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On ramification filtrations and p-adic differential equations, II: mixed characteristic case

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Abstract

Let K be a complete discrete valuation field of mixed characteristic $(0, p)$, with possibly imperfect residue field. We prove a Hasse–Arf theorem for the arithmetic ramification filtrations on G_K, except possibly in the absolutely unramified and non-logarithmic case, or the $p = 2$ and logarithmic case. As an application, we obtain a Hasse–Arf theorem for filtrations on finite flat group schemes over $\mathcal{O}_K$.

1. Introduction

1.1 Main results

This paper is a sequel to [Xia10], in which we proved a comparison theorem for the arithmetic ramification conductors defined by Abbes and Saito [AS02] and the differential ramification conductors defined by Kedlaya [Ked07]. In that paper, a key consequence was that one can use the Hasse–Arf theorem for the differential conductors to obtain a Hasse–Arf theorem for the arithmetic conductors in the equal characteristic $p > 0$ case.

In this paper, we combine the ideas from [Ked07, Xia10] with the techniques of non-archimedean differential modules in [KX10] to give a proof of the following Hasse–Arf theorem for the arithmetic ramification conductors in the mixed characteristic case.

Theorem. *Let K be a complete discrete valuation field of mixed characteristic* $(0, p)$ *and let G_K be its absolute Galois group. Let* $\mathrm{Fil}^\bullet G_K$ *and* $\mathrm{Fil}^\bullet_{\log} G_K$ *denote the ramification filtrations defined by Abbes and Saito [AS02].*

(1) *(Hasse–Arf theorem.) Let $\rho : G_K \to \mathrm{GL}(V_\rho)$ be a continuous representation of finite monodromy, where V_ρ is a finite-dimensional vector space over a field of characteristic zero. Then the Artin conductor* $\mathrm{Art}(\rho)$ *(defined using* $\mathrm{Fil}^\bullet G_K$*) is a non-negative integer if K is not absolutely unramified; the Swan conductor* $\mathrm{Swan}(\rho)$ *(defined using* $\mathrm{Fil}^\bullet_{\log} G_K$*) is a non-negative integer if $p > 2$, and* $\mathrm{Swan}(\rho) \in \frac{1}{2}\mathbb{Z}_{\geqslant 0}$ *if $p = 2$.*

(2) *The subquotients* $\mathrm{Fil}^a G_K/\mathrm{Fil}^{a+} G_K$ *for $a > 1$ and* $\mathrm{Fil}^a_{\log} G_K/\mathrm{Fil}^{a+}_{\log} G_K$ *for $a > 0$ of the ramification filtrations are trivial if $a \notin \mathbb{Q}$ and are abelian groups killed by p if $a \in \mathbb{Q}$, except in the absolutely unramified and non-logarithmic case.*

This theorem summarizes the results of Theorems 4.3.5, 4.5.14, and 4.7.3.

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We do not know whether $\mathrm{Swan}(\rho)$ may fail to be an integer when $p = 2$ in general; exclusion of the absolutely unramified and non-logarithmic case seems to be essential.

The theorem was conjectured implicitly in [AS02], and Abbes and Saito proved that the subquotients of the filtrations are abelian groups, except in the absolutely unramified and non-logarithmic case. After that, Hattori [Hat06, Hat08] gave some partial results on the first part of the theorem for the case where the corresponding field extension can be realized by a commutative finite flat group scheme. After the first draft of this paper was written, Saito [Sai] independently proved the second part of the theorem in the logarithmic case; it follows that $\mathrm{Swan}(\rho) \in \mathbb{Z}[1/p]$.

The technique used in this paper is very different from the approaches above, except that we need a small technical lemma (see § 3.4) borrowed from [AS03]. This paper shares some core ideas with the first paper in the series, [Xia10], but is logically independent of that paper.

1.2 Idea of the proof

To best convey the idea of the proof, assume that we are not in the excluded cases listed in the main theorem. We will come back to the reasons for excluding these cases later. We start with a naïve approach to the above theorem in the non-logarithmic case. One easily reduces the situation to the following case.

Let L/K be a finite totally ramified and wildly ramified Galois extension of complete discrete valuation fields of mixed characteristic $(0, p)$. Let $\mathcal{O}_K$, π_K and k denote the ring of integers, a uniformizer and the residue field, respectively. Assume that $\dim_{k^p} k < +\infty$. There are elements $b_1, \dots, b_m \in k$ such that $\bar{b}_1^{i_1} \cdots \bar{b}_m^{i_m}$, for $i_1, \dots, i_m \in \{0, \dots, p-1\}$, form a basis of k as a k^p-vector space; let $b_1, \dots, b_m$ be lifts of $\bar{b}_1, \dots, \bar{b}_m$ in $\mathcal{O}_K$. Our representation ρ is assumed to be absolutely irreducible, and it factors exactly through the Galois group $G_{L/K}$. We need to prove that $b(L/K) \cdot \dim \rho \in \mathbb{Z}$, where $b(L/K)$ is the ramification break, i.e. the maximal number b such that $\mathrm{Fil}^b G_{L/K} = G_L \mathrm{Fil}^b G_K / G_L \neq \{1\}$.

Step I: AS = TS *theorem* (make the Abbes–Saito space more functorial). Roughly speaking, the ramification break $b(L/K)$ is defined as follows. For the extension L/K and any rational number $a \in \mathbb{Q}_{>0}$, Abbes and Saito [AS02] defined a rigid analytic space AS^a together with a finite morphism $\Pi' : \mathrm{AS}^a \to A_K^{m+1}[0, |\pi_K|^a]$ (of degree $[L : K]$), where $A_K^{m+1}[0, |\pi_K|^a]$ denotes a (closed) polydisc over K of radius $|\pi_K|^a$. The ramification break $b(L/K)$ is the infimum among all $a \in \mathbb{Q}_{>0}$ such that the number of geometric connected components $\#\pi_0^{\mathrm{geom}}(\mathrm{AS}^a)$ is equal to $[L : K]$. A problem associated with this rigid analytic space is that it is not functorial under the operation of replacing K by a (not necessarily finite) complete extension K', which we shall refer to as a base change later on.

Pretend for the moment that we have a continuous homomorphism $\psi : \mathcal{O}_K \to \mathcal{O}_K[\![\delta_0, \dots, \delta_m]\!]$ such that $\psi(\pi_K) = \pi_K + \delta_0$ and $\psi(b_i) = b_i + \delta_i$ for $i = 1, \dots, m$. We define a new rigid analytic space, called the thickening space, to be

$$\mathrm{TS}^a_{L/K} = \mathrm{Spm}(L \otimes_{K,\psi} K\langle \pi_K^{-a}\delta_0, \dots, \pi_K^{-a}\delta_m\rangle) \xrightarrow{\ \Pi\ } A_K^{m+1}[0, |\pi_K|^a],$$

where Π is the projection to the second factor.

We can prove that $\mathrm{AS}^a \simeq \mathrm{TS}^a_{L/K}$ as rigid analytic K-spaces (see Theorem 3.3.3); this isomorphism does *not* respect the morphisms Π and Π' to the polydisc. The rigid analytic space $\mathrm{TS}^a_{L/K}$ also carries the information of the ramification break $b(L/K)$; together with Π, it is functorial under base change.

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Step II: *generic p^∞th roots* (a procedure to reduce to the perfect residue case). It is natural to make the following observation. Let a be a rational number slightly bigger than $b(L/K)$; then $\mathrm{TS}^a_{L/K}(=\mathrm{AS}^a)$ is geometrically the disjoint union of $[L:K]$ (poly)discs. What often happens is that if you increase the radius only on certain δ_i, then $\pi_0^{\mathrm{geom}}(\mathrm{TS}_{L/K})$ stays the same even when the radius goes beyond the cut-off point $|\pi_K|^{b(L/K)}$. In contrast, if one increases the radius along some other δ_i, $\pi_0^{\mathrm{geom}}(\mathrm{TS}_{L/K})$ will change as soon as the radius reaches $|\pi_K|^{b(L/K)}$. In the latter case, we say that the corresponding δ_i dominates. We remark that if we change the lift of $\bar{b}_j$ from b_j to $b_j+\pi_K$, then whether the 'uniformizer direction' δ_0 is dominant may change as well.

The ideal situation is when δ_0 is dominant. In this case, we can 'forget' about other directions, or, more concretely, we can make the residue field perfect by simply adding in all p-power roots of b_j for all j (and then completing). We will talk about this procedure in more detail in the next step. As remarked above, for this to happen, we need to find the 'correct lift' of each b_j. Following the idea of Borger [Bor04], we consider the notion of *generic rotation*. Let $x_1,\dots,x_m$ be transcendental over K, let K' be the completion of $K(x_1,\dots,x_m)$ with respect to the $(1,\dots,1)$-Gauss norm, and let $L'=K'L$. It easy to see that $b(L'/K')=b(L/K)$. The upshot is that if we set the p-basis of K' to be $\{b_1+x_1\pi_K,\dots,b_m+x_m\pi_K,x_1,\dots,x_m\}$, then the uniformizer direction is going to be dominant. So, if we set $\widetilde{K}$ to be the completion of the field obtained by adjoining to K' all p-power roots of $b_i+x_i\pi_K$ and x_i, we should have $b(\widetilde{K}L/\widetilde{K})=b(L/K)$ and are reduced to the classical situation because $\widetilde{K}$ has a perfect residue field.

Step III: *ramification break versus radii of convergence for differential modules* (where differential modules come into the picture). Since we 'pretended' earlier that we have a homomorphism ψ, the morphism $\Pi:\mathrm{TS}^a_{L/K}\to A_K^{m+1}[0,|\pi_K|^a]$ is étale; we can then push forward the ring of functions on $\mathrm{TS}^a_{L/K}$ to get a differential module $\mathcal{E}$ on the polydisc (compatible as a varies). Consider the naïve extension of scalars to $A_L^{m+1}[0,|\pi_K|^a]$. It is not hard to show that $\pi_0^{\mathrm{geom}}(\mathrm{TS}^a_{L/K})=[L:K]$ is almost equivalent to the differential module $\mathcal{E}$ being trivial over $A_L^{m+1}[0,|\pi_K|^a]$ (see Proposition 3.5.2).

A good thing about radii of convergence is that they are quite computable under base change. When making the base change from K to $\widetilde{K}$, we should have a Cartesian diagram

$$\begin{array}{ccc} \mathrm{TS}^a_{L/K} & \longleftarrow & \mathrm{TS}^a_{L\widetilde{K}/\widetilde{K}} \\ \downarrow{\scriptstyle \Pi} & & \downarrow{\scriptstyle \Pi_{\widetilde{K}}} \\ A_K^{m+1}[0,|\pi_K|^a] & \xleftarrow{f} & A_{\widetilde{K}}^{2m+1}[0,|\pi_K|^a] \end{array} \tag{1.2.1}$$

where f is induced by some map $f^*:\mathcal{O}_K[\![\delta_0,\dots,\delta_m]\!]\to\mathcal{O}_{\widetilde{K}}[\![\eta_0,\dots,\eta_{2m}]\!]$ characterized by $f^*\circ\psi=\psi_{\widetilde{K}}|_K:\mathcal{O}_K\to\mathcal{O}_{\widetilde{K}}[\![\eta_0,\dots,\eta_{2m}]\!]$. It is very easy to compare the radii of convergence of $\mathcal{E}$ with the radii of convergence of $f^*\mathcal{E}$, and the comparison of $b(L/K)$ and $b(L\widetilde{K}/\widetilde{K})$ follows.

Step IV: *logarithmic filtration* (a trick to deal with logarithmic filtration). We briefly discuss the idea behind the proof in the logarithmic case. We do not expect that we can always make the uniformizer direction 'log-dominant'. Instead, we expect a dichotomy:

- if the uniformizer direction is log-dominant, we are good anyway;
- if the uniformizer direction is not log-dominant, we expect that, after a large tame base change to $K_n=K(\pi_K^{1/n})$ and then a generic rotation for K_n as in Step II, $b(L'_n/K'_n)=nb_{\log}(L/K)$ and the uniformizer direction is *non-log*-dominant. Here the multiple n comes

from the normalization; the key is that after the follow-up generic rotation, the non-log ramification break is one less than the log ramification break.

Thus, we can always deduce that $n \cdot \mathrm{Swan}(\rho) \in \mathbb{Z}$ for $n \gg 0$ and $p \nmid n$. Taking two coprime numbers n_1 and n_2 will imply that $\mathrm{Swan}(\rho)$ is itself an integer.

We now come back to real life and discuss where the naïve approach fails and how we can fix it.

(1) The first thing to notice is that the desired homomorphism ψ *never* exists, as we cannot make $\psi(p) = p$ and $\psi(\pi_K) = \pi_K + \delta_0$ happen at the same time. As a remedy, we take ψ to be a function, which becomes a homomorphism if we modulo the ideal $I_K = p(\delta_0/\pi_K, \delta_1, \ldots, \delta_m)$ (Proposition 3.2.8). When K is absolutely unramified or, in other words, $v_K(p) = 1$, this condition is significantly weakened. This is the only hindrance to extending our main result to the absolutely unramified and non-logarithmic case (see also Remark 3.2.9).

We define the space $\mathrm{TS}^a_{L/K,\psi}$ by writing down the equations generating the extension $\mathcal{O}_L/\mathcal{O}_K$ and applying ψ termwise. When considering the effect of adding a generic pth root (instead of p^∞th root; see Remark 4.2.14), we similarly require that $f \circ \psi$ and $\psi_{\widetilde{K}}$ only agree modulo $I_{\widetilde{K}} = p(\eta_0/\pi_{\widetilde{K}}, \eta_1, \ldots, \eta_{2m})$. We have to carefully keep track of the error terms due to the non-homomorphism ψ and non-commutativity of $f \circ \psi$ and $\psi_{\widetilde{K}}$. In particular, if we still want (1.2.1) to be a Cartesian diagram, we need to modify $\mathrm{TS}^a_{L\widetilde{K}/\widetilde{K}}$ (see Theorem 4.3.4); this is the most difficult theorem of the paper. Luckily, the modification made here is not too serious, so that we still have AS = TS (Theorem 3.3.3) for the modified thickening space.

(2) Since we have a problem with defining ψ, the morphism $\Pi : \mathrm{TS}^a_{L/K,\psi} \to A^{m+1}_K[0, |\pi_K|^a]$ is only finite and étale if $a \geqslant b(L/K) - \epsilon$ for some $\epsilon > 0$. This is the only technical point for which we need to refer back to Abbes and Saito's approach, namely [AS02, Theorem 7.2] (and [AS03, Corollary 4.12] in the logarithmic case). This étaleness statement validates the construction of differential modules. The auxiliary étale locus given by ϵ enables us to find the exact loci where the intrinsic radii are maximal (or, equivalently, the loci where the differential module is trivial) and hence identify the ramification break.

(3) Since ψ fails to be a homomorphism, we have a minor technical issue when using differential modules. We have to study the generic radii of convergence over polydiscs instead of over one-dimensional discs (as was done in [Xia10]); this makes essential use of the recent results on p-adic differential modules from [KX10]. As a result, the proof in the logarithmic case is slightly more complicated, and for $p = 2$ we can only prove that Swan conductors lie in $\frac{1}{2}\mathbb{Z}$ instead of in $\mathbb{Z}$.

1.3 Who cares about the imperfect residue field case, anyway?

In algebraic geometry, if one wants to measure the ramification of an l-adic sheaf along a divisor, it is natural to pass to the completion at the generic point of the divisor; this would naturally give rise to a complete discrete valuation field with imperfect residue field, provided that the dimension of the divisor is not zero.

It is natural to ask how the ramification information varies from one divisor to another. Kedlaya started an interesting study in [Ked11] along this line, inspired by the semicontinuity results of André [And07] in complex algebraic geometry. In [Ked11], Kedlaya took an F-isocrystal on a smooth surface X overconvergent along the complement divisor D of simple normal crossings, in a compactification of X. If we blow up the intersection of two irreducible components of D, we may realize $\mathcal{F}$ over this new space and measure the Swan conductor along the exceptional divisor. This process can be iterated. Kedlaya proved in [Ked11] that, after suitable normalization, the Swan conductors along these exceptional divisors are interpolated by a continuous piecewise linear convex function. This result also holds for general smooth varieties of arbitrary dimension (see [Ked11]), as well as for lisse l-adic sheaves.

