ON FRÉCHET-DIFFERENTIABILITY OF NEMYTSKIJ OPERATORS ACTING IN HÖLDER SPACES

by MANFRED GOEBEL

(Received 2 May, 1989)

In any field of nonlinear analysis Nemytskij operators, the superposition operators generated by appropriate functions, play a crucial part. Their analytic properties depend on the postulated properties of the defining function and on the function space in which they are considered. A rich source for related questions is the monograph by J. Appell and P. P. Zabrejko [**2**] and the survey paper by J. Appell [**1**].

Nemytskij operators mapping a Hölder space $H^\nu[a, b]$, $0 < \nu \leq 1$, into another Hölder space $H^\mu[a, b]$, $0 < \mu \leq 1$, have interesting and sometimes surprising properties. Some hints in this direction can be found, particularly, in [**3**]. The purpose of this short note is to show that each function $f \in C^1(R)$ generates a Nemytskij operator $Fy(t) = f(y(t))$, which as a mapping in $H^\nu[a, b]$, $0 < \nu \leq 1$, is continuous, and that for $f \in C^2(R)$ the same Nemytskij operator is continuously Fréchet-differentiable. The results proved show that at least in the autonomous case considered the assumptions of the recent paper [**7**] can be relaxed. In [**1**, §6] a necessary and sufficient condition for Nemystskij operators acting in Hölder spaces to be Fréchet-differentiable can be found; our proof is independent of this criterion. An application of the results in an optimal control problem for a nonlinear singular integral equation is given in [**6**].

In what follows $\nu \in (0, 1]$ is fixed, $\|\cdot\|_\nu$ denotes the usual norm in $H^\nu[a, b]$,

$$\|y\|_\nu = \max_{t\in[a,b]} |y(t)| + \sup_{t,s\in[a,b]} \frac{|y(t) - y(s)|}{|t - s|^\nu}, \qquad y \in H^\nu[a, b],$$

and $\mathscr{L}(H^\nu[a, b])$ the set of all linear bounded operators mapping $H^\nu[a, b]$ into itself.

THEOREM 1. *If $f \in C^1(R)$, then the Nemytskij operator $Fy(t) = f(y(t))$ maps $H^\nu[a, b]$ continuously into itself.*

Proof. 1. Let $y \in H^\nu[a, b]$ be fixed. With the constants

$$\begin{aligned}
\alpha &= \min\{y(s) + \tau(y(t) - y(s)) \mid t, s \in [a, b], \tau \in [0, 1]\},\\
\beta &= \max\{y(s) + \tau(y(t) - y(s)) \mid t, s \in [a, b], \tau \in [0, 1]\},\\
\gamma &= \max\{|f'(t)| \mid t \in [\alpha, \beta]\}
\end{aligned}$$

we find

$$\begin{aligned}
|Fy(t) - Fy(s)| &= |f(y(t)) - f(y(s))|\\
&= |y(t) - y(s)| \left|\int_0^1 f'(y(s) + \tau(y(t) - y(s)))\, d\tau\right|\\
&\leq \gamma |y(t) - y(s)| \qquad \text{for all } t, s \in [a, b],
\end{aligned}$$

and therefore $Fy \in H^\nu[a, b]$.

2. We show the continuity of $F : H^\nu[a, b] \to H^\nu[a, b]$ at an arbitrary $y \in H^\nu[a, b]$. To this end we take a positive number δ_0 and define a function $h = h(s, t, u, v)$ by setting

$$h(s, t, u, v) = f(y(t) + u) - f(y(t)) - f(y(s) + v) + f(y(s))$$

https://doi.org/10.1017/S0017089500007965 Published online by Cambridge University Press

for each $\{s, t, u, v\} \in \Omega := [a, b] \times [a, b] \times [-\delta_0, \delta_0] \times [-\delta_0, \delta_0]$. According to V. A. Bondarenko and P. P. Zabrejko [**4**, Theorem 3] it is sufficient to prove that for arbitrary fixed $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) \in (0, \delta_0)$ such that

$$|h(s, t, u, v)| \le \varepsilon(|t-s|^{\nu} + \delta^{-1}|u-v|) \text{ for all } \{s, t, u, v\} \in \Omega \text{ with } |u|, |v| \le \delta. \quad (1)$$

