SEMILINEAR FOURTH ORDER BOUNDARY VALUE PROBLEMS

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We study a certain linear fourth order differential operator and show the existence of solutions to corresponding nonlinear problems. It will be shown that a maximum principle holds and that under certain conditions the linear operator has a positive principal eigenvalue with corresponding positive eigenfunction.

1. INTRODUCTION

Recently a number of papers have appeared dealing with the existence of solutions and, in some cases, with the uniqueness of solutions of fourth order boundary value problems arising from the study of elastic beams (see for example Aftabizadeh [1], Yang [12], Gupta [6], Agarwal [2], just to name a few). All of these papers treat what are called "nonresonance problems" near the first eigenvalue and most results are established for the boundary condition when the function and its second derivative vanish on the boundary.

In this paper we separate parts of the second derivative from the nonlinearity and make it part of the linear operator. Specifically, we consider boundary value problems of the form

(1) $$\begin{cases} u''''(t) + h(t)u''(t) + g(t, u(t), u'(t)) = f(t), \quad t \in (0, \pi), \\ u(0) = u(\pi) = 0 = u''(0) = u''(\pi). \end{cases}$$

The special case when $h(t) = -b \in \mathbf{R}$, $t \in (0, \pi)$, is of particular interest. The operator $u'''' - bu''$ occurs naturally in the context of the bending of a beam with axial load (see, for example, Hu [7], Washizu [11]). In terms of the deformation of beams, the boundary condition corresponds to a beam which is simply supported at both ends.

First we will study the linear problem $u'''' + hu''$. A maximum principle will be established which, in turn, will be used to prove further properties of the principal eigenvalue. In particular, we will show that under certain assumptions on h we have a positive operator.

Finally, we will prove some existence theorems, based in part on the results of the previous section. These results complement those of the papers cited above and, in some cases, extend them.

Let us make a final remark about our notation. $C^k([a,b])$ denotes the Banach space of k-times continuously differentiable functions with norm

$$|f|_k := \sup_{\substack{0 \leqslant i \leqslant k \\ t \in [a,b]}} \left| f^{(i)}(t) \right|.$$

For $p \in [1,\infty]$, $L^p(a,b)$ denotes the usual Lebesgue space with norm $\|\cdot\|_p$. However, the norm of the Hilbert space $L^2(a,b)$ is denoted simply by $\|\cdot\|$. For $p \in [1,\infty)$ and $k \in \mathbb{N}$, we let $W^{k,p}(a,b)$ denote the Sobolev space

$$W^{k,p}(a,b) := \{u \in L^p(a,b) \mid u^{(i)} \in L^p(a,b), \, i = 1,2,\ldots,k\}$$

with norm

$$\|u\|_{k,p} := \left(\sum_{i=0}^{k} \left(\left\| u^{(i)} \right\|_p \right)^p \right)^{1/p},$$

where, of course, $u^{(i)}$ denotes the ith distributional derivative of u. If $p = 2$, we set

$$H^k(a,b) := W^{k,2}(a,b)$$

with norm $\|\cdot\|_{k,2}$.

2. THE LINEAR PROBLEM

For a function $u \in W^{2,p}(0,\pi)$ let $B_1(u) := u \mid \partial(0,\pi) = u \mid \{0,\pi\}$, and for $v \in W^{4,p}(0,\pi)$ we let $B_2(v) := v'' \mid \partial(0,\pi) = v'' \mid \{0,\pi\}$.

In this section we will prove the existence of a principal eigenvalue of the linear problem. To do this, we will first prove a version of the maximum principle.

LEMMA 1. *Assume that a function $h: [0,\pi] \to \mathbb{R}$ satisfies $h(t) \leqslant 0$ on $[0,\pi]$ and is bounded on $[0,\pi]$. Suppose that $u(t)$ satisfies*

(2)
$$\begin{cases} u''''(t) + h(t)u''(t) \geqslant 0, & t \in (0,\pi) \\ B_1 u = 0, \quad u''(0) \leqslant 0, \quad u''(\pi) \leqslant 0. \end{cases}$$

If $u''(t)$ is continuous on $[0,\pi]$, then either

$$u(t) = 0 \text{ in } [0,\pi] \quad \text{or} \quad u(t) > 0 \text{ in } (0,\pi).$$

PROOF: Set $v(t) := u''(t)$. Then $v(t)$ satisfies

(3)
$$\begin{cases} v''(t) + h(t)v(t) \geqslant 0, & t \in (0,\pi) \\ v(0) \leqslant 0, \quad v(\pi) \leqslant 0. \end{cases}$$

Applying the standard maximum principle to this (see Protter-Weinberger [9]), we deduce that either

(4) $v(t) = 0$ in $[0, \pi]$ or $v(t) < 0$ in $(0, \pi)$.

