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ON SCREENABILITY AND METRIZABILITY OF MOORE SPACES

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After showing that each screenable Moore space is pointwise paracompact and that the converse is not true, Heath in [**4**] asked for a necessary and sufficient condition for a pointwise paracompact Moore space to be screenable. In [**12**], Traylor asked for a necessary and sufficient condition for a pointwise paracompact Moore space to be metrizable. It is the purpose of this paper to provide such conditions, and to establish relationships between those conditions and metrization problems in Moore spaces.

A Moore space S is a space (all spaces are T_1) in which there exists a sequence $G = (G_1, G_2, \ldots)$ of open coverings of S, called a development, which satisfies the first three parts of Axiom I in [**7**]. The statement that a collection H of subsets of the space S is point finite (point countable) means that no point of S belongs to infinitely (uncountably) many elements of H. The statement that a space S is pointwise paracompact means that for each open covering G of S there exists a point finite open covering H of S which refines G. It follows immediately that a pointwise paracompact Moore space has a point countable basis. However, Heath has stated in [**6**] that the converse is not true.

The statement that a collection H of subsets of the space S is σ-pairwise disjoint (σ-discrete) means that H is the union of countably many collections, each of which is pairwise disjoint (discrete). A space S is said to be screenable (strongly screenable) if and only if, for each open covering G of S, there exists a σ-pairwise disjoint (σ-discrete) open covering H of S which refines G. In [**1**], Bing established that in a Moore space S the following are equivalent: (1) S is metrizable; (2) S is normal and screenable; (3) S is strongly screenable.

Definition 1. A space S is said to be star-screenable (strongly star-screenable) if and only if, for each open covering G of S, there exists a σ-pairwise disjoint (σ-discrete) open covering H of S which refines $\{st(x,G)|x \in S\}$.

Notes. (1) Each strongly star-screenable space is star-screenable.
(2) Each separable space is strongly star-screenable.

THEOREM 2. *A space S is screenable (strongly screenable) if and only if S is star-screenable (strongly star-screenable) and for each open covering of S there exists a point countable refinement which is an open covering of S.*

Proof. The necessity follows immediately. Thus, suppose that S is star-screenable (strongly star-screenable), and that for each open covering of S

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there exists a point countable refinement which is an open covering of S. Then S is screenable (strongly screenable). For, suppose that G is an open covering of S. Let G' denote a point countable open refinement of G which covers S. Denote by $H = \bigcup_{i=1}^{\infty} H_i$ an open covering of S which refines $\{st(x, G')|x \in S\}$, where for each positive integer i, H_i is pairwise disjoint (discrete). For each positive integer i, consider H_i. If $h \in H_i$, let x_h denote a point of S such that $h \subset st(x_h, G')$, and denote by $\{g_1(x_h), g_2(x_h), \ldots\}$ the set of all elements of G' which contain x_h. Now, for each positive integer i and each positive integer j, let F_{ij} denote $\{g_j(x_h) \cap h|h \in H_i\}$. It follows that $F = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} F_{ij}$ is a σ-pairwise disjoint (σ-discrete) open covering of S which refines G.

COROLLARY 3. *A Moore space with a point countable basis is screenable (metrizable) if and only if it is star-screenable (strongly star-screenable).*

Definition 4. A space S is said to be star-screenable (strongly star-screenable) with respect to the subset M of S if and only if, for each collection G of open sets in S covering M, there exists a σ-pairwise disjoint (σ-discrete) collection H of open sets in S covering M and refining $\{st(x, G)|x \in M\}$.

A space S is subparacompact if and only if, for each open covering G of S, there exists a σ-discrete collection F of closed sets in S which refines G and covers S. Each Moore space is subparacompact [**1**].

THEOREM 5. *A subparacompact space S is screenable (strongly screenable) if and only if S is star-screenable (strongly star-screenable) with respect to each closed subset M of S.*

Proof. The necessity follows immediately. Thus, suppose that S is subparacompact and star-screenable (strongly star-screenable) with respect to each closed subset. Let G be an open covering of S. Since S is subparacompact there exists a collection $F = \bigcup_{i=1}^{\infty} F_i$ of closed sets in S which covers S and refines G, where for each positive integer i, F_i is discrete. For each i, consider F_i. Denote by D_i an open covering of F_i^* refining G such that each element of D_i contains only one element of F_i and intersects no other element of F_i. Now, S is star-screenable (strongly star-screenable) with respect to F_i^*. Thus, there exists a σ-pairwise disjoint (σ-discrete) open covering H_i of F_i^* which refines $\{st(x, D_i)|x \in F_i^*\}$. Note that H_i also refines D_i. For, if $h \in H_i$, then h is contained in $st(q, D_i)$ for some q in F_i^*. But q is contained in only one element of D_i; hence, h is contained in the element of D_i containing q. Thus, $H = \bigcup_{i=1}^{\infty} H_i$ is a σ-pairwise disjoint (σ-discrete) open covering of S which refines G. Thus, S is screenable (strongly screenable).