The Lagrange formula yields

$$\begin{aligned} h(s, t, u, v) &= (y(t) + u - y(s) - v)\int_0^1 f'(y(s) + v + \tau(y(t) + u - y(s) - v))\, d\tau \\ &\qquad - (y(t) - y(s))\int_0^1 f'(y(s) + \tau(y(t) - y(s)))\, d\tau \\ &= (y(t) - y(s))\int_0^1 [f'(y(s) + \tau(y(t) - y(s)) + v + \tau(u-v)) \\ &\qquad - f'(y(s) + \tau(y(t) - y(s)))]\, d\tau \\ &\qquad + (u - v)\int_0^1 f'(y(s) + \tau(y(t) - y(s)) + v + \tau(u-v))\, d\tau \\ &= (y(t) - y(s))\int_0^1 [g(s, t, u, v, \tau) - g(s, t, 0, 0, \tau)]\, d\tau \\ &\qquad - (u - v)\int_0^1 g(s, t, u, v, \tau)\, d\tau, \quad (2) \end{aligned}$$

where we have put

$$g(s, t, u, v, \tau) = f'(y(s) + \tau(y(t) - y(s)) + v + \tau(u - v)).$$

Since g is uniformly continuous on $\Omega \times [0, 1]$, there exists a $\delta_1(\varepsilon) \in (0, \delta_0)$ such that

$$|g(s, t, u, v, \tau) - g(s, t, 0, 0, \tau)| \le \varepsilon k^{-1}$$

for all $\{s, t, u, v, \tau\} \in \Omega \times [0, 1]$ with $|u|, |v| \le \delta_1(\varepsilon)$, where the constant $k > 0$ denotes the Hölder coefficient of y, and there exists a $\delta_2(\varepsilon) \in (0, \delta_0)$ such that

$$\delta |g(s, t, u, v, \tau)| \le \varepsilon$$

for all $\{s, t, u, v, \tau\} \in \Omega \times [0, 1]$ and for all $\delta \in (0, \delta_2(\varepsilon))$. Using both these inequalities in (2) we get (1) provided $\delta = \min\{\delta_1(\varepsilon), \delta_2(\varepsilon)\}$. ■

The proof just given is essentially based on [**4**]. Probably a direct proof along the lines of P. Drábek [**5**] is possible too. Since in that paper the situation is in a way analogous to the case considered here, I expect that our assumption $f \in C^1(R)$ cannot be weakened.

To prepare our main result we now consider the parameter integral

$$G(t) = \int_0^1 g(t, \tau)\, d\tau, \qquad t \in [a, b],$$

and prove the following lemma.

https://doi.org/10.1017/S0017089500007965 Published online by Cambridge University Press

LEMMA. *If the integrand* $g = g(t, \tau)$ *satisfies the assumptions*

(i) $g \in C([a, b] \times [0, 1])$,

(ii) $|g(t, \tau) - g(s, \tau)| \le c|t - s|^\nu$ *for all* $t, s \in [a, b]$ *and for all* $\tau \in [0, 1]$, *with* c *a positive constant,*

then

$$G \in H^\nu[a, b] \quad \text{with} \quad \|G\|_\nu \le \int_0^1 \|g(\cdot, \tau)\|_\nu \, d\tau.$$

Proof. The first statement is obvious. By definition of $\|\cdot\|_\nu$ we have

$$\|g(\cdot, \tau)\|_\nu \ge |g(t, \tau)| + \frac{|g(t, \tau) - g(s, \tau)|}{|t - s|^\nu} \qquad \text{for all } t, s \in [a, b] \text{ and for all } \tau \in [0, 1],$$

and, after integrating this inequality,

$$\begin{aligned} \int_0^1 \|g(\cdot, \tau)\|_\nu \, d\tau &\ge \int_0^1 |g(t, \tau)| \, d\tau + \frac{1}{|t - s|^\nu} \int_0^1 |g(t, \tau) - g(s, \tau)| \, d\tau \\ &\ge \left| \int_0^1 g(t, \tau) \, d\tau \right| + \frac{1}{|t - s|^\nu} \left| \int_0^1 [g(t, \tau) - g(s, \tau)] \, d\tau \right| \\ &= |G(t)| + \frac{|G(t) - G(s)|}{|t - s|^\nu} \end{aligned}$$

for all $t, s \in [a, b]$, from which the desired inequality follows. ■

THEOREM 2. *Let the Nemytskij operator* $Fy(t) = f(y(t))$ *be generated by* $f \in C^2(R)$. *Then at each point* $y \in H^\nu[a, b]$ *the operator* $F : H^\nu[a, b] \to H^\nu[a, b]$ *has a continuous Fréchet derivative* $F'(y)$ *given by*

$$F'(y)z(t) = f'(y(t))z(t) \qquad \text{for all } z \in H^\nu[a, b].$$

Proof. 1. We define a Nemytskij operator by setting $\tilde{F}y(t) = f'(y(t))$. Because of Theorem 1 we certainly have

$$F, \tilde{F} : H^\nu[a, b] \to H^\nu[a, b] \text{ are continuous.} \tag{3}$$

For any given $y \in H^\nu[a, b]$ we define another operator A_y by

$$A_y z(t) = \tilde{F}y(t)z(t), \qquad z \in H^\nu[a, b].$$

Since $H^\nu[a, b]$ is a Banach algebra (cf. Prößdorf [**8**, p. 93]), we have

$$A_y : H^\nu[a, b] \to H^\nu[a, b] \text{ with } \|A_y z\|_\nu \le \|\tilde{F}y\|_\nu \|z\|_\nu \qquad \text{for all } z \in H^\nu[a, b].$$