Now, if $v(t) = 0$ in $[0, \pi]$, then $u''(t) = 0$ in $[0, \pi]$ and $B_1 u = 0$. This clearly implies that $u(t) = 0$ in $[0, \pi]$.

If, on the other hand, $v(t) < 0$ in $(0, \pi)$, then $u(t)$ satisfies

(5) $\begin{cases} u''(t) < 0, & t \in (0, \pi), \\ u(0) = 0 = u(\pi). \end{cases}$

Again, by the standard maximum principle we obtain $u(t) > 0$ in $(0, \pi)$. This proves the lemma.   ☐

REMARK 1. If one of the inequalities in (2) does not reduce to an equality, then, of course, the conclusion of Lemma 1 is that $u(t) > 0$ in $(0, \pi)$.   ☐

The next result is a boundary point lemma.

LEMMA 2. *Let h be as in Lemma 1 and assume that $u \in W^{4,1}(0, \pi)$ satisfies $u(t) > 0$ in $(0, \pi)$ and*

(6) $\begin{cases} u''''(t) + h(t)u''(t) \geqslant 0, & t \in (0, \pi), \\ B_1 u = 0 = B_2 u. \end{cases}$

Then $u'(0) > 0$ and $u'(\pi) < 0$.

PROOF: Again, we set $v(t) := u''(t)$. Then

$$\begin{cases} v''(t) + h(t)v(t) \geqslant 0, & t \in (0, \pi), \\ v(0) = 0 = v(\pi), \end{cases}$$

implying that either $v(t) = 0$ in $[0, \pi]$ or $v(t) < 0$ in $(0, \pi)$. However, since $u(t) > 0$ in $(0, \pi)$ and $B_1 u = 0$, we must conclude that

$$v(t) = u''(t) < 0 \text{ in } (0, \pi).$$

Now, for each $r \in (0, \pi)$ there exists some $s(r) \in (0, r)$ such that

$$u(r) = u(0) + u'(0)r + u''(s(r))r^2/2.$$

Since $u(0) = 0$, we actually have

$$u(r) = u'(0)r + u''(s(r))r^2/2.$$

This shows that we cannot have $u'(0) < 0$. Suppose we had $u'(0) = 0$. Then

$$u(r) = u''(s(r))r^2/2.$$

Now $s(r) \in (0, r) \subseteq (0, \pi)$ and therefore $u''(s(r)) < 0$. This, again, contradicts our assumption that $u(t) > 0$ in $(0, \pi)$. We therefore must have $u'(0) > 0$.

The argument that $u'(\pi) < 0$ is similar. In this case we merely use the fact that

$$u(r) = u(\pi) + u'(\pi)(r - \pi) + u''(t(r))(r - \pi)^2/2$$

for some $t(r) \in (r, \pi)$.   ☐

Next, we will study the operator

$$L_h u := u'''' + h(t)u'',$$

where $\mathrm{dom}\,(L_h) := \{u \in H^4(0, \pi) \mid B_1 u = 0 = B_2 u\}$ and where $h \in L^\infty(0, \pi)$ satisfies $h(t) \leqslant 0$ in $[0, \pi]$.

Let $R(f) := v$, where v is the unique solution of problem

$$\begin{cases} v''(t) = h(t)v(t) = f(t), & t \in (0, \pi), \\ B_1 v = 0. \end{cases}$$

and set $S(f) := w$, where w is the unique solution of the problem

$$\begin{cases} w''(t) = f(t), & t \in (0, \pi), \\ B_1 w = 0. \end{cases}$$

Then R and S are the solution operators corresponding to those two problems, and it is well-known that $R, S \colon L^2(0, \pi) \to C^1([0, \pi])$ are compact linear operators. Also note that

$$L_h u = f \iff u = (S \circ R)(f).$$

Using regularity theory, we see that $T := S \circ R$ is a compact linear operator from $L^2(0, \pi)$ into $C^3([0, \pi])$.