An interesting question is: does the same phenomenon happen for a noetherian complete regular local ring $\mathcal{O}_K[\![t_1, \ldots, t_n]\!]$, where $\mathcal{O}_K$ is a complete discrete valuation ring of mixed characteristic?

Another application is to the study of finite flat group schemes via ramification filtration initiated by Abbes and Mokrane in [AM04]. Hattori conjectured that one can give a bound on the denominators of ramification breaks. This can be proved by an analogous Hasse–Arf theorem for finite flat group schemes. Thus, as a consequence of the main theorem of this paper, we obtain a Hasse–Arf theorem for finite flat group schemes in the mixed characteristic case by an argument originally due to Hattori.

1.4 Structure of the paper

In § 1, we recall some results on p-adic differential modules from [KX10].

In § 2, we set up the framework for proving the main result. The definition of ramification filtrations is reviewed in § 2.2.

In § 3.1, we introduce the standard Abbes–Saito spaces. In §§ 3.2–3.5, we define the function ψ mentioned earlier and construct the thickening spaces and associated differential modules; the aim is to translate the question about ramification breaks into a question about the intrinsic radii of convergence. In § 3.6, we discuss a variant of thickening spaces.

The proofs of the main results, Theorems 4.3.5, 4.5.14, and 4.7.3, occupy the whole of § 4. In the first three subsections, we deduce the Hasse–Arf theorem for non-logarithmic ramification filtration. In § 4.4, we apply the Hasse–Arf theorem for Artin conductors to obtain a Hasse–Arf theorem for finite flat group schemes. In § 4.5, we deduce the integrality of Swan conductors from that of Artin conductors by tame base change. In the final two subsections, we use a trick due to Kedlaya to prove that the subquotients of the logarithmic filtration (on the wild ramification group) are abelian groups killed by p.

1.5 Notation

Owing to the technical details involved, the notation in this paper is particularly complicated. Here we list a few important terms together with short explanations and the locations of first appearance. We hope that this will help to make the paper more accessible.

K complete discrete valuation field of mixed characteristic of absolute ramification degree β_K; $L =$ finite extension; $\theta = |\pi_K|$.

$\widetilde{K}$ (4.2.1) K with generic pth root added.

$\widetilde{K}_n$ (4.5.9) K with $\pi_K^{1/n}$ and generic pth roots added.

K_* (4.6.1) an 'Artin-Scheier' extension of K.

$\widetilde{K}_\gamma$ (4.7.1) K_* with generic pth roots added.

$J = \{1, \ldots, m\}$ and $J^+ = J \cup \{0\}$ are used to index a p-basis.

$b_1, \ldots, b_m$ or b_J (3.1.6) lifts of a p-basis of k.

$c_1, \ldots, c_m$ or c_J (3.1.6) lifts of a p-basis of l.

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$u_0, \ldots, u_m$ (3.1.6) proxies for c_{J+}.

$p_0, \ldots, p_m$ or p_{J+} (3.1.6) relations of the extension $\mathcal{O}_L$ over $\mathcal{O}_K$ with generators c_J and π_L.

N^a (3.1.5) set of elements of $\mathcal{O}_K[u_{J+}]$ with norm less than or equal to θ^a.

$\mathrm{AS}^a_{L/K(,\log)}$ and $\mathcal{O}^a_{\mathrm{AS},L/K(,\log)}$ (3.1.9) (standard) Abbes–Saito spaces and their rings of functions.

$\mathcal{R}_K = \mathcal{O}_K[\![\delta_0/\pi_K, \delta_J]\!]$ (3.2.4); similarly for $\mathcal{R}_{\widetilde{K}}$ (4.2.5).

$\psi_K : \mathcal{O}_K \to \mathcal{O}_K[\![\delta_{J+}]\!] \subseteq \mathcal{R}_K$ (3.2.1); similarly for $\psi_{\widetilde{K}}$ (4.2.5) and other fields.

$\mathcal{S}_K = \mathcal{R}_K\langle u_{J+}\rangle$ (3.2.12).

R_{J+} (3.2.12) elements of $(\delta_{J+})\mathcal{S}_K$ representing the error terms with error gauge $\geqslant \omega$.

$\mathrm{TS}^a_{L/K(,\log),R_{J+}}$ and $\mathcal{O}^a_{\mathrm{TS},L/K(,\log),R_{J+}}$ (3.2.13) thickening spaces and their rings of functions; similarly for the standard ones $\mathrm{TS}^a_{L/K(,\log),\psi}$ and $\mathcal{O}^a_{\mathrm{TS},L/K(,\log),\psi}$ (3.2.13).

$\Delta : \mathcal{S}_K/(\psi(p_{J+}) + R_{J+}) \to \mathcal{O}_K\langle u_{J+}\rangle/(p_{J+}) \xrightarrow{\sim} \mathcal{O}_L$ (3.1.6) and (3.2.16); $\overline{\Delta}$ is its reduction.

$\mathrm{ET}_{L/K,R_{J+}}$ or $\mathrm{ET}_{L/K}$ (3.4.1) étale locus over which the thickening space is étale.

$\mathfrak{c}_{0,I}, \mathfrak{c}_\Lambda, \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda, \mathfrak{p}_{0,I}, \mathfrak{p}_\Lambda, \mathfrak{S}_K, \mathfrak{R}_{0,I}, \mathfrak{R}_\Lambda, \mathfrak{N}^a, \ldots$ (§ 3.6) recursive versions of the above.

$\Delta : \mathfrak{S}_K/(\psi(\mathfrak{p}_{0,I}) + \mathfrak{R}_{0,I}, \psi(\mathfrak{p}_\Lambda) + \mathfrak{R}_\Lambda) \to \mathcal{O}_K\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle/(\mathfrak{p}_{0,I}, \mathfrak{p}_\Lambda) \xrightarrow{\to} \mathcal{O}_L$ (3.6.1) and (3.6.3).

$\tilde{\mathfrak{c}}_{0,I}, \tilde{\mathfrak{c}}_\Lambda, \tilde{\mathfrak{u}}_{0,I}, \tilde{\mathfrak{u}}_\Lambda, \tilde{\mathfrak{v}}, \tilde{\mathfrak{p}}_{0,I}, \tilde{\mathfrak{p}}_\Lambda, \tilde{\mathfrak{q}}, \mathfrak{S}_{\widetilde{K}}, \widetilde{\mathfrak{R}}_{0,I}, \widetilde{\mathfrak{R}}_\Lambda, \widetilde{\mathfrak{R}}_{\tilde{\mathfrak{q}}}, \ldots$ (proof of Theorem 4.2.9) recursive versions for $\widetilde{K}$.

2. Background review

2.1 Differential modules

We recall some recent results from the theory of p-adic differential modules. This subject was first studied by Christol, Dwork, Mebkhout and Robba [CD94, CM00, CR94]. Recently, Kedlaya and the author improved some of the techniques in [Ked10, KX10]. We record some useful results from these sources.

Convention 2.1.1. Throughout this paper, $p > 0$ will be a prime number. By a *p-adic field* we mean a field K of characteristic zero, complete with respect to a non-archimedean norm for which $|p| = 1/p$. In particular, the residue field of K has characteristic p.

Convention 2.1.2. For an index set J, we write e_J or (e_J) for a tuple $(e_j)_{j\in J}$. For another tuple b_J, we write $b_J^{e_J} = \prod_{j\in J} b_j^{e_j}$ if only finitely many of the e_j are non-zero. We also use $\sum_{e_J=0}^n$ to mean the sum over $e_j \in \{0, 1, \ldots, n\}$ for each $j \in J$, allowing only finitely many of them to be non-zero. To simplify the notation, we may suppress the range of the summation when it is clear. For a set A, we write $e_J \subset A$ or $(e_J) \subset A$ to mean that $e_j \in A$ for any $j \in J$.

Notation 2.1.3. From now on, let K be a p-adic field and fix an element $\pi_K \in K^\times$ of norm $\theta < 1$. When K has discrete valuation, we take π_K to be a uniformizer.

Notation 2.1.4. For an interval $I \subset [0, +\infty]$, we denote the n-dimensional polyannulus with radii in I by $A^n_K(I)$. (We do not impose any rationality condition on the endpoints of I, so this space should be viewed as an analytic space in the sense of Berkovich [Ber90].) If I is written explicitly in terms of its endpoints (e.g. as $[\alpha, \beta]$), we suppress the parentheses around I (and write, e.g., $A^n_K[\alpha, \beta]$).

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Notation 2.1.5. For a complete topological ring R, we use $R\langle u_1, \dots, u_m\rangle$ to denote the completion of the polynomial ring $R[u_1, \dots, u_m]$ with respect to the topology induced from R. When R is a complete $\mathcal{O}_K$-algebra, we let $R\langle \pi_K^{-a_1}\delta_1, \dots, \pi_K^{-a_m}\delta_m\rangle$ denote the formal substitution of $R\langle u_1, \dots, u_m\rangle$ via $u_j = \pi_K^{-a_j}\delta_j$ for $j = 1, \dots, m$, where $a_1, \dots, a_m \in \mathbb{R}$. In particular, $K\langle \pi_K^{-a_1}\delta_1, \dots, \pi_K^{-a_m}\delta_m\rangle$ is the ring of analytic functions on $A_K^1[0, \theta^{a_1}] \times \cdots \times A_K^1[0, \theta^{a_m}]$.

We use $K[\![T]\!]_0$ to denote the bounded power series ring consisting of formal power series $\sum_{i\in\mathbb{Z}_{\geqslant 0}} a_i T^i$ for which $a_i \in K$ and $|a_i|$ are bounded.

Notation 2.1.6. In this subsection, let $J = \{1, \dots, m\}$ and $J^+ = J \cup \{0\}$.

Definition 2.1.7. For $s_{J^+} \subset \mathbb{R}$, the $\theta^{s_{J^+}}$-*Gauss norm* on $K[\delta_{J^+}]$ is the norm given by

$$\left|\sum_{e_{J^+}} a_{e_{J^+}} \delta_{J^+}^{e_{J^+}}\right|_{s_{J^+}} = \max\{|a_{e_{J^+}}| \cdot \theta^{e_0 s_0 + \cdots + e_m s_m}\}.$$

It extends uniquely to $K(\delta_{J^+})$; we denote the completion by $F_{s_{J^+}}$. This Gauss norm also extends continuously to $K\langle \pi_K^{-a_0}\delta_0, \dots, \pi_K^{-a_m}\delta_m\rangle$ if $s_j \in [a_j, +\infty)$ for all $j \in J^+$. Hence, $K\langle \pi_K^{-a_0}\delta_0, \dots, \pi_K^{-a_m}\delta_m\rangle$ embeds into $F_{s_{J^+}}$.

Convention 2.1.8. Throughout this paper, all (relative) differentials and derivations are continuous and all connections are integrable. For simplicity, we may suppress the continuity and integrability.

Definition 2.1.9. Let F be a differential field of order one and characteristic zero, i.e. a field of characteristic zero equipped with a derivation ∂. Assume that F is complete for a non-archimedean norm $|\cdot|$. Let V be a differential module with differential operator ∂. The *spectral norm of* ∂ *on* V is defined to be

$$|\partial|_{\mathrm{sp},V} = \lim_{n\to+\infty} |\partial^n|_V^{1/n}.$$

One can show that $|\partial|_{\mathrm{sp},V} \geqslant |\partial|_{\mathrm{sp},F}$ (see [Ked10, Lemma 6.2.4]).

Define the *intrinsic* ∂*-radius* of V to be

$$\mathrm{IR}_\partial(V) = |\partial|_{\mathrm{sp},F}/|\partial|_{\mathrm{sp},V} \in (0,1].$$

Example 2.1.10. For $s_{J^+} \subset \mathbb{R}$, the spectral norms of ∂_{J^+} on $F_{s_{J^+}}$ are as follows:

$$|\partial_j|_{F_{s_{J^+}},\mathrm{sp}} = p^{-1/(p-1)}\theta^{-s_j} \quad \text{for } j \in J^+.$$

Remark 2.1.11. If F'/F is a complete extension and ∂ extends to F', then for any differential module V on F, $V \otimes F'$ is a differential module on F'. Moreover, if $|\partial|_{\mathrm{sp},F} = |\partial|_{\mathrm{sp},F'}$, we have $\mathrm{IR}_\partial(V) = \mathrm{IR}_\partial(V \otimes F')$.

Notation 2.1.12. Let $a_{J^+} \subset \mathbb{R}$ be a tuple and let $X = A_K^1[0, \theta^{a_0}] \times \cdots \times A_K^1[0, \theta^{a_m}]$ be the closed polydisc with radii $\theta^{a_{J^+}}$ and with δ_{J^+} as coordinates.

Notation 2.1.13. A *differential module* over X (relative to K) is a finite locally free coherent sheaf $\mathcal{E}$ on X together with an integrable connection

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Big(\bigoplus_{j\in J^+} \mathcal{O}_X \cdot d\delta_j\Big).$$

Let $\partial_{J^+} = \partial/\partial\delta_{J^+}$ be the dual basis of $d\delta_{J^+}$; these elements act commutatively on $\mathcal{E}$. A section $\mathbf{v}$ of $\mathcal{E}$ over X is said to be *horizontal* if $\partial_j(\mathbf{v}) = 0$ for all $j \in J^+$. Let $H^0_\nabla(X, \mathcal{E})$ denote the set of horizontal sections on $\mathcal{E}$ over X. A differential module is said to be *trivial* if there exists a set of horizontal sections which forms a basis of $\mathcal{E}$ as a free coherent sheaf.

Let $s_j \in [a_j, +\infty)$ for $j \in J^+$. For $j \in J^+$, let $\mathrm{IR}_j(\mathcal{E}; s_{J^+})$ denote the intrinsic ∂_j-radius $\mathrm{IR}_{\partial_j}(\mathcal{E} \otimes_{\mathcal{O}_X} F_{s_{J^+}})$. Let $\mathrm{IR}(\mathcal{E}; s_{J^+}) = \min_{j \in J^+}\{\mathrm{IR}_j(\mathcal{E}; s_{J^+})\}$ be the *intrinsic radius* of $\mathcal{E}$. If $s_{j'} = s$ for all $j' \in J$, we simply write $\mathrm{IR}_j(\mathcal{E}; s_0, \underline{s})$ and $\mathrm{IR}(\mathcal{E}; s_0, \underline{s})$ for the intrinsic ∂_j-radius and intrinsic radius, respectively. Moreover, if $s_0 = s$, we may further simplify the notation to $\mathrm{IR}_j(\mathcal{E}; \underline{s})$ and $\mathrm{IR}(\mathcal{E}; \underline{s})$.

Lemma 2.1.14. *Fix $j \in J^+$. There exists a unique continuous K-homomorphism $f^*_{\mathrm{gen},j} : F_{a_{J^+}} \to F_{a_{J^+}}[\![\pi_K^{-a_j} T_j]\!]_0$ such that $f^*_{\mathrm{gen},j}(\delta_{J^+\setminus\{j\}}) = \delta_{J^+\setminus\{j\}}$ and $f^*_{\mathrm{gen},j}(\delta_j) = \delta_j + T_j$.*

Proof. See [KX10, Lemma 1.2.12]. □

Lemma 2.1.15. *Write $F = F_{a_{J^+}}$ for short. The pullback $f^*_{\mathrm{gen},j}(\mathcal{E} \otimes_{\mathcal{O}_X} F)$ becomes a differential module over $A^1_F[0, \theta^{a_j})$ relative to F. Then, for any $r \in [0, 1]$, $\mathrm{IR}_j(\mathcal{E}; a_{J^+}) \geqslant r$ if and only if $f^*_{\mathrm{gen},j}(\mathcal{E} \otimes_{\mathcal{O}_X} F)$ is trivial over $A^1_F[0, r\theta^{a_j})$.*

Proof. This is essentially because the Taylor series $\sum_{n=0}^{\infty} \partial^n_{T_j}(\mathbf{v}) \cdot T_j^n/(n!) = \sum_{n=0}^{\infty} \partial_j^n(\mathbf{v}) \cdot T_j^n/(n!)$ converges when $|T_j| < r\theta^{a_j}$ for any section $\mathbf{v}$ if and only if $\mathrm{IR}_j(\mathcal{E}; a_{J^+}) \geqslant r$. For more details, see [KX10, Proposition 1.2.14]. □

We reproduce some basic properties of intrinsic radii, starting with the following off-centered tame base change, which is a fun exercise in [Ked10, ch. 9, Exercise 8]. For the sake of readers who may not be familiar with differential modules, we give a complete proof here.