THEOREM 6. *A Moore space S, in which there does not exist an uncountable collection of pairwise disjoint open sets, is separable if and only if it is star-screenable.*

Proof. Each separable space is strongly star-screenable.

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Thus, suppose that S is star-screenable. Let $G = (G_1, G_2, \ldots)$ be a development for S. For each positive integer i, let H_i denote a σ-pairwise disjoint open covering of S which refines $\{st(x, G_i)|x \in S\}$. If $h \in H_i$, let x_h denote a point of S such that $h \subset st(x_h, G_i)$. Since there does not exist an uncountable collection of pairwise disjoint open sets in S, it follows that $M_i = \{x_h|h \in H_i\}$ is countable for each i. Consider $M = \bigcup_{i=1}^{\infty} M_i$. It remains only to show that M is dense in S. Thus, suppose that $p \in S$ and that D is an open set containing p; then there exists a positive integer n such that each element of G_n containing p is contained in D. But p is contained in some element h of H_n and $h \subset st(x_h, G_n)$, where $x_h \in M_n$. Thus, some element of G_n contains both p and x_h, and it follows that $x_h \in D$. This completes the proof.

It is unknown whether each normal pointwise paracompact or each normal separable Moore space is metrizable. Since strong star-screenability is the property of separability needed for a pointwise paracompact Moore space to be metrizable, perhaps it can be shown that each normal separable Moore space is metrizable if and only if each normal strongly star-screenable Moore space is metrizable. The author does not know if each normal Moore space is strongly star-screenable. However, the following theorems relate normality, star-screenability, and metrizability in Moore spaces.

THEOREM 7. *Each normal star-screenable Moore space is strongly star-screenable.*

Proof. Suppose that G is an open covering of S. Denote by $H = \bigcup_{i=1}^{\infty} H_i$ an open covering of S which refines $\{st(x, G)|x \in S\}$, where for each positive integer i, H_i is pairwise disjoint. For each positive integer i, consider the open set H_i^*. Since S is a normal Moore space, there exists a countable collection, $\{F_{i1}, F_{i2}, \ldots\}$, of open sets such that for each positive integer j, $\overline{F_{ij}} \subset H_i^*$ and $H_i^* = \bigcup_{j=1}^{\infty} F_{ij}$. For each j, let $W_{ij} = \{h \cap F_{ij}|h \in H_i\}$. Note that W_{ij} is discrete for each j and that $H_i^* = \bigcup_{j=1}^{\infty} W_{ij}$.

Consider $W = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} W_{ij}$. It follows that W is a σ-discrete collection of open sets which covers S and refines $\{st(x, G)|x \in S\}$.

THEOREM 8. *There exists a nonnormal, nonseparable Moore space S which is strongly star-screenable.*

Proof. Consider the tangent circle space (i.e., Neimytzki plane). Let X denote the non-normal, separable Moore space whose points are the points of the upper half-plane together with the x-axis and having the development $G_1, G_2, \ldots$ such that for each positive integer n, G_n is the collection of sets which are either interiors of circles of radius less that $1/n$ lying wholly above the x-axis or the interiors of such circles tangent to the x-axis together with the point of tangency. Since X is separable, X is strongly star-screenable. Now, denote by S the non-separable Moore space S whose points are the same as those of X with the same definition of basic open sets for points on the x-axis but with singleton basic open sets for points above the x-axis.

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The space S is also strongly star-screenable. For, suppose that G is an open covering of S. Denote by U a collection of open sets in X covering the x-axis and refining G. Since X is strongly star-screenable and the x-axis is closed in X, there exists a σ-discrete collection F of open sets in X covering the x-axis and refining $\{st(x, U)|x \in U^*\}$. For each positive integer i, let H_i denote the collection of all singleton sets containing points in the plane above the line $y = 1/i$. It follows that $H = F \cup (\bigcup_{i=1}^{\infty} H_i)$ is a σ-discrete collection of open sets in S which covers S and refines $\{st(x, G)|x \in S\}$. This completes the proof.

THEOREM 9. *There exists a pointwise paracompact Moore space in which there does not exist an uncountable collection of pairwise disjoint open sets which is not star-screenable.*

Proof. In [**9**], Pixley and Roy gave an example of a nonseparable, pointwise paracompact Moore space in which there does not exist an uncountable collection of pairwise disjoint open sets. By Theorem 6, such a space could not be star-screenable.