Our goal is to show that T is a positive operator on some appropriate ordered Banach space. To this end, we set $X := C_0([0, \pi]) := \{u \in C([0, \pi]) \mid B_1 u = 0\}$ with positive cone

$$K := \{u \in C_0([0, \pi]) \mid u(t) \geqslant 0, \quad t \in [0, \pi]\}.$$

Since $\mathrm{int}(K) = \emptyset$, this setup is unsuitable. We therefore resort to a different Banach space in the following manner. Set $e(t) := \sin t$, $t \in [0, \pi]$, and define X_e and $\|\cdot\|_e$ by

$$X_e := \{u \in C_0([0, \pi]) \mid \exists a > 0 \text{ such that } |u(t)| < ae(t), \, t \in [0, \pi]\}$$

and

$$\|u\|_e := \inf\{a > 0 \mid |u(t)| \leqslant ae(t), \quad t \in [0, \pi]\}.$$

We also set

$$K_e := K \cap X_e.$$

It is well-known that $(X_e, \|\cdot\|_e)$ is a Banach space with normal order cone K_e and $\text{int}(K_e) \neq \emptyset$. For details see Amann [3], Deimling [5], Zeidler [13].

LEMMA 3. *The embedding* $(C_0^1([0, \pi]), |\cdot|_1) \to (X_e, \|\cdot\|_e)$ *is continuous.*

PROOF: Note that $2e(t) > 1$ on the interval $I_1 := \left[1/\sqrt{2}, \pi - 1/\sqrt{2}\right]$, $2e'(t) > 1$ on $I_2 := \left[0, (1/\sqrt{2})\right]$ and $2e'(t) < -1$ on $I_3 := \left[\pi - 1/\sqrt{2}, \pi\right]$. The first statement implies that $|u(t)| \leqslant 2|u|_1 e(t)$ on I_1, while the second and third statement imply that the same inequality also holds on I_2 and I_3, respectively. Therefore

$$|u(t)| \leqslant 2|u|_1 e(t), \qquad t \in [0, \pi],$$

which implies that

$$\|u\|_e \leqslant 2|u|_1.$$

This shows that the embedding is bounded (see Deimling [5]).   ☐

It is also easily seen that the embedding $(X_e, \|\cdot\|_e) \to (C_0(0, \pi), |\cdot|_0)$ is continuous. Now set $T_e := T \mid X_e$ and $T_0 := T \mid C_0([0, \pi])$. Then $T_e \colon X_e \to X_e$ and $T_0 \colon (C_0([0, \pi]), |\cdot|_0) \to (C_0^1([0, \pi]), |\cdot|_1)$ are compact. Since the linear operator T_e has the following factorisation

$$(X_e, \|\cdot\|_e) \to (C_0([0, \pi]), |\cdot|_0) \xrightarrow{T_0} (C_0^1([0, \pi]), |\cdot|_1) \to (X_e, \|\cdot\|_e),$$

it follows that $T_e : X_e \to X_e$ is compact.

LEMMA 4. *We have* $T_e(K_e \setminus \{0\}) \subseteq \text{int}(K_e)$.

PROOF: Observe that $u \in C_0([0, \pi])$ belongs to $\text{int}(K_e)$ if and only if $u(t) > 0$ in $(0, \pi)$ and $u'(\pi) < 0 < u'(0)$. Now, if $f \in K_e \setminus \{0\}$, then by Lemma 1 we conclude that $u := T_e(f)$ satisfies $u(t) > 0$ in $(0, \pi)$ and hence Lemma 2 implies that $u'(\pi) < 0 < u'(0)$. Therefore $u \in \text{int}(K_e)$.   ☐