Construction 2.1.16. Fix $n \in \mathbb{N}$ prime to p. Assume for the moment that $m = 0$ (and $a = a_0$), i.e. we consider the one-dimensional case $X = A^1_K[0, \theta^a]$. Fix $x_0 \in K$ such that $|x_0| = \theta^b > \theta^a$ ($b < a$). In particular, the point $\delta_0 = -x_0$ is not in the disc X. Write $K_n = K(x_0^{1/n})$, where we fix an nth root $x_0^{1/n}$ of x_0.

Consider the K-homomorphism $f^*_n : K\langle \pi_K^{-a}\delta_0\rangle \to K_n\langle \pi_K^{-a+b(n-1)/n}\eta_0\rangle$ sending δ_0 to

$$(x_0^{1/n} + \eta_0)^n - x_0 = x_0^{(n-1)/n}\eta_0\left(\sum_{i=0}^{n-1}\binom{n}{i+1}\left(\frac{\eta_0}{x_0^{1/n}}\right)^i\right),$$

where the term in parentheses on the right has norm 1 and is invertible because $|x_0^{1/n}| > |\eta_0|$. Hence f^*_n extends continuously to a homomorphism $F_a \to F'_{a-b(n-1)/n}$, where $F'_{a-b(n-1)/n}$ is the completion of $K_n(\eta_0)$ with respect to the $\theta^{a-b(n-1)/n}$-Gauss norm.

Also, f^*_n gives a morphism of rigid K-spaces $f_n : Z = A^1_{K_n}[0, \theta^{a-b(n-1)/n}] \to X = A^1_K[0, \theta^a]$. It is finite and étale because the branching locus is at $\delta_0 = -x_0$, outside the disc X. Thus, for a differential module $\mathcal{E}$ on X, its pullback $f^*_n\mathcal{E}$ is a differential module over Z via

$$f^*_n\mathcal{E} \xrightarrow{f^*_n\nabla} f^*_n(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X\, d\delta_0) \longrightarrow f^*_n\mathcal{E} \otimes_{\mathcal{O}_Z} \mathcal{O}_Z\, d\eta_0,$$

where the last homomorphism is given by $d\delta_0 \mapsto n(x_0^{1/n} + \eta_0)^{n-1} d\eta_0$.

Proposition 2.1.17. *With the above notation, we have*

$$\mathrm{IR}_{\partial_{\eta_0}}(f^*_n\mathcal{E}; a - b(n-1)/n) = \mathrm{IR}_{\partial_0}(\mathcal{E}; a).$$

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Proof. The proof is essentially the same as that of [Ked05, Lemma 5.11] or [Ked10, Proposition 9.7.6]. Lemma 2.1.14 gives the commutative diagram

$$\begin{array}{ccc} F_a & \xrightarrow{f^*_{\mathrm{gen},0}} & F_a[\![\pi_K^{-a}T_0]\!]_0 \\ \downarrow{\scriptstyle f_n^*} & & \downarrow{\scriptstyle \tilde{f}_n^*} \\ F'_{a-b(n-1)/n} & \xrightarrow{f'^*_{\mathrm{gen},0}} & F'_{a-b(n-1)/n}[\![\pi_K^{-a+b(n-1)/n}T'_0]\!]_0 \end{array}$$

where $\tilde{f}_n^*$ extends f_n^* by sending T_0 to $(x_0^{1/n}+\eta_0+T_0')^n-(x_0^{1/n}+\eta_0)^n$.

We claim that for $r\in[0,1]$, $\tilde{f}_n$ induces an isomorphism

$$F'_{a-b(n-1)/n}\times_{f_n^*,F_a}(A^1_{F_a}[0,r\theta^a))\cong A^1_{F'_{a-b(n-1)/n}}[0,r\theta^{a-b(n-1)/n}).$$

Indeed, if $|T_0'|<r\theta^{a-b(n-1)/n}<\theta^{b/n}$, then

$$\begin{aligned} |T_0| &= |(x_0^{1/n}+\eta_0+T_0')^n-(x_0^{1/n}+\eta_0)^n| \\ &= |nT_0'(x_0^{1/n}+\eta_0)^{n-1}|<r\theta^{a-b(n-1)/n}\cdot(\theta^{b/n})^{n-1}=r\theta^a. \end{aligned}$$

Conversely, if $|T_0|<r\theta^a$, we define the inverse map by the binomial series

$$T_0'=(x_0^{1/n}+\eta_0)\cdot\left[-1+\left(1+\frac{T_0}{(x_0^{1/n}+\eta_0)^n}\right)^{1/n}\right]=\sum_{i=1}^{\infty}\binom{1/n}{i}\frac{T_0^i}{(x_0^{1/n}+\eta_0)^{ni-1}}.$$

The series converges to an element with norm less than $r\theta^{a-b(n-1)/n}$.

Therefore, Lemma 2.1.15 implies that for $r\in[0,1]$,

$$\begin{aligned} &\mathrm{IR}_{\partial_0}(\mathcal{E};a)\geqslant r \\ &\iff f^*_{\mathrm{gen},0}(\mathcal{E}\otimes_{\mathcal{O}_X}F_a)\text{ is trivial over }A^1_{F_a}[0,r\theta^a) \\ &\iff \tilde{f}_n^*f^*_{\mathrm{gen},0}(\mathcal{E}\otimes_{\mathcal{O}_X}F_a)=f'^*_{\mathrm{gen},0}(f_n^*\mathcal{E}\otimes_{\mathcal{O}_Z}F'_{a-b(n-1)/n}) \\ &\qquad\text{is trivial over }A^1_{F'_{a-b(n-1)/n}}[0,r\theta^{a-b(n-1)/n}) \\ &\iff \mathrm{IR}_{\partial_{\eta_0}}(f_n^*\mathcal{E};a-b(n-1)/n)\geqslant r. \end{aligned}$$

The proposition follows. □

Similarly, we can study a type of off-centered Frobenius.

Construction 2.1.18. Let $b>0$ and $0<a<\min\{-\log_\theta p+b,pb\}$, and let $\beta\in K$ be an element of norm 1. Let L be the completion of $K(x)$ with respect to the θ^a-Gauss norm.

Let $f:Z=A^1_L[0,\theta^b]\to A^1_K[0,\theta^a]$ be the morphism given by $f^*:\delta_0\mapsto(\beta+\eta_0)^p-\beta^p+x$. By our choices of a and b, the leading term of $f^*(\delta_0)$ is x, which is transcendental over K. Hence f^* extends continuously to a homomorphism $F_a\to F_b'$, where F_b' is the completion of $L(\eta_0)$ with respect to the θ^b-Gauss norm. Moreover, $f^*\Omega^1_X\cong\Omega^1_Z$ because the branching locus is at $\eta_0=-\beta$, outside the disc. Thus $f^*\mathcal{E}$ becomes a differential module over $Z=A^1_L[0,\theta^b]$ via

$$f^*\mathcal{E}\xrightarrow{f^*\nabla}f^*(\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{O}_X\,d\delta_0)\longrightarrow f^*\mathcal{E}\otimes_{\mathcal{O}_Z}\mathcal{O}_Z\,d\eta_0,$$

where the second homomorphism is given by $d\delta_0\mapsto p(\beta+\eta_0)^{p-1}\,d\eta_0$.

Proposition 2.1.19. *Keeping the notation as above, we have*

$$\mathrm{IR}_{\partial_0}(f^*\mathcal{E};b)\geqslant\mathrm{IR}_{\partial_{\eta_0}}(\mathcal{E};a).$$

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Proof. As in Proposition 2.1.17, we start with the following commutative diagram from Lemma 2.1.14.

$$\begin{array}{ccc} F_a & \xrightarrow{f^*_{\mathrm{gen},0}} & F_a[\![\pi_K^{-a} T_0]\!]_0 \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle \tilde{f}^*} \\ F'_b & \xrightarrow{f'^*_{\mathrm{gen},0}} & F'_b[\![\pi_K^{-b} T'_0]\!]_0 \end{array}$$

where $\tilde{f}^*$ extends f^* by sending T_0 to $(\beta+\eta_0+T'_0)^p-(\beta+\eta_0)^p$.

For $r\in[0,1]$, by Lemma 2.1.20 below we have that $|T'_0|<r\theta^a$ implies $|T_0|<\max\{r^p\theta^{pa},p^{-1}r\theta^a\}<r\theta^b$.

Therefore, Lemma 2.1.15 implies that

$$\begin{aligned} &\mathrm{IR}_{\partial_0}(\mathcal{E};a)\geqslant r\\ &\quad\Longleftrightarrow f^*_{\mathrm{gen},0}(\mathcal{E}\otimes_{\mathcal{O}_X}F_a)\text{ is trivial over }A^1_{F_a}[0,r\theta^a)\\ &\quad\Longrightarrow \tilde{f}^*f^*_{\mathrm{gen},0}(\mathcal{E}\otimes_{\mathcal{O}_X}F_a)=f'^*_{\mathrm{gen},0}(f^*\mathcal{E}\otimes_{\mathcal{O}_Z}F'_b)\\ &\qquad\quad\text{is trivial over }A^1_{F'_b}[0,r\theta^b)\\ &\quad\Longleftrightarrow \mathrm{IR}_{\partial_{\eta_0}}(f^*\mathcal{E};b)\geqslant r. \end{aligned}$$

The proposition follows. □

Lemma 2.1.20 [Ked10, Lemma 10.2.2(a)]. *Let K be a non-archimedean field and let $b,T\in K$. For $r\in(0,1)$, if $|b-T|<r|b|$, then*

$$|b^p-T^p|\leqslant\max\{r^p|b|^p,p^{-1}r|b|^p\}.$$

Remark 2.1.21. A stronger form of Proposition 2.1.19 above for (straight) Frobenius can be found in [Ked10, Lemma 10.3.2] or [KX10, Lemma 1.4.11].

Now, we study the variation of intrinsic radii on polydiscs.

Definition 2.1.22. An *affine functional* on $\mathbb{R}^{m+1}$ is a function $\lambda:\mathbb{R}^{m+1}\to\mathbb{R}$ of the form $\lambda(x_0,\dots,x_m)=a_0x_0+\cdots+a_mx_m+b$ for some $a_0,\dots,a_m,b\in\mathbb{R}$. If $a_0,\dots,a_m\in\mathbb{Z}$, we say that λ is *transintegral* (short for 'integral after translation').

A subset $C\subseteq\mathbb{R}^{m+1}$ is *polyhedral* if there exist finitely many affine functionals $\lambda_1,\dots,\lambda_r$ such that

$$C=\{x\in\mathbb{R}^{m+1}:\lambda_i(x)\geqslant 0\text{ for }i=1,\dots,r\}.$$

If the λ_i can all be taken to be transintegral, we say that C is *transrational polyhedral*.

Proposition 2.1.23. *Let $a_{J^+}\subset\mathbb{R}$ be a tuple and let $X=A^1_K[0,\theta^{a_0}]\times\cdots\times A^1_K[0,\theta^{a_m}]$ be the polydisc with radii $\theta^{a_{J^+}}$ and coordinates δ_{J^+}. Let $\mathcal{E}$ be a differential module over X. Then the following properties hold.*

- (a) *(Continuity.) The function $\log_\theta\mathrm{IR}(\mathcal{E};s_{J^+})$ is continuous for $s_j\in[a_j,+\infty)$ and $j\in J^+$.*
- (b) *(Monotonicity.) Let $s_j\geqslant s'_j\geqslant a_j$ for all $j\in J^+$; then $\mathrm{IR}(\mathcal{E};s_{J^+})\geqslant\mathrm{IR}(\mathcal{E};s'_{J^+})$.*
- (c) *(Zero loci.) The subset $Z(\mathcal{E})=\{s_{J^+}\in[a_0,+\infty)\times\cdots\times[a_m,+\infty)\mid\mathrm{IR}(\mathcal{E};s_{J^+})=1\}$ is transrational polyhedral; moreover, it contains $[a'_0,+\infty)\times\cdots\times[a'_m,+\infty)$ for $a'_0,\dots,a'_m$ sufficiently large.*

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Proof. Statements (a) and (c) follow from [KX10, Theorem 3.3.9]; $Z(\mathcal{E})$ contains $[a'_0, +\infty) \times \cdots \times [a'_m, +\infty)$ for $a'_0, \ldots, a'_m$ sufficiently large because the intrinsic radii are always non-zero. For (b), by drawing zig-zag lines parallel to axes linking the two points s_{J^+} and s'_{J^+}, it suffices to consider the case where $s_j = s'_j$ for $j \in J^+ \backslash \{j_0\}$ and $s_{j_0} \geqslant s'_{j_0}$. In this case, we may perform a base change to the completion of $K(\delta_{J^+\backslash\{j_0\}})$ with respect to the $s_{J^+\backslash\{j_0\}}$-Gauss norm. The result follows from [KX10, Theorem 2.4.4(c)]. □

2.2 Ramification filtrations

In this subsection, we sketch Abbes and Saito's definition of ramification filtrations on the Galois group G_K of a complete discrete valuation field K of mixed characteristic $(0, p)$. For more details, the reader can consult [AS02, AS03].

In this subsection, we temporarily drop Notation 2.1.6.

Notation 2.2.1. For any complete discrete valuation field K of mixed characteristic $(0, p)$, we denote its ring of integers and residue field by $\mathcal{O}_K$ and k, respectively. Let π_K denote a uniformizer and $\mathfrak{m}_K$ the maximal ideal of $\mathcal{O}_K$ (generated by π_K). We normalize the valuation $v_K(\cdot)$ on K so that $v_K(\pi_K) = 1$. The *absolute ramification degree* is defined to be $\beta_K = v_K(p)$. We say that K is *absolutely unramified* if $\beta_K = 1$. For an element $a \in \mathcal{O}_K$, we write its reduction in k as $\bar{a}$; a is called a *lift* of $\bar{a}$.

We choose and fix an algebraic closure K^{alg} of K; all finite extensions of K are taken inside K^{alg}. Let G_K denote the absolute Galois group $\mathrm{Gal}(K^{\mathrm{alg}}/K)$. If L is a finite Galois extension of K, we denote the Galois group by $G_{L/K}$. We use $\mathbf{N}_{L/K}(x)$ to denote the norm of an element $x \in L$. If L is a (not necessarily algebraic) complete extension of K and is itself a discrete valuation field, we use $e_{L/K}$ to denote its *naïve ramification degree*, i.e. the index of the value group of K in that of L. We say that L/K is tamely ramified if $p \nmid e_{L/K}$ and the residue field extension l/k is *algebraic* and separable. If, moreover, $e_{L/K} = 1$, we say that L/K is unramified.

Notation 2.2.2. From now on, K will denote a complete discrete valuation field of mixed characteristic $(0, p)$, and L will be a finite Galois extension of K of naïve ramification degree $e = e_{L/K}$. Set $\theta = |\pi_K|$; this agrees with the convention in the previous subsection.

