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ON TWO OPEN PROBLEMS ABOUT STRONGLY CLEAN RINGS

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A ring is called strongly clean if every element is the sum of an idempotent and a unit which commute. In 1999 Nicholson asked whether every semiperfect ring is strongly clean and whether the matrix ring of a strongly clean ring is strongly clean. In this paper, we prove that if $R = \{m/n \in \mathbb{Q} : n \text{ is odd}\}$, then $M_2(R)$ is a semiperfect ring but not strongly clean. Thus, we give negative answers to both questions. It is also proved that every upper triangular matrix ring over the ring R is strongly clean.

Throughout this paper all rings are associative with unit. For a ring R, let $U(R)$ be the group of units of R, $M_n(R)$ the $n \times n$ matrix ring over R, and $T_n(R)$ the $n \times n$ upper triangular matrix ring over R, respectively. The identity matrix of $M_n(R)$ is denoted by I. $\mathbb{Q}$ means the field of rational numbers. A ring R is called clean if every element of R is a sum of an idempotent and a unit. The ring is called strongly clean if every element is the sum of an idempotent and a unit which commute. It is shown by Camillo and Yu [**2**, Theorem 9] that every semiperfect ring is clean. Han and Nicholson [**4**, Corollary 1] showed that every matrix ring $M_n(R)$ over a clean ring is again clean.

Nicholson asked whether every semiperfect ring is strongly clean [**5**, Question 5] and whether the matrix ring of a strongly clean ring is strongly clean [**5**, Question 3]. In this paper, we prove that if $R = \{m/n \in \mathbb{Q} : n \text{ is odd}\}$, then $M_2(R)$ is a semiperfect ring but not strongly clean. Thus, we answer the two questions above, both in the negative. Also we prove that every upper triangular matrix ring over the ring R is strongly clean. Thus, we obtain a new class of strongly clean rings.

EXAMPLE 1. Let $R = \{m/n \in \mathbb{Q} : n \text{ is odd}\}$. Then $M_2(R)$ is a semiperfect ring but it is not strongly clean.

PROOF: Since R is a commutative local ring, it is semiperfect and strongly clean. Since semiperfect rings are Morita invariant, $M_2(R)$ is semiperfect. By direct computation, we find all nontrivial idempotents in the matrix ring $M_2(R)$ are of the following types:

$$\begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}, \quad \text{where } a, b, c \in R \text{ and } bc = a - a^2.$$

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Consider $\begin{pmatrix} 8 & 6 \\ 3 & 7 \end{pmatrix} \in M_2(R)$. Since $\begin{pmatrix} 8 & 6 \\ 3 & 7 \end{pmatrix}$ and $\begin{pmatrix} 7 & 6 \\ 3 & 6 \end{pmatrix}$ are not units in $M_2(R)$, we can write

$$\begin{pmatrix} 8 & 6 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix} + \begin{pmatrix} 8-a & 6-b \\ 3-c & 6+a \end{pmatrix}.$$

where $a, b, c \in R$ and

$$bc = a - a^2 \tag{1}$$

Suppose that

$$\begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}\begin{pmatrix} 8-a & 6-b \\ 3-c & 6+a \end{pmatrix} = \begin{pmatrix} 8-a & 6-b \\ 3-c & 6+a \end{pmatrix}\begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}.$$

By comparing the $(1,1)$-entry and $(2,1)$-entry on both sides, we obtain

$$a(8-a) + b(3-c) = (8-a)a + (6-b)c \tag{2}$$

$$c(8-a) + (1-a)(3-c) = (3-c)a + (6+a)c \tag{3}$$

By (1), (2) and (3), we obtain

$$73a^2 - 73a + 18 = 0.$$

The equation has no solutions in R, so $M_2(R)$ is not strongly clean. ∎

REMARK 2. The above example gives negative answers to both questions of Nicholson.

By Nicholson [**5**], every strongly π-regular ring or local ring is strongly clean, and they seem to be all known examples of strongly clean rings up to now. Here we give a new class of strongly clean rings which are neither strongly π-regular nor local. A ring R is called uniquely clean if every element of R is a sum of an idempotent and a unit and the presentation is unique. This concept was introduced in [**1**]. Similarly, we can define uniquely strongly clean rings.

THEOREM 3. *Let R be a commutative local ring. Then R is uniquely clean ring if and only if $T_n(R)$ is uniquely strongly clean for every $n \geqslant 1$.*

PROOF: "$\Leftarrow$". Let $n = 1$. Then R is uniquely strongly clean. Since R is commutative, R is uniquely clean.