THEOREM 1. *The eigenvalue problem*

$$L_h u = \lambda u$$

has a simple real eigenvalue $\lambda_1 > 0$ *and the corresponding eigenfunction* $\varphi \in \text{int}(K_e)$. *Any other eigenvalue* λ *of* L_h *satisfies* $\lambda_1 < |\lambda|$ *and any eigenfunction corresponding to* λ *must change sign.*

PROOF: We note that $0 \notin \sigma(L)$ and therefore $\lambda \in \sigma(L)$ if and only if $1/\lambda \in \sigma(T_e)$. Since, by Lemma 4, T_e is strongly positive, we see that we can apply the Krein-Rutman

theorem (see Zeidler [13], Amann [3], Deimling [5]). From this, Theorem 1 follows immediately. ☐

Let us assume now that $h \in C^2([0, \pi])$ and let $u, v \in \text{dom}(L_h)$. A simple calculation shows that

$$\langle L_h u, v \rangle = \langle u, v'''' + (hv)'' \rangle.$$

Therefore $\text{dom}(L_h) \subseteq \text{dom}(L_h^*)$. In fact, the following is true.

LEMMA 5. *If $h \in C^2([0, \pi])$, then $L_h^* : \text{dom}(L_h^*) \subseteq L^2(0, \pi) \to L^2(0, \pi)$ is given by*

$$L_h^*(v) = v'''' + (hv)'',$$

where $\text{dom}(L_h^*) = \{v \in H^4(0, \pi) \mid B_1 v = 0 = B_2 v\}$.

PROOF: Let $v \in \text{dom}(L_h^*) \subseteq L^2(0, \pi)$. Then $v'''' + (hv)''$ is a distribution and therefore

$$[v'''' + (hv)''](f) = v(f'''') + v(hf'') = \langle v, L_h f \rangle = \langle L_h^* v, f \rangle$$

for all $f \in C_0^\infty(0, \pi)$. This implies that we in fact have

$$L_h^* v = v'''' + (hv)''.$$

In particular, $L_h^* v \in L^2(0, \pi)$ implies, by regularity, that $v \in H^4(0, \pi)$. Hence $\text{dom}(L_h^*) \subseteq H^4(0, \pi)$. Now, if $u \in \text{dom}(L_h)$ and $v \in \text{dom}(L_h^*)$, then

$$\langle L_h u, v \rangle = \langle u, L_h^* v \rangle.$$

Using integration by parts, we find that the left-hand side is given by

$$\langle L_h u, v \rangle = \left(u''' v + u' v'' + u' hv \right) \big|_0^\pi + \langle u, L_h^* v \rangle.$$

Therefore

$$\left(u''' v + u' v'' + u' hv \right) \big|_0^\pi = 0$$

for all $u \in \text{dom}(L_h)$. It is now easily seen, by constructing appropriate functions $u \in \text{dom}(L_h)$, that $B_1 v = 0 = B_2 v$. This shows that $\text{dom}(L_h^*) = \text{dom}(L_h)$ as claimed. ☐

Next, let $h \in C^2([0, \pi])$ satisfy $h(t) \leqslant 0$ on $[0, \pi]$. A further consequence of the Krein-Rutman theorem is that $1/\lambda_1$ is also a simple eigenvalue of T_e^* with corresponding strictly positive eigenfunction $\xi \in X_e^*$. Note also that λ_1 is an eigenvalue of L_h^* with corresponding eigenfunction $\theta \in \text{dom}(L_h^*)$ and that $\text{dom}(L_h^*) \subseteq X_e^*$.

Now let $w \in X_e$ and set $u := T_e w$. Then

$$\begin{aligned} \langle w, \theta \rangle &= \langle L_h u, \theta \rangle = \langle u, L_h^* \theta \rangle = \langle T_e w, \lambda_1 \theta \rangle = \langle \lambda_1 T_e w, \theta \rangle \\ &= \lambda_1 T_e^* \theta(w). \end{aligned}$$

Therefore $T_e^* \theta = 1/\lambda_1 \theta$ and so θ is strictly positive, that is,

$$\langle \theta, u \rangle > 0 \text{ for all } u \in K_e \setminus \{0\}.$$

This clearly implies that $\theta(t) > 0$ for all $t \in (0, \pi)$. Summarising, we have proved the following theorem.