Definition 2.2.3. Take $Z = (z_j)_{j \in J} \subset \mathcal{O}_L$ to be a finite set of elements generating $\mathcal{O}_L$ over $\mathcal{O}_K$, i.e. $\mathcal{O}_K[u_J]/\mathcal{I} \xrightarrow{\sim} \mathcal{O}_L$ mapping u_j to z_j for all $j \in J = \{1, \ldots, m\}$. Let $(f_i)_{i=1,\ldots,n}$ be a finite set of generators of $\mathcal{I}$. For $a \in \mathbb{Q}_{>0}$, define the *Abbes–Saito space* to be

$$\mathrm{AS}^a_{L/K,Z} = \{(u_1, \ldots, u_m) \in A^m_K[0,1] : |f_i(u_J)| \leqslant \theta^a \text{ for } 1 \leqslant i \leqslant n\}.$$

We denote the set of *geometric* connected components of $\mathrm{AS}^a_{L/K,Z}$ by $\pi_0^{\mathrm{geom}}(\mathrm{AS}^a_{L/K,Z})$. The *highest ramification break* $b(L/K)$ of the extension L/K is defined to be the minimal $b \in \mathbb{R}_{\geqslant 0}$ such that for any rational number $a > b$, $\#\pi_0^{\mathrm{geom}}(\mathrm{AS}^a_{L/K,Z}) = [L : K]$.

Definition 2.2.4. Keep the notation as above. Take a subset $P \subset Z$ and assume that P, and hence Z, contains π_L. Let $e_j = v_L(z_j)$, with $z_j \in P$. Take a lift $g_j \in \mathcal{O}_K[u_J]$ of $z_j^e/\pi_K^{e_j}$ for each $z_j \in P$; take a lift $h_{i,j} \in \mathcal{O}_K[u_J]$ of $z_j^{e_i}/z_i^{e_j}$ for each pair $(z_i, z_j) \in P \times P$. For $a \in \mathbb{Q}_{>0}$, define the *logarithmic Abbes–Saito space* to be

$$\mathrm{AS}^a_{L/K,\log,Z,P} = \left\{ (u_J) \in A^m_K[0,1] \;\middle|\; \begin{array}{ll} |f_i(u_J)| \leqslant \theta^a & \text{for } 1 \leqslant i \leqslant n \\ |u_j^e - \pi_K^{e_j} g_j| \leqslant \theta^{a+e_j} & \text{for all } z_j \in P \\ |u_j^{e_i} - u_i^{e_j} h_{i,j}| \leqslant \theta^{a+e_ie_j/e} & \text{for all } (z_i, z_j) \in P \times P \end{array} \right\}.$$

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Similarly, the *highest logarithmic ramification break* $b_{\log}(L/K)$ of the extension L/K is defined to be the minimal $b \in \mathbb{R}_{\geqslant 0}$ such that for any rational number $a > b$ we have $\#\pi_0^{\mathrm{geom}}(\mathrm{AS}^a_{L/K,\log,Z,P}) = [L:K]$.

We reproduce several statements from [AS02, AS03].

Proposition 2.2.5. *The Abbes–Saito spaces have the following properties.*

(1) *For $a \in \mathbb{Q}_{>0}$, the Abbes–Saito spaces $\mathrm{AS}^a_{L/K,Z}$ and $\mathrm{AS}^a_{L/K,\log,Z,P}$ do not depend on the choices of generators $(f_i)_{i=1,\dots,n}$ of $\mathcal{I}$ and lifts g_j and $h_{i,j}$ for $i, j \in P$ (see [AS02, § 3]).*

(1′) *If, in the definitions of both Abbes–Saito spaces, we choose polynomials $(f_i)_{i=1,\dots,n}$ as generators of $\mathrm{Ker}\,(\mathcal{O}_K\langle u_J\rangle \to \mathcal{O}_L)$ instead of $\mathrm{Ker}\,(\mathcal{O}_K[u_J] \to \mathcal{O}_L)$, the spaces do not change.*

(2) *If we use another pair of generating sets Z and P satisfying the same properties, then we have a canonical bijection on the sets of the geometric connected components $\pi_0^{\mathrm{geom}}(\mathrm{AS}^a_{L/K,Z})$ and $\pi_0^{\mathrm{geom}}(\mathrm{AS}^a_{L/K,\log,Z,P})$ for different generating sets, where $a \in \mathbb{Q}_{>0}$. In particular, both highest ramification breaks are well-defined [AS02, § 3].*

(3) *The highest ramification break (respectively, the highest logarithmic ramification break) gives rise to a filtration on the Galois group G_K consisting of normal subgroups $\mathrm{Fil}^a G_K$ (respectively, $\mathrm{Fil}^a_{\log} G_K$) for $a \geqslant 0$ such that $b(L/K) = \inf\{a \mid \mathrm{Fil}^a G_K \subseteq G_L\}$ (respectively, $b_{\log}(L/K) = \inf\{a \mid \mathrm{Fil}^a_{\log} G_K \subseteq G_L\}$); see [AS02, Theorems 3.3 and 3.11]. Moreover, for L/K being a finite Galois extension, both highest ramification breaks are rational numbers [AS02, Theorems 3.8 and 3.16].*

(4) *Let K'/K be a (not necessarily finite) extension of complete discrete valuation fields. If K'/K is unramified, then $\mathrm{Fil}^a G_{K'} = \mathrm{Fil}^a G_K$ for $a > 0$; see [AS02, Proposition 3.7]. If K'/K is tamely ramified with ramification index $e < \infty$, then $\mathrm{Fil}^{ea}_{\log} G_{K'} = \mathrm{Fil}^a_{\log} G_K$ for $a > 0$; see [AS02, Proposition 3.15].*

(4′) *More generally, let L/K be a finite algebraic extension and let K'/K be a complete extension of discrete valuation fields with the same valued group and linearly independent of L. Write $L' = K'K$. If $\mathcal{O}_{L'} = \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$, then $b(L/K) = b(L'/K')$; see [AM04, Lemme 2.1.5].*

(5) *For $a \geqslant 0$, define $\mathrm{Fil}^{a+} G_K = \overline{\bigcup_{b>a} \mathrm{Fil}^b G_K}$ and $\mathrm{Fil}^{a+}_{\log} G_K = \overline{\bigcup_{b>a} \mathrm{Fil}^b_{\log} G_K}$. Then, the subquotients $\mathrm{Fil}^a G_K / \mathrm{Fil}^{a+} G_K$ are abelian p-groups if $a \in \mathbb{Q}_{>1}$ and are 0 if $a \notin \mathbb{Q}$, except when K is absolutely unramified (see [AS02, Theorem 3.8] and [AS03, Theorem 1]). The subquotients $\mathrm{Fil}^a_{\log} G_K / \mathrm{Fil}^{a+}_{\log} G_K$ are abelian p-groups if $a \in \mathbb{Q}_{>0}$ and are 0 if $a \notin \mathbb{Q}$ (see [AS02, Theorem 3.16] and [AS03, Theorem 1]).*

(6) *For $a > 0$, $\mathrm{Fil}^{a+1} G_K \subseteq \mathrm{Fil}^a_{\log} G_K \subseteq \mathrm{Fil}^a G_K$ (see [AS02, Theorem 3.15(1)]).*

(7) *The inertia subgroup is $\mathrm{Fil}^a G_K$ for $a \in (0, 1]$ and the wild inertia subgroup is $\mathrm{Fil}^{1+} G_K = \mathrm{Fil}^{0+}_{\log} G_K$ (see [AS02, Theorems 3.7 and 3.15]).*

(8) *When the residue field k is perfect, the arithmetic ramification filtrations agree with the classical upper numbered filtration [Ser79] in the following way: $\mathrm{Fil}^a G_K = \mathrm{Fil}^{a-1}_{\log} G_K = G_K^{a-1}$ for $a \geqslant 1$, where G_K^a is the classical upper numbered filtration on G_K (see [AS02, § 6.1]).*

Proof. Only the proof of (1′) has not already appeared in the literature, but the proof of (1) can be used verbatim to prove this assertion. For a brief summary of the proofs of the other statements, one may consult [Xia10, Proposition 4.1.6]; although the statements there are stated for the equal characteristic case, the proofs work just fine. □

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Remark 2.2.6. To avoid confusion, we point out that in the proof of our main theorem, we do not need (5) and the second statement of (3) on the rationality of the breaks from the proposition above. Therefore, we will prove these properties along the way of proving the main theorem.

Remark 2.2.7. Recently, Saito [Sai] gave a proof of the fact that $\mathrm{Fil}^a_{\log} G_K/\mathrm{Fil}^{a+}_{\log} G_K$ are abelian groups killed by p for $a \in \mathbb{Q}_{>0}$. We will prove this independently in our main theorem (which in fact appeared before his preprint).

DEFINITION 2.2.8. For $b \geqslant 0$, we write $\mathrm{Fil}^b G_{L/K} = (G_L \mathrm{Fil}^b G_K)/G_L$ and $\mathrm{Fil}^b_{\log} G_{L/K} = (G_L \mathrm{Fil}^b_{\log} G_K)/G_L$. We call b a *non-logarithmic (respectively, logarithmic) ramification break* of L/K if $\mathrm{Fil}^b G_{L/K}/\mathrm{Fil}^{b+} G_{L/K}$ (respectively, $\mathrm{Fil}^b_{\log} G_{L/K}/\mathrm{Fil}^{b+}_{\log} G_{L/K}$) is non-trivial.

DEFINITION 2.2.9. By a representation of G_K we mean a continuous homomorphism $\rho : G_K \to \mathrm{GL}(V_\rho)$ where V_ρ is a finite-dimensional vector space over a field F of characteristic zero. We allow F to have a non-archimedean topology; hence the image of G_K may not be finite. We say that ρ has *finite monodromy* if the image of the inertia subgroup of G_K is finite.

DEFINITION 2.2.10. For a representation $\rho : G_K \to \mathrm{GL}(V_\rho)$ of G_K with finite monodromy, define the *Artin and Swan conductors* of ρ as

$$\mathrm{Art}(\rho) \stackrel{\mathrm{Def}}{=} \sum_{a \in \mathbb{Q}_{\geqslant 0}} a \cdot \dim(V_\rho^{\mathrm{Fil}^{a+} G_K}/V_\rho^{\mathrm{Fil}^a G_K}), \tag{2.2.11}$$

$$\mathrm{Swan}(\rho) \stackrel{\mathrm{Def}}{=} \sum_{a \in \mathbb{Q}_{\geqslant 0}} a \cdot \dim(V_\rho^{\mathrm{Fil}^{a+}_{\log} G_K}/V_\rho^{\mathrm{Fil}^a_{\log} G_K}). \tag{2.2.12}$$

In fact, they are finite sums.

CONJECTURE 2.2.13 (Hasse–Arf theorem). *Let K be a complete discrete valuation field of mixed characteristic $(0, p)$, and let $\rho : G_K \to \mathrm{GL}(V_\rho)$ be a representation with finite monodromy. Then:*

(1) $\mathrm{Art}(\rho)$ *and* $\mathrm{Swan}(\rho)$ *are non-negative integers;*

(2) *the subquotients* $\mathrm{Fil}^a G_K/\mathrm{Fil}^{a+} G_K$ *and* $\mathrm{Fil}^a_{\log} G_K/\mathrm{Fil}^{a+}_{\log} G_K$ *are abelian groups killed by p.*

In Theorems 4.3.5, 4.5.14, and 4.7.3, we will prove this conjecture except in the absolutely unramified and non-logarithmic case, or the $p = 2$ and logarithmic case.

PROPOSITION 2.2.14. *When the residue field k is perfect, Conjecture 2.2.13 is true.*

Proof. By Proposition 2.2.5(8), this result follows from the classical Hasse–Arf theorem [Ser79, § VI.2 Theorem 1]. □

3. Construction of spaces

In this section, we construct a series of rigid analytic spaces and study their relations; in particular, we prove that the Abbes–Saito spaces are the same as thickening spaces, and hence translate the question on ramification breaks to a question on generic radii of differential modules.

3.1 Standard Abbes–Saito spaces

In this subsection, we introduce the standard Abbes–Saito spaces by choosing a distinguished set of generators of $\mathcal{O}_L/\mathcal{O}_K$.

Definition 3.1.1. For a field k of characteristic p, a *p-basis* of k is a set $\bar{b}_J \subset k$ such that $\bar{b}_J^{e_J}$, where $e_j \in \{0, 1, \ldots, p-1\}$ for all $j \in J$ and $e_j = 0$ for all but finitely many j, form a basis of k as a k^p-vector space. For a complete discrete valuation field K of mixed characteristic $(0, p)$, a *p-basis* is a set of lifts $b_J \subset \mathcal{O}_K$ of a p-basis of the residue field k.

Hypothesis 3.1.2. Throughout this section, let K be a discrete valuation field of mixed characteristic $(0, p)$ with *separably closed* and *imperfect* residue field. Assume that K admits a *finite* p-basis. Also, let L/K be a wildly ramified Galois extension of naïve ramification degree $e = e_{L/K}$. In particular, L/K is totally ramified and $b(L/K) > 1$, $b_{\log}(L/K) > 0$.

Remark 3.1.3. In case there is confusion over the terminology here, by wildly ramified extension we mean a finite extension which is not tamely ramified, i.e. it *can* have a tamely ramified part.

This is a mild hypothesis because the conductors behave well under unramified base changes, and the tamely ramified case is well-studied.

Notation 3.1.4. For the rest of the paper, we reinstate Notation 2.1.6, namely, let $J = \{1, \ldots, m\}$ and $J^+ = J \cup \{0\}$. We will reserve j and m only for indexing p-bases and related variables, and $j = 0$ will refer to the uniformizer.

Notation 3.1.5. We define a norm on $\mathcal{O}_K[u_{J^+}]$ as follows: for $h = \sum_{e_{J^+}} \alpha_{e_{J^+}} u_{J^+}^{e_{J^+}}$, where $\alpha_{e_{J^+}} \in \mathcal{O}_K$, set $|h| = \max_{e_{J^+}}\{|\alpha_{e_{J^+}}| \cdot \theta^{e_0/e}\}$. For $a \in (1/e)\mathbb{Z}_{\geqslant 0}$, let N^a be the set of elements with norm less than or equal to θ^a; it is in fact an ideal.

The following construction provides a good set of generators for the extension $\mathcal{O}_L/\mathcal{O}_K$. Essentially, we just need some generators and relations with no redundancy which we can write down and work with.

Construction 3.1.6. Choose p-bases $b_J \subset \mathcal{O}_K$ and $c_J \subset \mathcal{O}_L$ of K and L, respectively. Let $\mathbf{k}_0 = k$ with p-basis $(\bar{b}_j)_{j \in J}$. By possibly rearranging the indexing in b_J, we can filter the extension l/k by subextensions $\mathbf{k}_j = k(\bar{c}_1, \ldots, \bar{c}_j)$ with p-bases $\{\bar{c}_1, \ldots, \bar{c}_j, \bar{b}_{j+1}, \ldots, \bar{b}_m\}$ for $j \in J$. Moreover, if $[\mathbf{k}_j : \mathbf{k}_{j-1}] = p^{r_j}$, then $\bar{c}_j^{p^{r_j}} \in \mathbf{k}_{j-1}$.

Write $\Delta : \mathcal{O}_K\langle u_{J^+}\rangle/\mathcal{I}_{L/K} \xrightarrow{\sim} \mathcal{O}_L$, mapping u_j to c_j for $j \in J$ and u_0 to π_L, where $\mathcal{I}_{L/K}$ is some proper ideal. Let $\overline{\Delta}$ be the composite of Δ with the reduction $\mathcal{O}_L \twoheadrightarrow l$. Hence,

$$\{u_{J^+}^{e_{J^+}} \mid e_j \in \{0, \ldots, p^{r_j} - 1\} \text{ for all } j \in J \text{ and } e_0 \in \{0, \ldots, e-1\}\} \tag{3.1.7}$$

forms a basis of $\mathcal{O}_K\langle u_{J^+}\rangle/\mathcal{I}_{L/K}$ as a free $\mathcal{O}_K$-module. We choose a set of generators p_{J^+} of $\mathcal{I}_{L/K}$ by writing each $u_j^{p^{r_j}}$ (for $j \in J$) or u_0^e (for $j = 0$) in terms of the basis (3.1.7). We say that p_j *corresponds* to c_j. Obviously, p_{J^+} generates $\mathcal{I}_{L/K}$. Moreover,

$$p_j \in u_j^{p^{r_j}} - \tilde{b}_j(u_1, \ldots, u_{j-1}) + N^{1/e} \cdot \mathcal{O}_K[u_{J^+}] \quad \text{for } j \in J,$$
$$p_0 \in u_0^e - d(u_1, \ldots, u_m)\pi_K + \pi_K N^{1/e} \cdot \mathcal{O}_K[u_{J^+}],$$

where $\tilde{b}_j(u_1, \ldots, u_{j-1}) \in \mathcal{O}_K[u_1, \ldots, u_{j-1}]$ with powers on u_i smaller than p^{r_i} for all $i = 1, \ldots, j-1$, and $d(u_1, \ldots, u_m) \in \mathcal{O}_K[u_1, \ldots, u_m]$ is a polynomial such that $d(c_1, \ldots, c_m) \in \mathcal{O}_L^\times$.