This implies

$$A_y \in \mathscr{L}(H^\nu[a, b]) \text{ with } \|A_y\|_{\mathscr{L}(H^\nu[a,b])} \le \|\tilde{F}y\|_\nu \qquad \text{for all } y \in H^\nu[a, b]$$

and, consequently,

$$\|A_y - A_z\|_{\mathscr{L}(H^\nu[a,b])} \le \|\tilde{F}y - \tilde{F}z\|_\nu \qquad \text{for all } y, z \in H^\nu[a, b].$$

Therefore, because of (3), $y \mapsto A_y$ is a continuous map from $H^\nu[a, b]$ into $\mathscr{L}(H^\nu[a, b])$.

2. It remains to show that $A_y = F'(y)$ for any fixed $y \in H^\nu[a, b]$. Again by means of

Lagrange's formula we get

$$\begin{aligned} &F(y+z)(t)-Fy(t)-A_y z(t)\\ &\quad = f(y(t)+z(t))-f(y(t))-f'(y(t))z(t)\\ &\quad = z(t)G_z(t), \end{aligned} \tag{4}$$

for all $z \in H^\nu[a, b]$ and for all $t \in [a, b]$, with the parameter integral

$$G_z(t)=\int_0^1 g(t, \tau)\, d\tau,$$

$$g(t, \tau)=f'(y(t)+\tau z(t))-f'(y(t))=\tilde{F}(y+\tau z)(t)-\tilde{F}y(t).$$

In virtue of (3), and because

$$\begin{aligned} &|F(y+\tau z)(t)-F(y+\tau z)(s)|\\ &\leq [|y(t)-y(s)|+\tau|z(t)-z(s)|]\int_0^1 |f''(y(s)+\tau z(s)+\sigma(y(t)+\tau z(t)-y(s)-\tau z(s)))|\, d\tau\\ &\leq k|t-s|^\nu \end{aligned}$$

for all $t, s \in [a, b]$ and for all $\tau \in [0, 1]$, where k is a positive constant depending on y and z only, the Lemma yields $G_z \in H^\nu[a, b]$ with

$$\|G_z\|_\nu \leq \int_0^1 \|\tilde{F}(y+\tau z)-\tilde{F}y\|_\nu\, d\tau \qquad \text{for all } z \in H^\nu[a, b]. \tag{5}$$

Now let $\varepsilon > 0$ be given. Then, by (3), there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\|\tilde{F}(y+\tau z)-\tilde{F}y\|_\nu < \varepsilon \qquad \text{for all } z \in H^\nu[a, b] \text{ with } \|z\|_\nu < \delta \text{ and for all } \tau \in [0, 1]. \tag{6}$$

Combining (4)–(6) we obtain

$$\|F(y+z)-Fy-A_y z\|_\nu \leq \|G_z\|_\nu \|z\|_\nu \text{ for all } z \in H^\nu[a, b] \text{ with } \|z\|_\nu < \delta,$$

which proves the statement. ■

REFERENCES

1. J. Appell, *The superposition operator in function spaces—a survey*, Preprint 141 (Mathematisches Institut Universität Augsburg, 1987).

2. J. Appell, P. P. Zabrejko, *Nonlinear superposition operators*, Cambridge Tracts in Mathematics 95 (Cambridge University Press, 1989).

3. J. Appell, E. De Pascale, P. P. Zabrejko, An application of B. N. Sadovskij's fixed point principle to nonlinear singular equations, *Z. Anal. Anwendungen* **6** (1987), 193–208.

4. V. A. Bondarenko, P. P. Zabrejko, The superposition operator in Hölder spaces (Russian), *Dokl. Akad. Nauk SSR* **222** (1975), 1265–1268.

5. P. Drábek, Continuity of Nemytskij's operator in Hölder spaces, *Comment. Math. Univ. Carolin.* **16** (1975) 37–57.

6. M. Goebel, D. Oestreich, Optimal control of a nonlinear singular integral equation arising in electrochemical machining, *Z. Anal. Anwendungen*, to appear.

https://doi.org/10.1017/S0017089500007965 Published online by Cambridge University Press

7. R. Nugari, Continuity and differentiability properties of the Nemitskii operator in Hölder spaces, *Glasgow Math. J.* **30** (1988), 59–65.

8. S. Prößdorf, *Einige Klassen singulärer Gleichungen* (Akademie-Verlag, 1974).

Martin-Luther-Universität
Sektion Mathematik
Universitätsplatz 6
G.D.R.-4020 Halle (Saale)