THEOREM 2. *Let $h \in C^2([0, \pi])$ satisfy $h(t) \leqslant 0$ for all $t \in [0, \pi]$. Then the eigenvalue problem*

$$L_h^* u = \lambda u$$

has the simple eigenvalue λ_1 (the principle eigenvalue of L_h) and a corresponding eigenfunction $\theta \in \operatorname{dom}(L_h^*)$ such that $\theta(t) > 0$ for all $t \in (0, \pi)$.

If the function h is constant, that is, if $h(t) = -b$, $t \in [0, \pi]$, for some constant $b \in \mathbb{R}^+$, then of course $\sigma(L_h)$ is easily calculated. In fact, in this case we have

$$\sigma(L_h) = \{n^4 + n^2 b \mid n \in \mathbb{N}\}.$$

It is very useful to know what happens to the principal eigenvalue as the function $h(t)$ changes. In other words, assume that $h, p \in L^\infty(0, \pi)$ satisfy $h(t) \leqslant p(t) \leqslant 0$ and let μ_1 denote the principal eigenvalue of L_p. In the next theorem we will show that $\mu_1 \leqslant \lambda_1$.

THEOREM 3. *Let λ_1 and μ_1 denote the principal eigenvalues of L_h and L_p, respectively. Here $h, p \in L^\infty(0, \pi)$ are such that $h(t) \lneqq p(t) \leqslant 0$. Then*

$$0 < \mu_1 < \lambda_1.$$

PROOF: Let $T_e, P_e \colon X_e \to X_e$ denote the restriction of the solution operator of L_h and L_p, respectively. Now let $w \in K_e \setminus \{0\}$ and set $u := T_e(w)$ and $v := P_e(w)$. Then u and v satisfy the homogeneous boundary conditions and the respective differential equations

(7) $$u'''' + hu'' = w$$

and

(8) $$v'''' + pv'' = w.$$

Therefore

$$(u - v)'''' + hu'' - pv'' = 0.$$

Applying the maximum principle (Lemma 1) to (8), we conclude that $v'' < 0$ in $(0, \pi)$ and hence $-hv'' \lneqq -pv''$. Now we have

$$(u - v)'''' + h(u - v)'' \lneqq 0.$$

Since $(u - v)$ also satisfies the homogeneous boundary conditions, we must have $(u - v) < 0$ in $(0, \pi)$ and therefore $u < v$ in $(0, \pi)$. We have shown that $w > 0$ implies $T_e(w) < P_e(w)$. Now we deduce from Corollary 7.28 in Zeidler [13] that the spectral radius of T_e is less than that of P_e, that is, $1/\lambda_1 < 1/\mu_1$. Therefore $\mu_1 < \lambda_1$.   □

Assume that $b \in \mathbf{R}^+$. Then the principal eigenvalue of

$$\begin{cases} u'''' - bu'' = \lambda_1 u \\ B_1 u = 0 = B_2 u \end{cases}$$

is $\lambda_1 = 1 + b$. If now $h \in L^\infty(0, \pi)$ is such that $b \lneqq h(t)$, $t \in [0, \pi]$, and if μ_1 denotes the principal eigengalue of

$$\begin{cases} u'''' - h(t)u'' = \mu u \\ B_1 u = 0 = B_2 u, \end{cases}$$

then it follows from the above discussion that $1 + b < \mu_1$.

Let again $h \in L^\infty(0, \pi)$ be such that $h(t) \leqslant 0$ on $[0, \pi]$. We know that if h is a constant, then L_h is a positive operator, that is,

$$\langle L_h u, u \rangle > 0 \ \text{ for all } \ u \in \mathrm{dom}\,(L_h) \setminus \{0\}.$$

One may wonder whether the same is true if h is not a constant. Below, we will give an example showing that L_h is not necessarily positive for any $h \in L^\infty(0, \pi)$ with $h(t) \leqslant 0$.

EXAMPLE. Let $h \in L^\infty(0, \pi)$ be defined by

$$h(t) := \begin{cases} -1000\pi, & \text{if } t \in I, \\ 0, & \text{otherwise,} \end{cases}$$

where $I := [0.4, 0.6] \cup [\pi - 0.6, \pi - 0.4]$. Also set

$$u(t) := [t(t - \pi)]^3, \qquad t \in [0, \pi].$$

Then $u \in \mathrm{dom}\,(L_h)$ and a simple, but tedious, calculation shows that

$$\langle L_h u, u \rangle < 0.$$

(Note that $\|u''\|_\infty < 40$ and on I we have $u(t) < -1$ and $u''(t) < -20$.)