Remark 3.1.8. It is not possible to avoid introducing $\tilde{b}_j(u_1, \ldots, u_{j-1})$ and $d(u_1, \ldots, u_m)$. Counterexamples were provided and communicated to the author by Shun Ohkubo; see [Xia10, Remark 3.3.6 and Example 3.3.10]. However, to best convey the idea of the proof, we invite the reader to pretend that these two elements are trivial, which is already quite general.

Definition 3.1.9. The *(standard) Abbes–Saito spaces* $\mathrm{AS}^a_{L/K}$ for $a \in \mathbb{Q}_{>1}$ and $\mathrm{AS}^a_{L/K,\log}$ for $a \in \mathbb{Q}_{>0}$ are defined by taking the generators to be $\{c_J, \pi_L\}$ and the relations to be p_{J^+}; see Proposition 2.2.5(1′). In particular, their rings of functions are

$$\mathcal{O}^a_{\mathrm{AS},L/K} = K\langle u_{J^+}, \pi_K^{-a} V_{J^+}\rangle/(p_0(u_{J^+}) - V_0, \dots, p_m(u_{J^+}) - V_m)$$

and

$$\mathcal{O}^a_{\mathrm{AS},L/K,\log} = K\langle u_{J^+}, \pi_K^{-a-1} V_0, \pi_K^{-a} V_J\rangle/(p_0(u_{J^+}) - V_0, \dots, p_m(u_{J^+}) - V_m).$$

3.2 The ψ-function and thickening spaces

In this subsection, we first define a function (*not* a homomorphism) $\psi : \mathcal{O}_K \to \mathcal{O}_K[\![\delta_0/\pi_K, \delta_J]\!]$, which is an approximation to the deformation of the uniformizer π_K and p-basis as in [Xia10, Theorem 3.2.7]. Then, we introduce the thickening spaces for the extension L/K (see [Xia10, § 3.1] for motivation).

As a reminder, we assume Hypothesis 3.1.2 for this section; we fix a finite p-basis (b_J) and a uniformizer π_K of K.

Construction 3.2.1. Let $r \in \mathbb{N}$ and $h \in \mathcal{O}_K^\times$. An *rth p-basis decomposition* of h involves writing h as

$$h = \sum_{e_J=0}^{p^r-1} b_J^{e_J} \Bigg(\sum_{n=0}^{\infty} \Bigg(\sum_{n'=0}^{\lambda_{r,e_J,n}} \alpha^{p^r}_{r,e_J,n,n'} \Bigg) \pi_K^n \Bigg) \tag{3.2.2}$$

for some $\alpha_{r,e_J,n,n'} \in \mathcal{O}_K^\times \cup \{0\}$ and some $\lambda_{r,e_J,n} \in \mathbb{Z}_{\geqslant 0}$. Such expressions always exist but are not unique. For $r' > r$, we can express each of $\alpha_{r,e_J,n,n'}$ in (3.2.2) using an $(r'-r)$th p-basis decomposition and then rearrange the formal sum to obtain an r'th p-basis decomposition. For $h \in \mathcal{O}_K^\times$, we say that an r'th p-basis decomposition is *compatible* with the rth p-basis decomposition in (3.2.2) if it can be obtained in the above manner.

We define the function $\psi : \mathcal{O}_K \to \mathcal{O}_K[\![\delta_{J^+}]\!]$ as follows: for each $h \in \mathcal{O}_K^\times \backslash \{1\}$, we fix a compatible system of rth p-basis decompositions for all $r \in \mathbb{N}$, and define

$$\psi(h) = \lim_{r \to +\infty} \sum_{e_J=0}^{p^r-1} (b_J + \delta_J)^{e_J} \Bigg(\sum_{n=0}^{\infty} \Bigg(\sum_{n'=0}^{\lambda_{r,e_J,n}} \alpha^{p^r}_{r,e_J,n,n'} \Bigg) (\pi_K + \delta_0)^n \Bigg); \tag{3.2.3}$$

this expression converges by the compatibility of the p-basis decompositions. Define $\psi(1) = 1$, which corresponds to the naïve compatible system of p-basis decompositions of the element 1. For $h \in \mathcal{O}_K \backslash \{0\}$, write $h = \pi_K^s h_0$ for $s \in \mathbb{N}$ and $h_0 \in \mathcal{O}_K^\times$. Define $\psi(h) = (\pi_K + \delta_0)^s \psi'(h_0)$, where $\psi'(h_0)$ is the limit as in (3.2.3) with respect to a compatible system of p-basis decompositions of h_0 (which does not have to be the same as the one that defines $\psi(h_0)$). Finally, we define $\psi(0) = 0$.

Most of the time, it is more convenient to view ψ as a function on $\mathcal{O}_K$ which takes values in the larger ring $\mathcal{O}_K[\![\delta_0/\pi_K, \delta_J]\!]$.

We can extend ψ naturally to polynomial rings or formal power series rings with coefficients in $\mathcal{O}_K$ by applying ψ termwise.

Notation 3.2.4. For the rest of the paper, let $\mathcal{R}_K = \mathcal{O}_K[\![\delta_0/\pi_K, \delta_J]\!]$.

Caution 3.2.5. The map ψ is *not* a homomorphism, nor is it canonically defined. This is because one cannot 'deform' the uniformizer in the mixed characteristic case. Moreover, since K will

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not be absolutely unramified in applications, a p-basis may not deform freely either. However, Proposition 3.2.8 below says that ψ is approximately a homomorphism.

Remark 3.2.6. In the p-basis decomposition (3.2.2), we allow extra freedom given by n'. So, we have the freedom of writing $1+p$ as itself or as $1+1+\cdots+1$. This is one of the places where the above ambiguity arises. Allowing this extra freedom in n' is in fact not necessary, except in Construction 4.6.11 where we need the diagram (4.6.12) to *commute*.

Definition 3.2.7. For two $\mathcal{O}_K$-algebras R_1 and R_2 and an ideal I of R_2, an *approximate homomorphism modulo* I is a function $f: R_1 \to R_2$ such that for $h_1 \in \pi_K^{a_1} R_1$ and $h_2 \in \pi_K^{a_2} R_2$ with $a_1, a_2 \in \mathbb{Z}_{\geqslant 0}$, we have $\psi(h_1h_2) - \psi(h_1)\psi(h_2) \in \pi_K^{a_1+a_2} I$ and $\psi(h_1+h_2) - \psi(h_1) - \psi(h_2) \in \pi_K^{\min\{a_1,a_2\}} I$.

Moreover, if R_1' and R_2' are two $\mathcal{O}_K$-algebras, a diagram of functions

$$\begin{array}{ccc} R_1' & \xrightarrow{f'} & R_2' \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle g'} \\ R_1 & \xrightarrow{f} & R_2 \end{array}$$

is said to be *approximately commutative modulo* I if for any $h \in \pi_K^a R_1'$ we have $g'(f'(h)) - f(g(h)) \in \pi_K^a I$.

Proposition 3.2.8. *For $h \in \mathcal{O}_K$, we have $\psi(h) - h \in (\delta_{J+}) \cdot \mathcal{O}_K[\![\delta_{J+}]\!]$; and, modulo the ideal $I_K = p(\delta_0/\pi_K, \delta_J)\mathcal{R}_K$, $\psi(h)$ does not depend on the choice of the compatible system of p-basis decompositions. Moreover, ψ is an approximate homomorphism modulo I_K.*

Proof. First, $\psi(h) - h \in (\delta_{J+}) \cdot \mathcal{O}_K[\![\delta_{J+}]\!]$ is obvious from the construction. Next, we observe that when $p^r > \beta_K$, in any rth p-basis decomposition for $h \in \mathcal{O}_K^\times$, the sum $\sum_{n'=0}^{\lambda_{(r),e_J,n}} \alpha_{(r),e_J,n,n'}^{p^r} \pi_K^n$ for any e_J and n in (3.2.2) is well-defined modulo p. So the ambiguity of defining ψ lies in I_K.

For $h_1, h_2 \in \mathcal{O}_K^\times$, the formal sum or product of compatible systems of p-basis decompositions of h_1 and h_2 is just some compatible system of p-basis decompositions of h_1+h_2 or h_1h_2. Thus, $\psi(h_1)+\psi(h_2)$ and $\psi(h_1)\psi(h_2)$ are the same as $\psi(h_1+h_2)$ and $\psi(h_1h_2)$ modulo I_K. The statement for general elements in $\mathcal{O}_K$ follows from this. □

Remark 3.2.9. From Proposition 3.2.8, we see that the ideal case is where $\beta_K \gg 1$. In contrast, when $\beta_K = 1$, $I_K = (\delta_0, p\delta_J)$. The above proposition does not give us much information about ψ. This is why we are not able to prove Conjecture 2.2.13 in the absolutely unramified and non-logarithmic case. This reflects the constraints in [AS03] from a different point of view, where Abbes and Saito formulated the dichotomy as follows:

$$\Omega^1_{\mathcal{O}_K/\mathbb{Z}_p} \otimes_{\mathcal{O}_K} k = \begin{cases} \bigoplus_{j\in J} k \cdot db_j & \text{if } \beta_K = 1, \\ \bigoplus_{j \in J} k \cdot db_j \oplus k \cdot d\pi_K & \text{if } \beta_K > 1. \end{cases}$$

Hypothesis 3.2.10. For the rest of the section, assume that K is not absolutely unramified, that is, $\beta_K \geqslant 2$.

Lemma 3.2.11. *Let $h \in \mathcal{O}_K$. Write $dh = \bar{h}_0 d\pi_K + \bar{h}_1 db_1 + \cdots + \bar{h}_m db_m$ when viewed as a differential in $\Omega^1_{\mathcal{O}_K/\mathbb{Z}_p} \otimes_{\mathcal{O}_K} k$. Then $\psi(h) - h \equiv \bar{h}_0\delta_0 + \cdots + \bar{h}_m\delta_m$ modulo $(\pi_K) + (\delta_0/\pi_K, \delta_J)^2$ in $\mathcal{R}_K$.*

Proof. For an rth p-basis decomposition (with $r \geqslant 1$) as in (3.2.2), we have, modulo the ideal $(\pi_K) + (\delta_{J^+})(\delta_0/\pi_K, \delta_J)$,

$$\psi(h) - h \equiv \sum_{e_J=0}^{p^r-1} \sum_{n=0}^{\infty} \sum_{n'=0}^{\lambda_{(r),e_J,n}} ((b_J + \delta_J)^{e_J} \alpha_{(r),e_J,n,n'}^{p^r} (\pi_K + \delta_0)^n - b_J^{e_J} \alpha_{(r),e_J,n,n'}^{p^r} \pi_K^n)$$

$$\equiv \sum_{e_J=0}^{p^r-1} \sum_{n=0}^{\infty} \sum_{n'=0}^{\lambda_{(r),e_J,n}} \alpha_{(r),e_J,n,n'}^{p^r} b_J^{e_J} \pi_K^n \left(\frac{n\delta_0}{\pi_K} + \frac{e_1\delta_1}{b_1} + \cdots + \frac{e_m\delta_m}{b_m} \right) \equiv \bar{h}_0\delta_0 + \cdots + \bar{h}_m\delta_m.$$

Taking the limit does not break the congruence relation. □

DEFINITION 3.2.12. Write $\mathcal{S}_K = \mathcal{R}_K\langle u_{J^+}\rangle$. For $\omega \in (1/e)\mathbb{N} \cap [1, \beta_K]$, we say that a set of elements $(R_{J^+}) \subset (\delta_{J^+}) \cdot \mathcal{S}_K$ *has error gauge* $\geqslant\omega$ if $R_0 \in (N^\omega\delta_0, N^{\omega+1}\delta_J) \cdot \mathcal{S}_K$ and $R_j \in (N^{\omega-1}\delta_0, N^\omega\delta_J) \cdot \mathcal{S}_K$ for all $j \in J$. We say that (R_{J^+}) is *admissible* if it has error gauge $\geqslant$1.

DEFINITION 3.2.13. Let $a \in \mathbb{Q}_{>1}$. We define the *standard (non-logarithmic) thickening space (of level a)* $\mathrm{TS}^a_{L/K,\psi}$ of L/K to be the rigid space associated to

$$\mathcal{O}^a_{\mathrm{TS},L/K,\psi} = K\langle\pi_K^{-a}\delta_{J^+}\rangle\langle u_{J^+}\rangle/(\psi(p_{J^+})).$$

For $(R_{J^+}) \subset (\delta_{J^+}) \cdot \mathcal{S}_K$ admissible, we define the *(non-logarithmic) thickening space (of level a)* $\mathrm{TS}^a_{L/K,R_{J^+}}$ to be the rigid space associated to

$$\mathcal{O}^a_{\mathrm{TS},L/K,R_{J^+}} = K\langle\pi_K^{-a}\delta_{J^+}\rangle\langle u_{J^+}\rangle/(\psi(p_{J^+}) + R_{J^+}).$$

Similarly, for $a \in \mathbb{Q}_{>0}$, we define the *standard logarithmic thickening space (of level a)* $\mathrm{TS}^a_{L/K,\log,\psi}$ of L/K to be the rigid space associated to

$$\mathcal{O}^a_{\mathrm{TS},L/K,\log,\psi} = K\langle\pi_K^{-a-1}\delta_0, \pi_K^{-a}\delta_J\rangle\langle u_{J^+}\rangle/(\psi(p_{J^+})).$$

For $(R_{J^+}) \subset (\delta_{J^+}) \cdot \mathcal{S}_K$ admissible, we define the *logarithmic thickening space (of level a)* $\mathrm{TS}^a_{L/K,\log,R_{J^+}}$ to be the rigid space associated to

$$\mathcal{O}^a_{\mathrm{TS},L/K,\log,R_{J^+}} = K\langle\pi_K^{-a-1}\delta_0, \pi_K^{-a}\delta_J\rangle\langle u_{J^+}\rangle/(\psi(p_{J^+}) + R_{J^+}).$$

Let $\mathrm{TS}_{L/K,R_{J^+}} = \bigcup_{a\in\mathbb{Q}_{>0}} \mathrm{TS}^a_{L/K,\log,R_{J^+}}$. Then we have the following natural Cartesian diagram for $a \in \mathbb{Q}_{>0}$.

$$\begin{array}{ccccc}
\mathrm{TS}^{a+1}_{L/K,R_{J^+}} & \hookrightarrow & \mathrm{TS}^a_{L/K,\log,R_{J^+}} & \hookrightarrow & \mathrm{TS}_{L/K,R_{J^+}} \\
\downarrow \Pi & & \downarrow \Pi & & \downarrow \Pi \\
A_K^{m+1}[0,\theta^{a+1}] & \hookrightarrow & A_K^1[0,\theta^{a+1}] \times A_K^m[0,\theta^a] & \hookrightarrow & A_K^1[0,\theta) \times A_K^m[0,1)
\end{array}$$

Here Π denotes the natural projection to the polydiscs with coordinates δ_{J^+}.

Remark 3.2.14. The error gauge is supposed to measure how 'standard' a thickening space is. Unfortunately, a standard thickening space itself depends on a very non-canonical function ψ. The upshot is that, by Proposition 3.2.8, the notion of having error gauge $\geqslant\omega$ does not depend on the choice of ψ if $\omega \in [1, \beta_K]$; note that the terms in p_0 are all divisible by π_K, except u_0^e.

