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MULTIPLICATIVE FUNCTIONS AND RAMANUJAN'S τ-FUNCTION

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Abstract

It is proved that $(|\tau(n)|n^{-11/2})^{\delta}$ has a mean-value for $0 < \delta < 2$, where $\tau(n)$ is Ramanujan's function from modular arithmetic. Some further results are conjectured.

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Ramanujan's τ-function is defined according to the identity

$$\sum_{n=1}^{\infty} \tau(n)x^n = x\prod_{j=1}^{\infty}(1 - x^j)^{24}.$$

Our purpose is to prove the following

THEOREM. *Let* $0 < \delta \leqslant 2$. *Then*

$$\lim_{x\to\infty} x^{-1}\sum_{n<x}\left(\frac{|\tau(n)|}{n^{11/2}}\right)^{\delta} = A_\delta$$

exists and is finite. In particular

$$\lim_{x\to\infty} x^{-13/2}\sum_{n<x}|\tau(n)| = 2A_1/13$$

exists. Moreover, either every A_δ *with* $0 < \delta < 2$ *is zero, or the series*

$$\sum_p \frac{1}{p}\left(\frac{|\tau(p)|}{p^{11/2}} - 1\right)^2,$$

taken over the prime-numbers, converges.

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REMARKS. The existence of the limit A_δ is only new if $0 < \delta < 2$. It follows from a result of Rankin (1934) that A_2 exists and is non-zero.

We deduce Theorem 1 from the following result, which is of independent interest.

THEOREM 2. *Let $g(n)$ be a non-negative multiplicative arithmetic function which has a mean-value. Then $g(n)^\delta$ has a mean-value for each δ, $0 < \delta < 1$. Moreover, if any of these latter mean-values is non-zero, then the series*

$$\sum p^{-1}\left(\sqrt{g(p)} - 1\right)^2$$

converges.

REMARKS. In this formulation we interpret 0^δ to be *zero*. A function $g(n)$ is said to be *arithmetic* if it is defined on the positive integers, *multiplicative* if it satisfies $g(ab) = g(a)g(b)$ whenever the integers a and b are mutually prime, and to have a *mean-value* if

$$\lim_{x\to\infty} x^{-1} \sum_{n \leqslant x} g(n)$$

exists and is finite.

Our proof of Theorem 2 makes use of a number of results from the author's paper Elliott (1980b)–this journal. We shall refer to it as E. We here note that on pages 180, 195 and 202 of that paper the exponent $-mit$ should be replaced by $-m(it + 1)$. In Lemma 8 of E the condition (31) may be omitted (see Lemma 1 below). Moreover, the alternate proof of Theorem 1 (of E) which is mentioned at the foot of page 179 is due to Daboussi, and not to Daboussi and Delange, as was asserted.

LEMMA 1. *Let $g(n)$ be a multiplicative function for which the series*

$$\sum_{|g(p)-1|<1/2} \frac{|g(p)-1|^2}{p}, \qquad \sum_{|g(p)-1|>1/2} \frac{|g(p)-1|^\alpha}{p}, \qquad \sum_{p,\, m \geqslant 2} \frac{|g(p^m)|}{p^m}$$

converge, $\alpha > 1$.

Then

$$\{x\Lambda(\log x)\}^{-1} \sum_{n \leqslant x} g(n) \to J, \qquad x \to \infty,$$

where

$$\Lambda(u) = \exp\left(\sum_p p^{-1-1/u}(g(p) - 1)\right)$$

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is a slowly-oscillating function of $\exp(u)$, *and the constant* J *is given by*

$$J = \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \dots\right)\exp\left(\frac{1 - g(p)}{p}\right).$$

PROOF. A proof of this result when $\alpha = 2$ is indicated in Elliott (1980a), Chapter 10. The present Lemma 1 is the same as Lemma 8 of E with the superfluous condition (32) of that formulation omitted.

LEMMA 2. *The inequality* $|y^\delta - 1| \leqslant 3|y - 1|$ *holds uniformly for* $0 < \delta \leqslant 1$, $0 \leqslant y \leqslant 2$.