However, if we impose some conditions on $h(t)$, then L_h will be a positive operator.

THEOREM 4. *Let $h \in H^2(0, \pi)$ and $b \in \mathbb{R}$ be such that $h(t) \leqslant -b$, $t \in [0, \pi]$. Then*

$$\langle L_h u, u \rangle \geqslant (1 + b - \|h'\|_\infty) \|u'\|^2 .$$

PROOF: Note that for $u \in \text{dom}\,(L_h)$ we have

$$\|u\| \leqslant \|u'\| \quad \text{and} \quad \|u'\| \leqslant \|u''\| ,$$

and for every $f \in L^2(0, \pi)$ we have

$$\|f\|^2 \leqslant \sqrt{\pi} \, \|f^2\| .$$

Now

$$
\begin{aligned}
\langle Lu, u \rangle &= \int_0^\pi (u'')^2 dt - \int_0^\pi h(u')^2 dt - \int_0^\pi h' u u' dt \\
&\geqslant \|u''\|^2 + b \int_0^\pi (u')^2 dt - \|h'\|_\infty \|u\| \|u'\| \\
&\geqslant \|u''\|^2 + b \|u'\|^2 - \|h'\|_\infty \|u'\|^2 \\
&\geqslant (1 + b - \|h'\|_\infty) \|u'\|^2 .
\end{aligned}
$$

This proves the theorem.   ☐

If h is constant, it is clear that the operator L_h is self-adjoint. As one might suspect, this is the only case when L_h is self-adjoint.

THEOREM 5. *Let $h \in L^\infty(0, \pi)$. Then L_h is self-adjoint if and only if h is constant almost everywhere.*

PROOF: It suffices to show that the self-adjointness of L_h implies that h is constant almost everywhere. To that end, we assume that L_h is self-adjoint. Then we certainly must have $\langle L_h u, v \rangle = \langle u, L_h u \rangle$ for all $u, v \in \text{dom}\,(L_h)$ and therefore

$$\int_0^\pi h[u'' v - u v''] dt = 0, \quad \text{for all} \quad u, v \in \text{dom}\,(L_h).$$

Let g be an arbitrary function in $H_0^1(0, \pi)$. Then there exists a sequence $\{g_k\} \subseteq C_0^\infty(0, \pi)$ such that

$$g_k \to g \quad \text{in} \quad H_0^1(0, \pi) \quad \text{as} \quad k \to \infty.$$

Fix $k \in \mathbb{N}$ and set $v(t) := \sin t$ and

$$u(t) := \sin t \int_0^t \frac{g_k(s)}{\sin^2 s} ds.$$

It then follows that $u, v \in \operatorname{dom}(L_h)$ and

$$u''v - uv'' = g'_k \ \text{ in } \ [0, \pi].$$

Therefore

$$\int_0^\pi h(t) g'_k(t) dt = 0$$

for all $k \in \mathbb{N}$. From this we deduce that

$$\int_0^\pi h(t) g'(t) dt = 0$$

for all $g \in H_0^1(0, \pi)$. Finally, by the lemma of Du Bois Raymond from the calculus of variations we conclude that h is constant almost everywhere.   □

In the next section we will show how some of the results from this section can be used to extend, at least in some special cases, certain types of existence results.

3. NONLINEAR PROBLEMS

In this section we will prove the existence of solutions of certain nonlinear problems. Our results compliment those of Gupta [6] and, in some special cases, improve them.

Many of the existence results (and also multiplicity results) which are known for elliptic boundary value problems can be carried over to problems such as (1). This is because one often makes use only of certain abstract properties of the elliptic problems. The case when $h(t) \leqslant 0$, $t \in [0, \pi]$, is particularly nice because in this case L_h has a positive principal eigenvalue with corresponding positive eigenfunction φ satisfying $\varphi'(\pi) < 0 < \varphi'(0)$ (see Theorem 1). This is very useful in many cases.