Remark 3.2.15. The reason for introducing non-standard thickening spaces (or, rather, thickening spaces which do not have error gauge $\geqslant\beta_K$) is, as we will show later, that adding

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a generic pth root results in the error gauge of (R_{J^+}) dropping by one; the comparison theorem, Theorem 3.3.3, guarantees that as long as the (R_{J^+}) are admissible (i.e. $\beta_K \geqslant 1$), the thickening spaces still compute the same ramification break. In the same vein, if $\beta_K = 1$, we cannot afford to drop the error gauge; this is why we are not able to prove Conjecture 2.2.13 in the absolutely unramified and non-logarithmic case (see also Remark 3.2.9).

Notation 3.2.16. Let $(R_{J^+}) \subset (\delta_{J^+}) \cdot \mathcal{S}_K$ be admissible. With an abuse of notation, we shall still use Δ to denote the composite

$$\mathcal{S}_K/(\psi(p_{J^+}) + R_{J^+}) \xrightarrow{\mathrm{mod}(\delta_0/\pi_K, \delta_J)} \mathcal{O}_K\langle u_{J^+}\rangle/(p_{J^+}) \xrightarrow[\simeq]{\Delta} \mathcal{O}_L.$$

We remark that $\psi(p_{J^+}) - p_{J^+} + R_{J^+}$ are in fact contained in the ideal of $\mathcal{S}_K$ generated by δ_{J^+}. We denote the composition of Δ and the reduction $\mathcal{O}_L \twoheadrightarrow l$ by $\overline{\Delta}$.

Lemma 3.2.17. *Let $(R_{J^+}) \subset (\delta_{J^+}) \cdot \mathcal{S}_K$ be admissible. Then*

$$\{u_{J^+}^{e_{J^+}} \mid e_j \in \{0, \ldots, p^{r_j} - 1\} \text{ for all } j \in J \text{ and } e_0 \in \{0, \ldots, e-1\}\} \tag{3.2.18}$$

is a basis of $\mathcal{S}_K/(\psi(p_{J^+}) + R_{J^+})$ over $\mathcal{R}_K$. As a consequence, it also gives a basis of $\mathcal{O}^a_{\mathrm{TS}, L/K, R_{J^+}}$ over $K\langle \pi_K^{-a}\delta_{J^+}\rangle$ for $a \in \mathbb{Q}_{>1}$ and a basis of $\mathcal{O}^a_{\mathrm{TS}, L/K, \log, R_{J^+}}$ over $K\langle \pi_K^{-a-1}\delta_0, \pi_K^{-a}\delta_J\rangle$ for $a \in \mathbb{Q}_{>0}$. In particular, the morphism $\Pi : \mathrm{TS}_{L/K, R_{J^+}} \to A^1_K[0, \theta) \times A^m_K[0, 1)$ is finite and flat.

Proof. Given an element $h \in \mathcal{S}_K/(\psi(p_{J^+}) + R_{J^+})$, we first take a representative $\tilde{h} \in \mathcal{S}_K$. Then we simplify it by iteratively replacing u_0^e and $u_j^{p^{r_j}}$ by, respectively, $u_0^e - \psi(p_0) - R_0$ and $u_j^{p^{r_j}} - \psi(p_j) - R_j$ for $j \in J$. This procedure converges and gives an element with power of u_0 smaller than e and power of u_j smaller than p^{r_j} for $j \in J$. □

3.3 AS = TS theorem

In [Xia10], the essential step linking the arithmetic conductors and the differential conductors is the comparison theorem (see [Xia10, Theorem 4.3.6]), which asserts that the lifted Abbes–Saito spaces are isomorphic to the thickening spaces. In the mixed characteristic case, we do not have to lift the Abbes–Saito spaces. Instead, in this subsection, we prove a (slightly more general) comparison theorem over the base field K.

Remember that we continue to assume Hypotheses 3.1.2 and 3.2.10. We start with a lemma.

Lemma 3.3.1. *Let $(R_{J^+}) \subset (\delta_{J^+}) \cdot \mathcal{S}_K$ be admissible. Then*

$$\det\left(\frac{\partial(\psi(p_i) - p_i + R_i)}{\partial \delta_j}\right)_{i,j \in J^+}\Bigg|_{\delta_{J^+}=0} \in (\mathcal{O}_K\langle u_{J^+}\rangle/(p_{J^+}))^\times = \mathcal{O}_L^\times.$$

Proof. The proof is quite similar to that of [Xia10, Lemmas 4.3.1 and 4.3.3]. We also remark that the proof becomes very technical in order to deal with the appearance of $\tilde{b}_j(u_1, \ldots, u_{j-1})$ and $d(u_1, \ldots, u_m)$ and, partially, R_{J^+} (see Remark 3.1.8). If we could have taken $\tilde{b}_j(u_1, \ldots, u_{j-1})$ and $d(u_1, \ldots, u_m)$ to be 1 and $R_{J^+} = 0$, the lemma is almost immediate because the leading term in each $\psi(p_i) - p_i$ is just δ_i, and the matrix becomes the identity matrix modulo π_L.

It is enough to prove that the matrix is of full rank modulo π_L. By Lemma 3.2.11 and the admissibility of R_{J^+}, modulo π_L, the first row will be all zero except for the first element, which is $d(\bar{c}_1, \ldots, \bar{c}_m) \in \kappa_L^\times$. Hence, we need only look at

$$\left(\frac{\partial(\psi(p_i) - p_i)}{\partial \delta_j}\right)_{i,j\in J} \bmod (\pi_L, \delta_0/\pi_K, \delta_J) = \left(\frac{\partial(\psi(\tilde{b}_i) - \tilde{b}_i)}{\partial \delta_j}\right)_{i,j\in J} \bmod (\pi_L, \delta_0/\pi_K, \delta_J), \tag{3.3.2}$$

where $\tilde{b}_i = \tilde{b}_i(u_1, \ldots, u_{i-1})$ in Construction 3.1.6. Let $\bar{\alpha}_{ij} \in l$ denote the entries in the matrix on the right-hand side of (3.3.2), where we make the identification $\mathcal{O}_K\langle u_{J+}\rangle/(p_{J+}, u_0) \xrightarrow{\sim} l$. Under this identification, $\tilde{b}_i$ will become $\bar{c}_i^{p^{r_i}}$ for all $i \in J$. It suffices to show that the ith row is l-linearly independent of the first $i-1$ rows, for all i. If we set

$$\bar{c}_i^{p^{r_i}} = \sum_{e_1=0}^{p^{r_0}-1} \cdots \sum_{e_{i-1}=0}^{p^{r_{i-1}}-1} \bar{\lambda}_{e_1,\ldots,e_{i-1}} \bar{c}_1^{e_1} \cdots \bar{c}_{i-1}^{e_{i-1}}$$

where $\bar{\lambda}_{e_1,\ldots,e_{i-1}} \in k$, then we would have, modulo π_K,

$$\tilde{b}_i(u_1, \ldots, u_{j-1}) \equiv \sum_{e_1=0}^{p^{r_0}-1} \cdots \sum_{e_{i-1}=0}^{p^{r_{i-1}}-1} \bar{\lambda}_{e_1,\ldots,e_{i-1}} u_1^{e_1} \cdots u_{i-1}^{e_{i-1}}.$$

Hence, if we set $d\bar{\lambda}_{e_1,\ldots,e_{i-1}} = \bar{\mu}_{e_1,\ldots,e_{i-1},1} d\bar{b}_1 + \cdots + \bar{\mu}_{e_1,\ldots,e_{i-1},m} d\bar{b}_m$, then by Lemma 3.2.11 we get

$$\begin{aligned}\bar{\alpha}_{i1}\, d\bar{b}_1 + \cdots + \bar{\alpha}_{im}\, d\bar{b}_m &= \sum_{e_1=0}^{p^{r_0}-1} \cdots \sum_{e_{i-1}=0}^{p^{r_{i-1}}-1} u_1^{e_1} \cdots u_{i-1}^{e_{i-1}} (\bar{\mu}_{e_1,\ldots,e_{i-1},1}\, d\bar{b}_1 + \cdots + \bar{\mu}_{e_1,\ldots,e_{i-1},m}\, d\bar{b}_m) \\ &\equiv d(\bar{c}_i^{p^{r_i}}) \text{ modulo } (d\bar{c}_1, \ldots, d\bar{c}_{i-1})\end{aligned}$$

in $\Omega^1_{\mathbf{k}_{i-1}/\mathbb{F}_p}$; it is, in fact, non-trivial because $d\bar{c}_1, \ldots, d\bar{c}_m$ form a basis for $\Omega^1_{\kappa_L/\mathbb{F}_p}$ and hence there should not be any auxiliary relations among $d\bar{c}_1, \ldots, d\bar{c}_m$ in $\Omega^1_{\mathbf{k}_i/\mathbb{F}_p}$. But we know that the sums $\bar{\alpha}_{i'1} d\bar{b}_1 + \cdots + \bar{\alpha}_{i'm} d\bar{b}_m$ for $i' < i$ all lie in the subspace of $\Omega^1_{\mathbf{k}_{i-1}/\mathbb{F}_p}$ generated by $d\bar{c}_1, \ldots, d\bar{c}_{i-1}$. Hence the ith row of the matrix in (3.3.2) is ($\mathbf{k}_{i-1}$-)linearly independent of the first $i-1$ rows. The lemma follows. □

THEOREM 3.3.3. *If $(R_{J+}) \subset (\delta_{J+}) \cdot \mathcal{S}_K$ is admissible, we have the following isomorphisms of K-algebras:*

$$\mathcal{O}^a_{\mathrm{AS},L/K} \simeq \mathcal{O}^a_{\mathrm{TS},L/K,R_{J+}} \quad \text{if } a \in \mathbb{Q}_{>1},$$
$$\mathcal{O}^a_{\mathrm{AS},L/K,\log} \simeq \mathcal{O}^a_{\mathrm{TS},L/K,\log,R_{J+}} \quad \text{if } a \in \mathbb{Q}_{>0}.$$

Example 3.3.4. Before proving the theorem, we illustrate the idea using an example.

Assume $p > 2$. Let K be the completion of $\mathbb{Q}_p(\zeta_p)(b)$ with respect to the 1-Gauss norm on $b\,(= b_1)$; we take $\pi_K = \zeta_p - 1$. (Strictly speaking, Hypothesis 3.1.2 requires K to have separably closed residue field; in fact, however, Theorem 3.3.3 holds without this assumption.) Let $L = K((b\pi_K)^{1/p})((b + \pi_K)^{1/p})$; it is a Galois extension with inseparable residue field extension and naïve ramification degree p. We take the uniformizer of L to be $\pi_L = (b\pi_K)^{1/p}$ and we take $c = (b + \pi_K)^{1/p}$; these generate the extension $\mathcal{O}_L/\mathcal{O}_K$ with relations $p_0(u_0, u_1) = p_0(u_0) = u_0^p - b\pi_K$ and $p_1(u_0, u_1) = p_1(u_1) = u_1^p - b - \pi_K$. For $a > 0$, the Abbes–Saito space is given by

$$\mathcal{O}^a_{\mathrm{AS},L/K} = K\langle u_0, u_1, \pi_K^{-a} V_0, \pi_K^{-a} V_1\rangle/(u_0^p - b\pi_K - V_0, u_1^p - b - \pi_K - V_1).$$

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We take the function $\psi : \mathcal{O}_K \to \mathcal{O}_K[\![\delta_0, \delta_1]\!]$ so that $\psi(b) = b + \delta_1$ and $\psi(b\pi_K) = (b + \delta_1)(\pi_K + \delta_0)$. Then the standard thickening space is given by

$$\mathcal{O}^a_{\mathrm{TS},L/K,\psi} = K\langle u_0, u_1, \pi_K^{-a}\delta_0, \pi_K^{-a}\delta_1\rangle/(u_0^p - (b+\delta_1)(\pi_K+\delta_0), u_1^p - b - \delta_1 - \pi_K - \delta_0).$$

We will identify these two algebras by matching u_0 and u_1 from the two algebras. To do this, we first construct a (continuous) homomorphism $\chi_1 : \mathcal{O}^a_{\mathrm{AS},L/K} \to \mathcal{O}^a_{\mathrm{TS},L/K,\psi}$ such that $\chi_1(u_0) = u_0$ and $\chi_1(u_1) = u_1$; then we are forced to send V_0 to $\chi_1(u_0^p - b\pi_K) = \pi_K\delta_1 + b\delta_0 + \delta_0\delta_1$ and V_1 to $\chi_1(u_1^p - b - \pi_K) = \delta_0 + \delta_1$. For χ_1 to be well-defined, we need to check convergence, which is quite obvious from the way it is written in this particular example.

Conversely, we want to construct the inverse (continuous) homomorphism $\chi_2 : \mathcal{O}^a_{\mathrm{TS},L/K,\psi} \to \mathcal{O}^a_{\mathrm{AS},L/K}$. Again, we need $\chi_2(u_1) = u_1$ and $\chi_2(u_2) = u_2$. It is less obvious where we need to send δ_0 and δ_1. But we know that the images $\chi_2(\delta_0)$ and $\chi_2(\delta_1)$ must satisfy

$$b\chi_2(\delta_0) + \pi_K\chi_2(\delta_1) = \chi_2(u_0^p - b\pi_K - \delta_0\delta_1) = V_0 - \chi_2(\delta_0)\chi_2(\delta_1)$$

and

$$\chi_2(\delta_0) + \chi_2(\delta_1) = \chi_2(u_1^p - b - \pi_K) = V_1.$$

Thinking of these as a system of linear equations, we have

$$\begin{pmatrix}\chi_2(\delta_0)\\ \chi_2(\delta_1)\end{pmatrix} = \begin{pmatrix} b & \pi_K \\ 1 & 1\end{pmatrix}^{-1}\begin{pmatrix} V_0 - \chi_2(\delta_0)\chi_2(\delta_1) \\ V_1\end{pmatrix}. \tag{3.3.5}$$

We can determine the value of $\chi_2(\delta_0)$ and $\chi_2(\delta_1)$ by iteratively plugging the left-hand side of (3.3.5) into its right-hand side. In our special case, one can check by hand that this process will converge eventually to two elements of $\mathcal{O}^a_{\mathrm{AS},L/K}$, which will be the images of $\chi_2(\delta_0)$ and $\chi_2(\delta_1)$, respectively. For the general case, however, it is better to employ a 'fixed-point theorem' argument.

We now prove Theorem 3.3.3.

Proof. The proof is similar to that of [Xia10, Theorem 4.3.6]. We will match up u_{J^+} in the two rings.

We first observe that

$$\{u_{J^+}^{e_{J^+}} \mid e_j \in \{0, \ldots, p^{r_j} - 1\} \text{ for all } j \in J \text{ and } e_0 \in \{0, \ldots, e-1\}\} \tag{3.3.6}$$

forms a basis of $\mathcal{O}^a_{\mathrm{AS},L/K}$ (respectively, $\mathcal{O}^a_{\mathrm{AS},L/K,\log}$) over $K\langle \pi_K^{-a}V_{J^+}\rangle$ (respectively, $K\langle \pi_K^{-a-1}V_0, \pi_K^{-a}V_J\rangle$) as a finite free module. Given

$$h = \sum_{e_{J^+}, e'_{J^+}} \alpha_{e_{J^+}, e'_{J^+}} u_{J^+}^{e_{J^+}} V_{J^+}^{e'_{J^+}} \in \mathcal{O}^a_{\mathrm{AS},L/K} \quad (\text{respectively, } \mathcal{O}^a_{\mathrm{AS},L/K,\log})$$

written in terms of this basis, where $\alpha_{e_{J^+}, e'_{J^+}} \in K$, we define

$$|h|_{\mathrm{AS},a} = \max_{e_{J^+}, e'_{J^+}} \{|\alpha_{e_{J^+}, e'_{J^+}}| \cdot \theta^{ae'_0 + \cdots + ae'_m + e_0/e}\}$$

$$\left(\text{respectively, } |h|_{\mathrm{AS},\log,a} = \max_{e_{J^+}, e'_{J^+}} \{|\alpha_{e_{J^+}, e'_{J^+}}| \cdot \theta^{(a+1)e'_0 + ae'_1 + \cdots + ae'_m + e_0/e}\}\right).$$

It is clear that $\mathcal{O}^a_{\mathrm{AS},L/K}$ (respectively, $\mathcal{O}^a_{\mathrm{AS},L/K,\log}$) is complete and submultiplicative for this norm (i.e. $|h_1h_2|_{\mathrm{AS},a} \leqslant |h_1|_{\mathrm{AS},a}|h_2|_{\mathrm{AS},a}$ and $|h_1h_2|_{\mathrm{AS},\log,a} \leqslant |h_1|_{\mathrm{AS},\log,a}|h_2|_{\mathrm{AS},\log,a}$); the

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requirement $a > 1$ in the non-logarithmic case guarantees that upon replacing u_0^e by $u_0^e - p_0 - V_0$, the norm does not increase.