PROOF. If $0 \leqslant y \leqslant 1$ then $1 - y^\delta \leqslant 1 - y^2 = (1 + y)(1 - y) \leqslant 2(1 - y)$. If $1 < y \leqslant 2$ then $y^\delta - 1 \leqslant y^2 - 1 = (y + 1)(y - 1) \leqslant 3(y - 1)$.

PROOF OF THEOREM 2. We need only consider the case when for some value of δ, $0 < \delta < 1$, $g(n)^\delta$ does not have the mean-value zero, that is

$$\limsup_{x \to \infty} x^{-1} \sum_{n \leqslant x} g(n)^\delta > 0.$$

In particular, the value

$$A = \lim_{x \to \infty} x^{-1} \sum_{n \leqslant x} g(n)$$

which exists by hypothesis, must be non-zero.

In the notation of E page 181, the function $g(n)$ satisfies hypothesis H, and from Lemma 1 of that paper we obtain the convergence of the series

$$\sum_{|g(p) - 1| > 1/2} \frac{1}{p}, \qquad \sum_{|g(p) - 1| \leqslant 1/2} \frac{(g(p) - 1)^2}{p}.$$

From Lemma 4 of E, with the notation $h(n) = g(n)^\delta$, $\alpha = 1/\delta$, we obtain the convergence of the series

$$\sum_{p,\, m \geqslant 2} p^{-m} g(p^m), \qquad \sum_p p^{-1} |g(p)^\delta - 1|^{1/\delta}.$$

Note that if $g(p) > 3/2$ then

$$\left(g(p)^\delta - 1\right)^{1/\delta} \geqslant \left(g(p)^\delta \{1 - (2/3)^\delta\}\right)^{1/\delta} = c(\delta) g(p)$$

for a certain positive constant $c(\delta)$, so that the series

$$\sum_{g(p) > 3/2} p^{-1} g(p)$$

converges.

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An integration by parts shows that as $s \to 1+$,

$$\sum_{n=1}^{\infty} g(n)n^{-s} \sim A(s-1)^{-1}.$$

Since the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \sim (s-1)^{-1}$$

as $s \to 1+$,

$$\lim_{s \to 1+} \zeta(s)^{-1} \sum_{n=1}^{\infty} g(n)n^{-s}$$

exists and is non-zero.

We view this last ratio in terms of the corresponding Euler product(s):

$$\prod_{p} (1 - p^{-s})^{-1}(1 + g(p)p^{-s} + \dots).$$

We put into a product Π_2 those terms corresponding to primes p for which $|p(p) - 1| > \frac{1}{2}$. From our above results this product is seen to be absolutely convergent (with a non-zero value) if $s \geqslant 1$. The remaining terms we put into a product Π_1 which we rearrange into the form

$$\Pi_1 = \prod_{|g(p)-1|<1/2} (1 + p^{-s}\{g(p) - 1\} + \psi(p)),$$

where

$$\psi(p) = \sum_{m=2}^{\infty} \{g(p^m) - g(p^{m-1})\}p^{-ms}.$$

Note that for a suitably chosen q,

$$\sum_{p>q} |\psi(p)| \leqslant \sum_{\substack{p>q \\ g(p)<3/2}} \left(g(p)p^{-2} + 2\sum_{m=2}^{\infty} g(p^m)p^{-m} \right) < \frac{1}{4}.$$

Hence

$$\sum_{\substack{p>q \\ |g(p)-1|<1/2}} |\log(1 + p^{-s}\{g(p) - 1\} + \psi(p)) - p^{-s}\{g(p) - 1\} - \psi(p)|$$

$$\leqslant \sum_{\substack{p>q \\ |g(p)-1|<1/2}} (p^{-1}|g(p) - 1| + |\psi(p)|)^2 \sum_{j=0}^{\infty} \left(\frac{3}{4}\right)^j$$

$$\leqslant \sum_{|g(p)-1|<1/2} 8p^{-2}(g(p) - 1)^2 + \sum_{p>q} 2|\psi(p)| < \infty.$$