"$\Rightarrow$". When $n = 1$, the claim holds trivially. Let $n \geqslant 2$ and let $A \in T_n(R)$. Write

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} = \begin{pmatrix} A_1 & \alpha \\ 0 & a_{nn} \end{pmatrix}.$$

By induction hypothesis, A_1 can be uniquely expressed as $A_1 = E + U$ where $E^2 = E \in T_{n-1}(R)$ and U is a unit in $T_{n-1}(R)$ and $EU = UE$. Moreover, $a_{nn} = e + u$ where $e^2 = e \in R$ and u is a unit in R. Thus,

$$A = \begin{pmatrix} E+U & \alpha \\ 0 & e+u \end{pmatrix} = \begin{pmatrix} E & \alpha_1 \\ 0 & e \end{pmatrix} + \begin{pmatrix} U & \alpha_2 \\ 0 & u \end{pmatrix}.$$

Let $F = \begin{pmatrix} E & \alpha_1 \\ 0 & e \end{pmatrix}$, $V = \begin{pmatrix} U & \alpha_2 \\ 0 & u \end{pmatrix}$. Then $V \in T_n(R)$ is a unit. We next show that there exist α_1, α_2 such that $F^2 = F$ and $FV = VF$. It is clear that

$$F^2 = F \Leftrightarrow E\alpha_1 + \alpha_1 e = \alpha_1 \tag{4}$$

$$FV = VF \Leftrightarrow E\alpha_2 + \alpha_1 u = U\alpha_1 + \alpha_2 e \tag{5}$$

Note that $\alpha = \alpha_1 + \alpha_2$.

CASE 1. $a_{nn} \in R$ is a unit. Since R is uniquely clean, $e = 0$ and $u = a_{nn}$. In this case, (4) becomes $E\alpha_1 = \alpha_1$ and (5) becomes $E(\alpha - \alpha_1) + \alpha_1 u = U\alpha_1$. Then

$$\begin{aligned} E\alpha &= (U + E - uI)\alpha_1 = (U + E - uI)E\alpha_1 \\ &= (UE + E - uE)\alpha_1 = (U + (1-u)I)E\alpha_1 = (U + (1-u)I)\alpha_1. \end{aligned}$$

Since R is uniquely clean, if u is a unit, then $1 - u$ is not a unit, otherwise $1 - u = 0 + (1 - u) = 1 + (-u)$ implies that $1 = 0$. Let

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1,n-1} \\ 0 & u_{22} & \cdots & u_{2,n-1} \\ \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & u_{n-1,n-1} \end{bmatrix}.$$

Since u_{ii} is a unit, $u_{ii} + (1 - u)$ is also a unit. Hence $U + (1 - u)I$ is a unit. So we can let $\alpha_1 = (U + (1-u)I)^{-1}E\alpha$ and $\alpha_2 = \alpha - (U + (1-u)I)^{-1}E\alpha$.

CASE 2. a_{nn} is not a unit in R. Then $e = 1$ and $u = a_{nn} - 1$. In this case, (4) becomes $E\alpha_1 = 0$, and (5) becomes $E(\alpha - \alpha_1) + \alpha_1 u = U\alpha_1 + (\alpha - \alpha_1)$. Hence,

$$\begin{aligned} (E - I)\alpha &= \big(U - uI - (I - E)\big)\alpha_1 = \big(U - uI - (I - E)\big)(I - E)\alpha_1 \\ &= \big[(U - uI)(I - E) - (I - E)\big]\alpha_1 = (U - uI - I)(I - E)\alpha_1 \\ &= (U - uI - I)\alpha_1 = (U - (u+1)I)\alpha_1. \end{aligned}$$

Since $u + 1 = a_{nn}$ is not a unit. Let

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1,n-1} \\ 0 & u_{22} & \cdots & u_{2,n-1} \\ \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & u_{n-1,n-1} \end{bmatrix}.$$

Then u_{ii} is a unit. So is $u_{ii} - (u+1)$. Hence, $U - (u+1)I$ is a unit. So let $\alpha_1 = \big(U - (u+1)I\big)^{-1}(E - I)\alpha$ and $\alpha_2 = \alpha - \big(U - (u+1)I\big)^{-1}(E - I)\alpha$.

Now to show that $T_n(R)$ is uniquely strongly clean, let $A = F + V = D + N$ where $D^2 = D$, $N \in T_n(R)$ is a unit and $DN = ND$. Write $D = \begin{pmatrix} D_1 & \beta \\ 0 & d \end{pmatrix}$ and $N = \begin{pmatrix} N_1 & \gamma \\ 0 & t \end{pmatrix}$. Then $D_1^2 = D_1$, $d^2 = d$, N_1, t are units, and $D_1N_1 = N_1D_1$, and $a_{nn} = d + t$, and $A_1 = D_1 + N_1$. By induction hypothesis, D_1, N_1 are unique, and d, t are unique. So, from the above proof, β, γ are unique. Thus, D, N are unique. $\square$

Corollary 4. *Let $R = \{m/n \in \mathbb{Q} : n$ is odd $\}$. Then $T_n(R)$ is a uniquely strongly clean ring for every $n \geqslant 1$.*

In [5, Proposition 2(3)] Nicholson showed that if $2 \in U(R)$, then R is strongly clean if and only if every element is the sum of a unit and a square root of 1 which commute. In fact $2 \in U(R)$ is also necessary.

Proposition 5. *A ring R is strongly clean and $2 \in U(R)$ if and only if every element is the sum of a unit and a square root of 1 which commute.*

Proof: We only need prove that $2 \in U(R)$ is necessary. Let $a \in R$ and $a = x + u$ where $x^2 = 1$, $u \in U(R)$ and $xu = ux$. Similarly, $x = y + v$ where $y^2 = 1$, $v \in U(R)$ and $yv = vy$. Thus $x^2 = (y+v)^2 = y^2 + 2yv + v^2$, then $2y = -v \in U(R)$. Hence $2 \in U(R)$. $\square$

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