Similarly, by Lemma 3.2.17, (3.3.6) also forms a basis of $\mathcal{O}^a_{\mathrm{TS},L/K,R_{J^+}}$ (respectively, $\mathcal{O}^a_{\mathrm{TS},L/K,\log,R_{J^+}}$) over $K\langle\pi_K^{-a}\delta_{J^+}\rangle$ (respectively, $K\langle\pi_K^{-a-1}\delta_0, \pi_K^{-a}\delta_J\rangle$) as a finite free module. Given

$$h = \sum_{e_{J^+},e'_{J^+}} \alpha_{e_{J^+},e'_{J^+}} u_{J^+}^{e_{J^+}} \delta_{J^+}^{e'_{J^+}} \in \mathcal{O}^a_{\mathrm{TS},L/K,R_{J^+}} \quad (\text{respectively, } \mathcal{O}^a_{\mathrm{TS},L/K,\log,R_{J^+}})$$

written in terms of this basis, where $\alpha_{e_{J^+},e'_{J^+}} \in K$, we define

$$|h|_{\mathrm{TS},a} = \max_{e_{J^+},e'_{J^+}} \{|\alpha_{e_{J^+},e'_{J^+}}| \cdot \theta^{ae'_0+\cdots+ae'_m+e_0/e}\}$$
$$\left(\text{respectively, } |h|_{\mathrm{TS},\log,a} = \max_{e_{J^+},e'_{J^+}} \{|\alpha_{e_{J^+},e'_{J^+}}| \cdot \theta^{(a+1)e'_0+ae'_1+\cdots+ae'_m+e_0/e}\}\right).$$

It is clear that $\mathcal{O}^a_{\mathrm{TS},L/K,R_{J^+}}$ (respectively, $\mathcal{O}^a_{\mathrm{TS},L/K,\log,R_{J^+}}$) is complete and submultiplicative for this norm. The requirement $a > 1$ in the non-logarithmic case guarantees that upon replacing u_0^e by $u_0^e - \psi(p_0) - R_0$, the norm does not increase.

Define a continuous homomoprhism $\chi_1 : \mathcal{O}^a_{\mathrm{AS},L/K} \to \mathcal{O}^a_{\mathrm{TS},L/K,R_{J^+}}$ (respectively, $\chi_1 : \mathcal{O}^a_{\mathrm{AS},L/K,\log} \to \mathcal{O}^a_{\mathrm{TS},L/K,\log,R_{J^+}}$) by sending u_{J^+} to u_{J^+} and hence V_j to $p_j(u_{J^+}) = p_j(u_{J^+}) - \psi(p_j(u_{J^+})) - R_j$ for all $j \in J^+$. We need to verify the convergence conditions for all V_j. Indeed, Proposition 3.2.8 and the admissibility of R_{J^+} imply that

$$|p_j - \psi(p_j)|_{\mathrm{TS},a} \leqslant \theta^a \quad \text{and} \quad |R_j|_{\mathrm{TS},a} \leqslant \theta^a \quad \text{for all } j \in J^+$$
$$\left(\text{respectively, } |p_j - \psi(p_j)|_{\mathrm{TS},\log,a} \leqslant \begin{cases} \theta^{a+1} & j = 0 \\ \theta^a & j \in J \end{cases} \text{ and } |R_j|_{\mathrm{TS},\log,a} \leqslant \begin{cases} \theta^{a+1+1/e} & j = 0 \\ \theta^{a+1/e} & j \in J \end{cases}\right).$$

Now we define the inverse χ_2 of χ_1. Obviously, we should send u_{J^+} back to u_{J^+}. We need to define $\chi_2(\delta_{J^+})$ properly. Let $A = (A_{ij})_{i,j\in J^+}$ denote the unique matrix in $\mathcal{O}_K[\![u_{J^+}]\!]$ such that

$$A \equiv \left(\frac{\partial(\psi(p_i) + R_i)}{\partial \delta_j}\right)_{i,j\in J^+} \bmod (\delta_{J^+}) \cdot \mathcal{S}_K.$$

By Lemma 3.3.1, the image of A in $\mathrm{Mat}_{m+1}(\mathcal{O}_K\langle u_{J^+}\rangle/(p_{J^+})) = \mathrm{Mat}_{m+1}(\mathcal{O}_L)$, denoted by $\overline{A}$, is invertible. Let B denote the $(m+1)\times(m+1)$ matrix with coefficients in $\bigoplus_{e_0=0}^{e-1}\bigoplus_{e_1=0}^{p^{r_1}-1}\cdots\bigoplus_{e_m=0}^{p^{r_m}-1}\mathcal{O}_K u_{J^+}^{e_{J^+}}$ whose image in $\mathrm{Mat}_{m+1}(\mathcal{O}_K\langle u_{J^+}\rangle/(p_{J^+}))$ is the *inverse* of $\overline{A}$. Then we have

$$BA - I \in \mathrm{Mat}_{m+1}((p_{J^+}) \cdot \mathcal{O}_K\langle u_{J^+}\rangle), \tag{3.3.7}$$

where I is the $(m+1)\times(m+1)$ identity matrix.

Define the subset

$$\Lambda = \{{}^t(x_0, \ldots, x_m) \in (\mathcal{O}^a_{\mathrm{AS},L/K})^{\oplus(m+1)} : |x_j|_{\mathrm{AS},a} \leqslant \theta^a \ \forall j \in J^+\}$$
$$(\text{respectively, } \Lambda = \{{}^t(x_0, \ldots, x_m) \in (\mathcal{O}^a_{\mathrm{AS},L/K,\log})^{\oplus(m+1)} :$$
$$|x_0|_{\mathrm{AS},\log,a} \leqslant \theta^{a+1} \text{ and } |x_j|_{\mathrm{AS},\log,a} \leqslant \theta^a \ \forall j \in J\}).$$

It carries a norm $|\cdot|_\Lambda$ defined by taking the maximum of $|\cdot|_{\mathrm{AS},a}$ (respectively, $|\cdot|_{\mathrm{AS},\log,a}$) over its entries. Consider the function $\mathbf{F}:\Lambda\to\Lambda$ given by

$$\mathbf{F}\begin{pmatrix}x_0\\ \vdots\\ x_m\end{pmatrix}=\begin{pmatrix}x_0\\ \vdots\\ x_m\end{pmatrix}-B\begin{pmatrix}(\psi(p_0)+R_0)(u_{J^+},x_{J^+})\\ \vdots\\ (\psi(p_m)+R_m)(u_{J^+},x_{J^+})\end{pmatrix} \tag{3.3.8}$$

$$=(I-BA)\begin{pmatrix}x_0\\ \vdots\\ x_m\end{pmatrix}$$

$$-B\left(\begin{pmatrix}(\psi(p_0)+R_0)(u_{J^+},x_{J^+})-p_0\\ \vdots\\ (\psi(p_m)+R_m)(u_{J^+},x_{J^+})-p_m\end{pmatrix}-A\begin{pmatrix}x_0\\ \vdots\\ x_m\end{pmatrix}\right)-B\begin{pmatrix}V_0\\ \vdots\\ V_m\end{pmatrix}, \tag{3.3.9}$$

where $(\psi(p_j)+R_j)(u_{J^+},x_{J^+})$ is the formal substitution of x_j for δ_j for any $j\in J^+$.

To see that $\mathbf{F}$ is well-defined, we need to bound the norms of each term in (3.3.9) when ${}^t(x_0,\ldots,x_m)\in\Lambda$. By (3.3.7), $I-BA$ (viewed as an element in $\mathcal{O}^a_{\mathrm{AS},L/K}$ or, respectively, $\mathcal{O}^a_{\mathrm{AS},L/K,\log}$) has norm less than or equal to θ^a. Hence, in the non-logarithmic case, the first term of (3.3.9) has norm less than or equal to θ^{2a}; in the logarithmic case, the first term of (3.3.9) has norm less than or equal to θ^{2a}, except for the first row, which has norm less than or equal to θ^{2a+1}. By the definition of A, the second term of (3.3.9) has entries in $(\delta_{J^+})^2\mathcal{S}_K$, except for the first row, which is in $(\delta_{J^+})^2\mathcal{S}_K\cap(x_0^2,\pi_K x_0)\mathcal{S}_K$ (because of how p_0 is defined). Hence, in the non-logarithmic case, this term has norm less than or equal to θ^{2a-1}; in the logarithmic case, this term has norm less than or equal to θ^{2a}, except for the first row, which has norm less than or equal to $\min\{\theta^{a+2},\theta^{2a}\}\leqslant\theta^{a+2}$.

Hence, we see clearly that $\mathbf{F}$ does map Λ into Λ. Moreover, we observe that $\mathbf{F}$ is contractive, that is, there exists $\varepsilon\in(0,1)$ (in fact, $\varepsilon=\theta^{a-1}$ in the non-logarithmic case and $\varepsilon=\theta^{\min\{a,1\}}$ in the logarithmic case) such that for $\mathbf{x}={}^t(x_0,\ldots,x_m)$ and $\mathbf{y}={}^t(y_0,\ldots,y_m)\in\Lambda$, we have

$$|\mathbf{F}(\mathbf{x})-\mathbf{F}(\mathbf{y})|_\Lambda<\varepsilon|\mathbf{x}-\mathbf{y}|_\Lambda\qquad(\text{respectively, }|\mathbf{F}(\mathbf{x})-\mathbf{F}(\mathbf{y})|<\varepsilon|\mathbf{x}-\mathbf{y}|_\Lambda).$$

Therefore, $\mathbf{F}$ has a unique fixed point in Λ, denoted by $\mathbf{x}={}^t(x_0,\ldots,x_m)\in\Lambda$.

Now, we define a continuous homomorphism $\tilde{\chi}_2:K\langle u_{J^+},\pi_K^{-a}\delta_{J^+}\rangle\to\mathcal{O}^a_{\mathrm{AS},L/K}$ (respectively, $\tilde{\chi}_2:K\langle u_{J^+},\pi_K^{-a-1}\delta_0,\pi_K^{-a}\delta_J\rangle\to\mathcal{O}^a_{\mathrm{AS},L/K,\log}$) by $\tilde{\chi}_2(u_j)=u_j$ for $j\in J^+$ and $\tilde{\chi}_2(\delta_j)=x_j$.

We now check that $\tilde{\chi}_2(\psi(p_j)+R_j)=0$ for all $j\in J^+$. Indeed, by (3.3.8) we have

$$B\begin{pmatrix}\tilde{\chi}_2(\psi(p_0)+R_0)\\ \vdots\\ \tilde{\chi}_2(\psi(p_m)+R_m)\end{pmatrix}=B\begin{pmatrix}(\psi(p_0)+R_0)(u_{J^+},x_{J^+})\\ \vdots\\ (\psi(p_m)+R_m)(u_{J^+},x_{J^+})\end{pmatrix}=\begin{pmatrix}x_0\\ \vdots\\ x_m\end{pmatrix}-\mathbf{F}\begin{pmatrix}x_0\\ \vdots\\ x_m\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}.$$

Hence, $\tilde{\chi}_2$ factors through a continuous homomorphism $\chi_2:\mathcal{O}^a_{\mathrm{TS},L/K,R_{J^+}}\to\mathcal{O}^a_{\mathrm{AS},L/K}$ (respectively, $\chi_2:\mathcal{O}^a_{\mathrm{TS},L/K,\log,R_{J^+}}\to\mathcal{O}^a_{\mathrm{AS},L/K,\log}$).

Finally, we claim that χ_2 and χ_1 are inverse to each other. One may check this directly from the definition. Alternatively, observe that by our definition, they are inverse to each other on a dense subset $K[u_{J^+}]$ (the density is proved in Lemma 3.3.11 below); therefore, they have to be inverse to each other, and give an isomorphism between the ring of functions on the Abbes–Saito space and the ring of functions on the thickening space. □

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Remark 3.3.10. An alternative way of understanding this theorem is to think of the thickening spaces as perturbations of the morphisms $\mathrm{AS}^a_{L/K} \to A^{m+1}_K[0, \theta^a]$ and $\mathrm{AS}^a_{L/K,\log} \to A^1_K[0, \theta^{a+1}] \times A^m_K[0, \theta^a]$. Abbes–Saito spaces will behave better under base change using the new morphisms.

LEMMA 3.3.11. *Let $(R_{J^+}) \subset (\delta_{J^+}) \cdot \mathcal{S}_K$ be admissible. Then $K[u_{J^+}]$ is dense in $\mathcal{O}^a_{\mathrm{TS},L/K,R_{J^+}}$ and $\mathcal{O}^a_{\mathrm{AS},L/K}$ for $a \in \mathbb{Q}_{>1}$, and in $\mathcal{O}^a_{\mathrm{TS},L/K,\log,R_{J^+}}$ and $\mathcal{O}^a_{\mathrm{AS},L/K,\log}$ for $a \in \mathbb{Q}_{>0}$.*

Proof. Since $V_j = p_j(u_{J^+}) \in K[u_{J^+}]$ for all $j \in J^+$, the density of $K[u_{J^+}]$ in $\mathcal{O}^a_{\mathrm{AS},L/K}$ and $\mathcal{O}^a_{\mathrm{AS},L/K,\log}$ is obvious from the definition. We now prove the density for the thickening spaces. It is enough to show that δ_{J^+} can be well-approximated by elements of $K[u_{J^+}]$. We keep the notation as in the proof of Theorem 3.3.3. Consider a variant of (3.3.9):

$$\begin{pmatrix}\delta_0\\ \vdots\\ \delta_m\end{pmatrix} = (I - BA)\begin{pmatrix}\delta_0\\ \vdots\\ \delta_m\end{pmatrix} - B\left(\begin{pmatrix}(\psi(p_0)+R_0)-p_0\\ \vdots\\ (\psi(p_m)+R_m)-p_m\end{pmatrix} - A\begin{pmatrix}\delta_0\\ \vdots\\ \delta_m\end{pmatrix}\right) - B\begin{pmatrix}p_0\\ \vdots\\ p_m\end{pmatrix}. \quad (3.3.12)$$

Note that $I - BA \in \mathrm{Mat}_{m+1}((p_{J^+}) \cdot \mathcal{O}_K\langle u_{J^+}\rangle)$ implies that the first term in the right-hand side of (3.3.12) has representatives in $(\delta_0/\pi_K, \delta_J)^2\mathcal{S}_K$ under the quotient $\mathcal{S}_K \to \mathcal{S}_K/(\psi(p_{J^+}) + R_{J^+})$. The second term in the right-hand side of (3.3.12) is already written in terms of elements in $(\delta_0/\pi_K, \delta_J)^2\mathcal{S}_K$. The third term in the right-hand side of (3.3.12) is a vector of elements in $K[u_{J^+}]$.

So, this means that we can approximate δ_{J^+} using $K[u_{J^+}]$ up to elements in $(\delta_0/\pi_K, \delta_J)^2\mathcal{S}_K$. We can use the same approximation to approximate $\delta_j\delta_{j'}$ for $j, j' \in J$ in the previous approximation and hence get an approximation of δ_{J^+} by elements in $K[u_{J^+}]$ up to $(\delta_0/\pi_K, \delta_J)^3\mathcal{S}_K$. Iterating this construction, we see that $K[u_{J^+}]$ is dense in $\mathcal{O}^a_{\mathrm{TS},L/K,R_{J^+}}$ for $a \in \mathbb{Q}_{>1}$ and dense in $\mathcal{O}^a_{\mathrm{TS},L/K,\log,R_{J^+}}$ for $a \in \mathbb{Q}_{>0}$. □

3.4 Étaleness of thickening spaces

In this subsection, we will study a variant of [AS02, Theorem 7.2] and [AS03, Corollary 4.12].