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Taking logarithms we deduce the finite existence of

$$\lim_{s\to 1+} \sum_{|g(p)-1|\leqslant 1/2} p^{-s}(g(p)-1).$$

Applying the Hardy–Littlewood tauberian Theorem (Hardy (1949), Elliott, (1979), Chapter 2) or a method of Daboussi and Delange (see E Lemma 9) we obtain the convergence of the series

$$\sum_{|g(p)-1|\leqslant 1/2} p^{-1}(g(p)-1).$$

We now apply our present Lemma 1 to the function $g(n)^\delta$, using $\alpha = 1/\delta$. If $|g(p)^\delta - 1| \leqslant \frac{1}{2}$ then $|g(p)-1| < d < 1$ for a certain (positive) number d. From Lemma 2,

$$\sum_{|g(p)^\delta-1|\leqslant 1/2} p^{-1}|g(p)^\delta - 1|^2 \leqslant 3 \sum_{|g(p)-1|<d} p^{-1}|g(p)-1|^2$$

$$\leqslant 3 \sum_{|g(p)-1|\leqslant 1/2} p^{-1}|g(p)-1|^2 + 3 \sum_{|g(p)-1|>1/2} p^{-1} < \infty.$$

The remaining conditions of Lemma 1 are readily seen to be satisfied and

$$\{x\Lambda(\log x)\}^{-1} \sum_{n\leqslant x} g(n)^\delta \to J, \qquad x\to\infty,$$

where

$$\Lambda(u) = \exp\left(\sum_p p^{-1-1/u}(g(p)^\delta - 1)\right).$$

Since

$$\sum_{|g(p)-1|>1/2} p^{-1}|g(p)^\delta - 1| \leqslant \sum_{|g(p)|<1/2} p^{-1} + \sum_{g(p)>3/2} p^{-1}g(p),$$

and when $|g(p)-1| \leqslant \frac{1}{2}$

$$g(p)^\delta - 1 = \{1-(1-g(p))\}^\delta - 1 = \delta(1-g(p)) + O(|1-g(p)|^2),$$

the series

$$\sum p^{-1}(g(p)^\delta - 1)$$

converges. A simple modification of Abel's well known theorem for power series now gives the finite existence of lim $\Lambda(u)$ as $u\to\infty$, and so the existence of the mean-value for $g(n)^\delta$.

The final assertion of Theorem 2 follows, in the present circumstances, from the inequalities

$$(\sqrt{g}-1)^2 \leqslant \begin{cases} g & \text{if } g > \frac{3}{2}, \\ 3(g-1)^2 & \text{if } \frac{1}{2} \leqslant g \leqslant \frac{3}{2}, \\ 1 & \text{if } g < \frac{1}{2}. \end{cases}$$

REMARKS. The methods of E will allow the complete characterization of multiplicative functions which satisfy hypothesis H with some $\alpha > 1$. We note here that in addition to the conditions given in Lemmas 1 and 4 of E, the function $w(x)$ which occurs on page 185 of that paper is to satisfy (16) there, and to be bounded *above* uniformly for all $x \geqslant 1$.

PROOF OF THEOREM 1. It was conjecture by Ramanujan and proved by Mordell (1917) that $\tau(n)$ is multiplicative. With $g(n) = (|\tau(n)|n^{-11/2})^2$ we may deduce Theorem 1 from Theorem 2 and Rankin's (1934) result that A_2 exists.

CONCLUDING REMARKS. It was proved by Deligne that $|\tau(p)| < 2p^{11/2}$. If we write $\tau(p)p^{-11/2} = 2 \operatorname{Cos} \theta_p$ then θ_p is real and may be taken in the interval $0 \leqslant \theta_p \leqslant \pi$.

Let us for the moment assume the validity of the Sato–Tate conjecture that as p varies the θ_p are distributed over this interval with a probability density $2(\operatorname{Sin} \theta)^2/\pi$. Then

$$\sum_{p \leqslant x} \frac{1}{p}\left(\frac{|\tau(p)|}{p^{11/2}} - 1\right)^2 \sim c \log \log x, \qquad x \to \infty,$$

with the constant

$$c = \frac{2}{\pi}\int_0^\pi (2|\operatorname{Cos} \theta| - 1)^2 (\operatorname{Sin} \theta)^2 \, d\theta = 2.$$

One would accordingly conjecture that every A_δ with $0 < \delta < 2$ has the value zero.

