.

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5. The number of invariant terms. In conclusion we establish

THEOREM 3. *For a prime $p > 3$ the number of invariant terms is equal to* $\frac{1}{6}(p^2 + 6p + 17)$.

Proof. It must first be indicated that all pairs (n, r) selected by Theorem 2 lead to different polynomial terms. We can exclude from further consideration the pair (0, 0) since its appearance in Theorem 2 (b and c_1) produces the polynomials x_2^p, and x_3^p. An examination of the controlling inequalities indicates that all other pairs (n, r) are distinct since (b) and (c_1) differ in n, (b) and (c_2) differ in r, as do (c_1) and (c_2). Then since n is the exponent of x_1 and r is the exponent of x_4, the distinct pairs (n, r) produce distinct polynomial terms.

To count the number of terms in (b) (Theorem 2) for $p = 6\alpha + 1$ we enumerate the pairs (K, L) where $0 \leqslant K \leqslant \frac{1}{3}(p - 1)$ and $0 \leqslant L \leqslant \frac{1}{2}(3K + 1)$ as in the accompanying table.

K	0	1	2	3	...	$\frac{1}{3}(p-1)$
L	0	0, 1, 2	0, 1, 2, 3	0, 1, 2, 3, 4, 5	...	$0, \ldots, \frac{1}{2}(p-1)$

The number of terms is thus

$$[1 + 3 + 4 + 6 + 7 + \ldots + \tfrac{1}{2}(p-1) + \tfrac{1}{2}(p+1)]$$
$$= [1 + 4 + 7 + \ldots + \tfrac{1}{2}(p+1)] + [3 + 6 + 9 + \ldots + \tfrac{1}{2}(p-1)].$$

By the ordinary formulas for an arithmetic series the number of terms is equal to $(p^2 + 6p + 5)/12$.

By a similar analysis, the inequality (c_1) contributes $(p^2 + 6p + 5)/16$ terms if α is odd, and $(p^2 + 6p + 9)/16$ terms if α is even. In a like manner (c_2) contributes $(p^2 + 6p + 5)/48$ if α is odd and $(p^2 + 6p - 7)/48$ terms if α is even. In either case the number of terms contributed by both (c_1) and (c_2) is $(p^2 + 6p + 5)/12$ since

$$(p^2 + 6p + 5)/16 + (p^2 + 6p + 5)/48 = (p^2 + 6p + 5)/12,$$
$$(p^2 + 6p + 9)/16 + (p^2 + 6p - 7)/48 = (p^2 + 6p + 5)/12.$$

The total number of terms thus obtained which are invariant under T (when $p = 6\alpha + 1$) is thus shown to be $\frac{1}{6}(p^2 + 6p + 17)$. A similar analysis for $p = 6\alpha - 1$ can be shown to yield the same result.

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