The statement that the development $G = (G_1, G_2, \ldots)$ for the Moore space S satisfies Axiom C at the point p of S means that, for each open set D containing p, there exists a positive integer n such that each element of G_n which intersects an element of G_n containing p is contained in D. If G is a development for the Moore space S then $C(G)$, the set of all points in S at which G satisfies Axiom C, is, if nonempty, an inner limiting, metrizable subset of S [**8**; **13**]. It is not known whether each normal Moore space has a dense metrizable subspace.
The following two theorems, however, replace normality with strong star-screenability in the best results obtained in this area.

THEOREM 10. (Procter [**10**].) *If G is a development for the strongly star-screenable Moore space S, then there exists a development G' for S such that $C(G')$ is 2-dense in S with respect to G (i.e., for each point p of S and each positive integer n, there exist two intersecting elements of G_n which contain both p and a point of $C(G')$ in their union).*

Proof. Let $G = (G_1, G_2, \ldots)$ denote a development for the strongly star-screenable Moore space S. For each positive integer i, let H_i denote a σ-discrete open covering of S which refines $\{st(x, G_i)|x \in S\}$. Let $H = \bigcup_{i=1}^{\infty} H_i$ and denote by K a subset of S which contains one point from each element of H. Note that K is the union of countably many point sets such that each is covered by a discrete collection of open sets intersecting it at only one point. It follows from the proof of [**2**, Theorem 5] that there exists a development G' for S which satisfies Axiom C at each point of K.
It remains only to show that $C(G')$ is 2-dense in S with respect to G. But if $p \in S$ and n is a positive integer, some element h of H_n contains p and is contained in $st(q, G_n)$, for some $q \in S$. Thus, since h contains a point of K, there exist elements g_1 and g_2 of G_n, each containing q, which contain both p and a point of K (hence, of $C(G')$) in their union. This completes the proof.

A space S is collectionwise Hausdorff provided that, for each discrete subset K of S, there exists a collection U of pairwise disjoint open sets covering K such that each element of U contains only one element of K.

THEOREM 11. (Fitzpatrick [**3**].) *If S is a collectionwise Hausdorff, strongly star-screenable Moore space, then for each subset M of S there exists a development G for S such that $C(G)$ is dense in S and $C(G) \cap M$ is dense in M.*

Proof. Suppose that M is a subset of the collectionwise Hausdorff, strongly star-screenable Moore space S. Then from the proof of [**3**, Theorem 1], it follows that there exists a dense subset K of S such that $K \cap M$ is dense in M and $K = \bigcup_{i=1}^{\infty} K_i$, where for each positive integer i, K_i is a discrete point set. Since S is collectionwise Hausdorff, for each i, there exists a collection F_i of pairwise disjoint open sets covering K_i such that each element of F_i contains only one point of K_i. Let $G' = (G_1', G_2', \ldots)$ denote a development for S, and for each positive integer i, let $H_i = \bigcup_{n=1}^{\infty} H_{in}$ denote an open covering of S which refines $\{st(x, G_i') | x \in S\}$, where for each positive integer n, H_{in} is discrete. Now, for each positive integer i, consider K_i. For each positive integer j, let K_{ij} denote the set of all points p in K_i such that each element of G_j' containing p is contained in the element of F_i which contains p. Note that $K_i = \bigcup_{j=1}^{\infty} K_{ij}$. For each positive integer n, denote by K_{ijn} the subset of K_{ij} contained in H_{jn}^*, and note that $K_{ij} = \bigcup_{n=1}^{\infty} K_{ijn}$. Also, it follows that for each j and for each n, no element of H_{jn} contains two elements of K_{ijn}. For, suppose that some $h \in H_{jn}$ contains points p_1 and p_2 of K_{ijn}. Then there exists a point $x \in S$ such that h is contained in $st(x, G_j')$. Thus, there exist intersecting elements g_1 and g_2 of G_j' which contain p_1 and p_2, respectively. But g_1 is contained in the element of F_i which contains p_1 and g_2 is contained in the element of F_i which contains p_2, and these two elements do not intersect.

Thus,

$$K = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcup_{n=1}^{\infty} K_{ijn}$$

is the union of countably many point sets such that each is covered by a discrete collection of open sets intersecting it at only one point. Again, it follows from the proof of [**2**, Theorem 5] that there exists a development G for S such that K is contained in $C(G)$. This completes the proof.

Questions. (1) Is each normal, collectionwise Hausdorff Moore space strongly star-screenable?

(2) Is each collectionwise Hausdorff, strongly star-screenable Moore space normal?

(3) What "weakening" of normality is a necessary and sufficient condition for a star-screenable Moore space to be strongly star-screenable?

(4) In [**11**], the author defined *wd*-normality and *sd*-normality, and established theorems concerning metrizable subspaces of Moore spaces with these

properties. What implications, if any, hold between these properties and (strong) star-screenability?

(5) In [**5**], Heath showed that each regular strongly complete, separable, semi-metrizable space is metrizable. Does it follow that each regular strongly complete, strongly star-screenable semi-metrizable space is metrizable?

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