Perhaps for each δ, $0 < \delta < 2$, we have

$$\sum_{n \leqslant x}\left(\frac{|\tau(n)|}{n^{11/2}}\right)^\delta = O\big(x(\log x)^{-h(\delta)}\big), \qquad x \geqslant 2,$$

with

$$h(\delta) = \frac{2}{\pi}\int_0^\pi \{1 - (2|\operatorname{Cos} \theta|)^\delta\}(\operatorname{Sin} \theta)^2 \, d\theta.$$

If now some A_δ with $\delta > 2$ were to exist, then since $A_2 \neq 0$ Theorem 2 would assert the convergence of

$$\sum \frac{1}{p}\left(\left(\frac{|\tau(p)|}{p^{11/2}}\right)^{\delta/2} - 1\right)^2.$$

This, also, is incompatible with the Sato–Tate conjecture. Very likely no (finite) mean-value A_δ with $\delta > 2$ can exist.

As to a finer behaviour of $|\tau(n)|$, let us assume that

$$\sum_{\substack{p<x \\ \left|\theta_p - \frac{\pi}{2}\right| < w}} \frac{1}{p} \leqslant \frac{c \log\log x}{(-\log w)^4}$$

holds uniformly for $x \geqslant 2$, $x^{-\lambda} < w \leqslant \pi/4$, for some fixed $\lambda > 0$. This asserts a local upper bound involving the distribution of the θ_p near to $\pi/2$ which although crude has a good uniformity. It is related to the Sato–Tate conjecture somewhat in the manner that the Brun–Titchmarsh upper bound from sieve theory is related to the classical prime number theorem.

Let us for the moment assume that $\tau(n)$ is never zero, and define the *additive function* $f(n) = \log|\tau(n)|n^{-11/2}$. Thus $f(ab) = f(a) + f(b)$ whenever a, b are mutually prime positive integers.

Our assumptions up until now then allow the proof that as $x \to \infty$

$$\frac{1}{\log\log x}\sum_{p<x}\frac{f(p)}{p} \to \frac{2}{\pi}\int_0^{\pi}(\log 2|\mathrm{Cos}\,\theta|)(\mathrm{Sin}\,\theta)^2\,d\theta = -\frac{1}{2},$$

$$\frac{1}{\log\log x}\sum_{p<x}\frac{f(p)^2}{p} \to \frac{2}{\pi}\int_0^{\pi}(\log 2|\mathrm{Cos}\,\theta|)^2(\mathrm{Sin}\,\theta)^2\,d\theta = \mu^2$$

for some $\mu > 0$. One can now treat $f(n)$ within the framework of the probabilistic theory of numbers, as if it were of the class H of Kubilius (Kubilius (1964), Elliott (1980a), Chapter 12). The relevant step being justified by Lemma (11.1) of Elliott (1980a). Hence we should obtain

$$\nu_x\left(n;\ \frac{|\tau(n)|}{n^{11/2}} \leqslant \frac{e^{z\mu\sqrt{\log\log x}}}{\sqrt{\log x}}\right) \Rightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-t^2/2}\,dt, \qquad x \to \infty,$$

the $\Rightarrow$ denoting weak convergence. In the present circumstance this amounts to proper convergence for each z. The symbol on the left hand side of this limiting relation denotes the frequency

$$\frac{\text{Number of integers } n \leqslant x \text{ for which } |\tau(n)|n^{-11/2} \leqslant \ldots}{\text{Number of integers } n \leqslant x}.$$

If $\tau(n)$ vanishes sometimes, one would expect the series

$$\sum_{\tau(p)=0}\frac{1}{p}$$

to converge. Otherwise $\tau(n) = 0$ would hold on a sequence of integers of asymptotic density *one*; almost always. (See, for example, Elliott (1979), Chapter 7.) If this last is not the case, then

$$\lim_{x\to\infty} x^{-1}\sum_{n\leqslant x,\ \tau(n)\neq 0} 1 = B > 0$$

would hold and the above assertion concerning the limiting behaviour of $|\tau(n)|n^{-11/2}$ could still be made provided that in the frequency one counted only integers for which $\tau(n) \neq 0$. This result may then be established (conditionally upon the above assumptions) by means of a finite probability model for non-negative multiplicative functions, constructed as in Chapter 3 of Elliott (1979).

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