Below we will give an example of one such result for a so-called resonance problem.

THEOREM 6. *Let $h(t) := -b \in \mathbb{R}$ for all $t \in [0, \pi]$ and assume that $g: [0, \pi] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying:*

   (i)   *for every $R > 0$ there exists a function $f_R \in L^1(0, \pi)$ such that*

$$|g(t, s)| \leqslant f_R(t)$$

   for all $t \in [0, \pi]$ and all $s \in \mathbb{R}$ with $|s| \leqslant R$;

   (ii) *there exist functions $c, d \in L^\infty(0, \pi)$ such that*

$$-1 - b \leqslant \liminf_{s \to +\infty} \frac{g(t, s)}{s} \leqslant \limsup_{s \to +\infty} \frac{g(t, s)}{s} \leqslant c(t)$$

$$-1 - b \leqslant \liminf_{s \to -\infty} \frac{g(t, s)}{s} \leqslant \limsup_{s \to -\infty} \frac{g(t, s)}{s} \leqslant d(t)$$

   uniformly almost everywhere in $t \in (0, \pi)$.

Let $f \in L^1(0, \pi)$ be such that

$$\int_0^\pi g(t, -\infty) \sin t \, dt < \int_0^\pi f(t) \sin t \, dt < \int_0^\pi g(t, +\infty) \sin t \, dt,$$

where $g(t, -\infty) := \limsup_{s \to -\infty} [g(t, s) + (1 + b)s]$ and $g(t, +\infty) := \liminf_{s \to +\infty} [g(t, s) + (1 + b)s]$.
Then the problem

(9)
$$\begin{cases} u''''(t) - bu''(t) + g(t, u(t)) = f(t), & t \in (0, \pi), \\ B_1 u = 0 = B_2 u, \end{cases}$$

has a solution $u \in W^{4,1}(0, \pi)$.

PROOF: Since $h(t)$ is constant, it follows that L_h is self-adjoint. A simple application of spectral analysis shows that if $u \in \text{dom}(L_h)$ is a solution of

(10)
$$L_h u + \mu u^+ - \nu u^- = 0,$$

where $\mu, \nu \in L^\infty(0, \pi)$ are such that $(\mu(t), \nu(t)) \in [-1 - b, c(t)] \times [-1 - b, d(t)]$ for almost everywhere $t \in (0, \pi)$, then $u \in \ker(L_h - (1 + b))$. Using this and the fact that the differential operator L_h gives rise to a differential operator in $L^1(0, \pi)$, that is,

$$L : \text{dom}(L) \subseteq L^1(0, \pi) \to L^1(0, \pi).$$

This operator has a compact resolvent as a map from $L^1(0, \pi)$ into $(C^{-1}([0, \pi]), |\cdot|_1)$. Actually, we have more regularity (see Section 2), but this is all we need. We now can apply Theorem 3 in [8] to obtain a solution.   ⬚

REMARK 2. (a) Theorem 6 also holds true for an arbitrary function $h \in L^\infty(0, \pi)$ if it can be shown that a solution $u \in \text{dom}(L_h)$ of (10) must lie in $\ker(L_h - \lambda_1)$. Of course, $1 + b$ must be replaced by λ_1 and $\sin t$ by a positive eigenfunction θ corresponding to the principal eigenvalue λ_1. Again, this follows from a simple application of Theorem 3 in [8].

(b) Similar results also hold for higher eigenvalues. One can directly apply Theorem 2 of [8] or use the results and ideas of [4].   ⬚

Our next theorem is a so-called nonresonance result. Here we allow h to be non-constant but assume some regularity on h.

THEOREM 7. Let $h \in H^2(0, \pi)$ and $b \in \mathbf{R}^+$ be such that $h(t) \leqslant -b$ for all $t \in [0, \pi]$. Suppose $g : [0, \pi] \times \mathbf{R}^2 \to \mathbf{R}$ is a Carathéodory function satisfying:

(i)   for some $a_1, a_2, a_3, a_4 \in L^\infty(0, \pi)$ we have

$$g(t, s, r)s \geqslant a_1(t)s^2 + a_2(t)sr + a_3(t)s + a_4(t)r$$

for all $(t, s, r) \in [0, \pi] \times \mathbf{R}^2$;

(ii)   there exist constants $b_1, b_2 \in \mathbf{R}^+$ and a function $b_0 \in L^1(0, \pi)$ such that

$$|g(t, s, r)| \leqslant b_0(t) + b_1 |s| + b_2 |r|$$

for all $(t, r, s) \in [0, \pi] \times \mathbf{R}^2$.

If the condition

(11)
$$-\frac{\pi^2}{4} \int_0^\pi a_1^-(t) dt > -1 - b + \|h'\|_\infty + \frac{\pi}{2} \|a_2\|$$

holds, where $a_1^-(t) := \min\{a_1(t), 0\}$, then for every $f \in L^1(0, \pi)$, the problem

(12)
$$\begin{cases} u''''(t) + h(t)u''(t) + g(t, u(t), u'(t)) = f(t), & t \in (0, \pi), \\ B_1 u = 0 = B_2 u, \end{cases}$$

has a solution $u \in W^{4,1}(0, \pi)$.

PROOF: Again, let $L: \mathrm{dom}(L) \subseteq L^1(0, \pi) \to L^1(0, \pi)$ denote the differential operator corresponding to (12) (clearly $L_h \subseteq L$). Let $N: H^1(0, \pi) \to L^1(0, \pi)$ be defined by

$$(Nu)(t) := f(t) - g(t, u(t), u'(t)), \qquad t \in (0, \pi).$$

We then employ the Leray-Schauder degree to solve the problem

$$Lu = Nu,$$

which is equivalent to (12). To this end, we consider the homotopy

(13)
$$Lu = rNu, \qquad r \in [0, 1].$$

First we shall show that the set of all solutions to (13) is bounded with respect to the $\|\cdot\|_{2,2}$-norm.

Now
$$\langle Lu, u \rangle = r \langle Nu, u \rangle$$

and from Theorem 4 we have

$$\langle Lu, u \rangle \geqslant \pi^{-1/2} \left(1 + b - \sqrt{\pi} \|h'\|_\infty\right) \|u'\|^2.$$

Again, using $\|u\| \leqslant \|u'\|$, $\|u\|^2 \leqslant \sqrt{\pi} \|u^2\|$ and condition (i) above, we obtain

$$r \langle Nu, u \rangle \leqslant \frac{\pi^2}{4} \|u'\|^2 \int_0^\pi a_1^-(t) dt + \frac{\pi}{2} \|a_2\| \|u'\|^2 + l(u),$$

where in $l(u)$ we have collected all lower order terms. Then

$$\lim_{\|u'\| \to \infty} \frac{l(u)}{\|u'\|^2} = 0.$$

Finally, we have

$$(1 + b - \|h'\|_\infty) \leqslant \frac{\pi^2}{4} \int_0^\pi a_1^-(t)dt + \frac{\pi}{2} \|a_2\| + \frac{l(u)}{\|u'\|^2}.$$

It follows from this and (11) that for some $M \in \mathbf{R}^+$ we have

$$\|u'\| \leqslant M$$

for all solutions of (13). Since $\|u\| \leqslant \|u'\|$, we see that all solutions of (13) are bounded with respect to the $\|\cdot\|_{2,2}$-norm.

It is now easily seen that $\|Nu\|_1$ is bounded for all solutions of (13). So, if we set $v := Lu$, then $\|v\|_1$ is also bounded. Without loss of generality we may assume that L is invertible. Then (13) is equivalent to

(14) $$v = rNL^{-1}v.$$

We have shown that for some $K \in \mathbf{R}^+$ we have

$$\|v\|_1 \leqslant K$$

for every solution of (14). Since $NL^{-1} \colon L^1(0,\pi) \to L^1(0,\pi)$ is compact, the assertion follows.   □

REMARK 3. (a) It should be noted that condition (11) allows some flexibility between $a_1(t)$, $a_2(t)$ and $h(t)$, since it does not involve the $\|\cdot\|_\infty$-norms of a_1 and a_2.

(b) Following ideas from [6, Theorem 2.1], it is also possible to allow the nonlinearity g to depend on u''.

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