Remember that Hypotheses 3.1.2 and 3.2.10 are still in force.

DEFINITION 3.4.1. Let $(R_{J^+}) \subset (\delta_{J^+}) \cdot \mathcal{S}_K$ be an admissible subset. Let $\mathrm{ET}_{L/K,R_{J^+}}$ be the rigid analytic subspace of $A^1_K[0, \eta) \times A^m_K[0, 1)$ over which the morphism Π defined in Definition 3.2.13 is étale. When there is no confusion over the choice of R_{J^+}, or if the choice is not important, we abbreviate $\mathrm{ET}_{L/K,R_{J^+}}$ to $\mathrm{ET}_{L/K}$.

THEOREM 3.4.2. *Let $b(L/K)$ be the highest non-logarithmic ramification break of L/K. There exists $\epsilon \in (0, b(L/K) - 1)$ such that $b(L/K) - \epsilon \in \mathbb{Q}$ and, for any admissible $(R_{J^+}) \subset (\delta_{J^+}) \cdot \mathcal{S}_K$, $A^{m+1}_K[0, \theta^{b(L/K)-\epsilon}] \subseteq \mathrm{ET}_{L/K,R_{J^+}}$.*

Proof. The proof is essentially the same as that for [AS02, Proposition 7.5]. The essential point is the 'congruence' $\partial(\psi(p_i) + R_i)/\partial u_j \equiv \partial(p_i)/\partial u_j$ over the said locus. For the reader's convenience, we include the proof here.

Recall from [AS02, Proposition 7.3] that

$$\Omega^1_{\mathcal{O}_L/\mathcal{O}_K} = \bigoplus_{i=1}^{r} \mathcal{O}_L/\pi_L^{\alpha_i}\mathcal{O}_L \quad \text{with } \alpha_i < e(b(L/K) - \epsilon) \quad (3.4.3)$$

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for some $\epsilon > 0$ and $r \in \mathbb{N}$. It does not hurt to take $\epsilon < b(L/K) - 1$ and $b(L/K) - \epsilon \in \mathbb{Q}$. Let $\mathcal{J} = (\partial(\psi(p_i) + R_i)/\partial u_j)_{i,j \in J^+}$ be the Jacobian matrix of $\mathrm{TS}^a_{L/K,R_{J^+}}$ over $A_K^{m+1}[0, \theta^a]$, whose entries are elements in $\mathcal{O} = \mathcal{O}_K\langle u_{J^+}, \pi_K^{-a}\delta_{J^+}\rangle/(\psi(p_i) + R_i)$.

Let $a = b(L/K) - \epsilon \in \mathbb{Q}$. Suppose that $\mathbf{x} \in A_K^1[0, \theta^a]$ is a K^{alg}-point at which $\det(\mathcal{J})$ vanishes; it gives a homomorphism $\mathcal{O}^a_{\mathrm{TS},L/K,R_{J^+}} \to K^{\mathrm{alg}}$. We let x_{J^+} and ν_{J^+} denote the images of u_{J^+} and δ_{J^+}, respectively; we have $x_j, \nu_j \in \mathcal{O}_{K^{\mathrm{alg}}}$ and $|\nu_j| \leqslant \theta^a$, for all $j \in J^+$. Hence, we have $|p_j(x_{J^+})| \leqslant \theta^a$ for all $j \in J^+$.

Now, we have the following two $\mathcal{O}_K$-algebra *homomorphisms*.

$$\begin{aligned}\varphi : \mathcal{O}_L = \mathcal{O}_K[u_0, \dots, u_m]/(p_0, \dots, p_m) &\longrightarrow \mathcal{O}_{K^{\mathrm{alg}}}/\pi_K^a\mathcal{O}_{K^{\mathrm{alg}}}\\ h(u_{J^+}) &\longmapsto h(x_{J^+}),\\ \mathrm{ev}_{\mathbf{x}} : \mathcal{O} = \mathcal{O}_K\langle u_{J^+}, \pi_K^{-a}\delta_{J^+}\rangle/(\psi(p_i) + R_i) &\longrightarrow \mathcal{O}_{K^{\mathrm{alg}}}\\ h(u_{J^+}, \delta_{J^+}) &\longmapsto h(x_{J^+}, \nu_{J^+}).\end{aligned}$$

Here φ is well-defined because $|p_j(x_{J^+})| \leqslant \theta^a$.

We consider the following commutative diagram of linear maps.

$$\begin{array}{ccccccc}\mathcal{O}^{\oplus(m+1)} & \xrightarrow{\mathrm{ev}_{\mathbf{x}}} & \mathcal{O}_{K^{\mathrm{alg}}}^{\oplus(m+1)} & \xrightarrow{\bmod \pi_K^a} & (\mathcal{O}_{K^{\mathrm{alg}}}/\pi_K^a\mathcal{O}_{K^{\mathrm{alg}}})^{\oplus(m+1)} & \xleftarrow{\varphi} & \mathcal{O}_L^{\oplus(m+1)}\\ \downarrow{\scriptstyle \mathcal{J}} & & \downarrow{\scriptstyle \mathrm{ev}_{\mathbf{x}}(\mathcal{J})} & & \downarrow{\scriptstyle (\partial p_i/\partial u_j)_{i,j\in J^+} \bmod \pi_K^a} & & \downarrow{\scriptstyle (\partial p_i/\partial u_j)_{i,j\in J^+}}\\ \mathcal{O}^{\oplus(m+1)} & \xrightarrow{\mathrm{ev}_{\mathbf{x}}} & \mathcal{O}_{K^{\mathrm{alg}}}^{\oplus(m+1)} & \xrightarrow{\bmod \pi_K^a} & (\mathcal{O}_{K^{\mathrm{alg}}}/\pi_K^a\mathcal{O}_{K^{\mathrm{alg}}})^{\oplus(m+1)} & \xleftarrow{\varphi} & \mathcal{O}_L^{\oplus(m+1)}\end{array} \tag{3.4.4}$$

Here, commutativity is clear except for the one in the middle, which follows from the simple but key fact that $|\nu_{J^+}| \leqslant \pi_K^a \implies \mathrm{ev}_{\mathbf{x}}(\mathcal{J}) \equiv (\partial p_i/\partial u_j)_{i,j \in J^+} \bmod \pi_K^a$.

Now, on the one hand, (3.4.3) implies that the cokernel of the right vertical arrow in (3.4.4) is isomorphic to $\bigoplus_{i=1}^r \mathcal{O}_L/\pi_L^{\alpha_i}\mathcal{O}_L$. Since $ea > \alpha_i$ for any i, the cokernel of the third vertical arrow in (3.4.4) is isomorphic to $\bigoplus_{i=1}^r \mathcal{O}_{K^{\mathrm{alg}}}/\pi_L^{\alpha_i}\mathcal{O}_{K^{\mathrm{alg}}}$.

On the other hand, we have assumed that $\det(\mathrm{ev}_{\mathbf{x}}(\mathcal{J})) = 0$; this implies that the cokernel of the second vertical arrow in (3.4.4) has a torsion-free constituent. Therefore, we know that the the cokernel of the third arrow must have a direct summand isomorphic to $\mathcal{O}_{K^{\mathrm{alg}}}/\pi_K^a\mathcal{O}_{K^{\mathrm{alg}}}$; this contradicts the claim in the previous paragraph. So we have étaleness as stated. ◻

Remark 3.4.5. Theorem 3.4.2 (as well as Theorem 3.4.7 later) states that the étale locus $\mathrm{ET}_{L/K,R_{J^+}}$ is a bit larger than the locus where $\mathrm{TS}^a_{L/K,R_{J^+}}$ (respectively, $\mathrm{TS}^a_{L/K,\log,R_{J^+}}$) becomes a geometrically disjoint union of $[L:K]$ discs. This is crucial for the proof of Corollary 3.5.4.

The following lemma is an easy fact about logarithmic relative differentials. This is not a good place to introduce the theory of logarithmic structure. For a systematic account of logarithmic structures and log-schemes, one may consult [KS04, § 4] and [Kat89].

Lemma 3.4.6. *If we provide $\mathcal{O}_L$ and $\mathcal{O}_K$ with the canonical log-structures $\pi_L^{\mathbb{N}} \hookrightarrow \mathcal{O}_L$ and $\pi_K^{\mathbb{N}} \hookrightarrow \mathcal{O}_K$, respectively, then the logarithmic relative differentials are such that*

$$\Omega^1_{\mathcal{O}_L/\mathcal{O}_K}(\log/\log) = \bigoplus_{j\in J} \mathcal{O}_L\, du_j \oplus \mathcal{O}_L \frac{du_0}{u_0} \Big/ \left(d(p_J), \frac{d(p_0)}{\pi_K}, \frac{d\pi_K}{\pi_K}, dx \text{ for } x \in \mathcal{O}_K \right).$$

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Theorem 3.4.7. *Let $b_{\log}(L/K)$ be the highest logarithmic ramification break of L/K. Then there exists $\epsilon \in (0, b_{\log}(L/K))$ such that $b_{\log}(L/K) - \epsilon \in \mathbb{Q}$ and, for any admissible $(R_{J^+}) \subset (\delta_{J^+}) \cdot \mathcal{S}_K$, we have $A_K^1[0, \theta^{b_{\log}(L/K)+1-\epsilon}] \times A_K^m[0, \theta^{b_{\log}(L/K)-\epsilon}] \subseteq \mathrm{ET}_{L/K,R_{J^+}}$.*

Proof. The proof is similar to that of Theorem 3.4.2, except that we need to invoke [AS03, Proposition 4.11(2)] to give a bound on $\Omega^1_{\mathcal{O}_L/\mathcal{O}_K}(\log/\log)$; the explicit description of $\Omega^1_{\mathcal{O}_L/\mathcal{O}_K}(\log/\log)$ in Lemma 3.4.6 singles out δ_0 and gives rise to the smaller radius θ^{a+1}. □

3.5 Construction of differential modules

In this subsection, we set up the framework for interpreting ramification filtrations by differential modules.

We remind the reader that we are still assuming Hypotheses 3.1.2 and 3.2.10.

Construction 3.5.1. Let $(R_{J^+}) \subset (\delta_{J^+}) \cdot \mathcal{S}_K$ be admissible. By Lemma 3.2.17, $\Pi : \Pi^{-1}(\mathrm{ET}_{L/K}) \to \mathrm{ET}_{L/K}$ is finite and étale. We call $\mathcal{E} = \Pi_*(\mathcal{O}_{\Pi^{-1}(\mathrm{ET}_{L/K})})$ *a differential module associated to* L/K; it is defined over $\mathrm{ET}_{L/K}$, and the differential module structure is given by

$$\nabla : \mathcal{E} \to \Pi_*(\Omega^1_{\Pi^{-1}(\mathrm{ET}_{L/K})/K}) \simeq \mathcal{E} \otimes_{\mathcal{O}_{\mathrm{ET}_{L/K}}} \Omega^1_{\mathrm{ET}_{L/K}/K} = \mathcal{E} \otimes_{\mathcal{O}_{\mathrm{ET}_{L/K}}} \bigg(\bigoplus_{j\in J^+} \mathcal{O}_{\mathrm{ET}_{L/K}}\, d\delta_j \bigg).$$

Thus, we can define the actions of differential operators $\partial_j = \partial/\partial\delta_j$ for $j \in J^+$ on $\mathcal{E}$ and talk about intrinsic radii $\mathrm{IR}(\mathcal{E}; s_{J^+})$ as in Notation 2.1.13 if $A_K^1[0, \theta^{s_0}] \times \cdots \times A_K^1[0, \theta^{s_m}] \subseteq \mathrm{ET}_{L/K}$.

Proposition 3.5.2. *The following statements are equivalent for $a \in \mathbb{Q}_{>1}$ (respectively, for $a \in \mathbb{Q}_{>0}$).*

(1) *The highest non-logarithmic (respectively, logarithmic) ramification break satisfies $b(L/K) \leqslant a$ (respectively, $b_{\log}(L/K) \leqslant a$).*

(2) *For any (some) admissible $(R_{J^+}) \subset \mathcal{S}_K$ and any rational number $a' > a$,*

$$\#\pi_0^{\mathrm{geom}}(\mathrm{TS}^{a'}_{L/K,R_{J^+}}) = [L : K] \quad (\textit{respectively, } \#\pi_0^{\mathrm{geom}}(\mathrm{TS}^{a'}_{L/K,\log,R_{J^+}}) = [L : K]).$$

(3) *For any (some) admissible $(R_{J^+}) \subset \mathcal{S}_K$, $A_K^{m+1}[0, \theta^a] \subseteq \mathrm{ET}_{L/K,R_{J^+}}$ (respectively, $A_K^1[0, \theta^{a+1}] \times A_K^m[0, \theta^a] \subseteq \mathrm{ET}_{L/K,R_{J^+}}$) and the intrinsic radius of $\mathcal{E}$ over $A_K^{m+1}[0, \theta^a]$ (respectively, $A_K^1[0, \theta^{a+1}] \times A_K^m[0, \theta^a]$) is maximal:*

$$\mathrm{IR}(\mathcal{E}; \underline{a}) = 1 \quad (\textit{respectively, } \mathrm{IR}(\mathcal{E}; a+1, \underline{a}) = 1).$$

Proof. The proof is similar to that of [Xia10, Theorem 3.4.5].

(1) $\Longleftrightarrow$ (2) is immediate from Theorem 3.3.3.

(2) $\Longrightarrow$ (3): for any rational number $a' > a$, (2) implies that for some finite extension K' of K, $\mathrm{TS}^{a'}_{L/K,R_{J^+}} \times_K K'$ (respectively, $\mathrm{TS}^{a'}_{L/K,\log,R_{J^+}} \times_K K'$) has $[L : K]$ connected components and is hence forced to be $[L : K]$ copies of $A_{K'}^{m+1}[0, \theta^{a'}]$ (respectively, $A_{K'}^1[0, \theta^{a'+1}] \times A_{K'}^m[0, \theta^{a'}]$) because Π is finite and flat; in particular, Π is étale there. Therefore, $\mathcal{E} \otimes_K K'$ is a trivial differential module over $A_{K'}^{m+1}[0, \theta^{a'}]$ (respectively, $A_{K'}^1[0, \theta^{a'+1}] \times A_{K'}^m[0, \theta^{a'}]$). As a consequence,

$$\mathrm{IR}(\mathcal{E}; \underline{a'}) = \mathrm{IR}(\mathcal{E} \otimes K'; \underline{a'}) = 1 \quad (\text{respectively, } \mathrm{IR}(\mathcal{E}; a'+1, \underline{a'}) = \mathrm{IR}(\mathcal{E} \otimes_K K'; a'+1, \underline{a'}) = 1).$$

Statement (3) follows from the continuity of intrinsic radii in Proposition 2.1.23(a), upon taking a' to be sufficiently close to a.

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(3) $\Longrightarrow$ (2): statement (3) implies that, for any rational number $a' > a$, $\mathcal{E}$ is a trivial differential module on $A_K^{m+1}[0, \theta^{a'}]$ (respectively, $A_K^1[0, \theta^{a'+1}] \times A_K^m[0, \theta^{a'}]$). Indeed, we have a bijection

$$H^0_\nabla(A_K^{m+1}[0, \theta^{a'}], \mathcal{E}) \xrightarrow{\cong} \mathcal{E}|_{\delta_{J^+}=0} \quad \text{(respectively, } H^0_\nabla(A_K^1[0, \theta^{a'+1}] \times A_K^m[0, \theta^{a'}], \mathcal{E}) \xrightarrow{\cong} \mathcal{E}|_{\delta_{J^+}=0}\text{)}, \tag{3.5.3}$$

whose inverse is given by a Taylor series. (The convergence of the Taylor series is guaranteed by the condition on the intrinsic radii.) This is in fact a ring isomorphism by basic properties of Taylor series. The left-hand side of (3.5.3) is a subring of $\mathcal{O}^{a'}_{\mathrm{TS},L/K,R_{J^+}}$ (respectively, $\mathcal{O}^{a'}_{\mathrm{TS},L/K,\log,R_{J^+}}$), while the right-hand side is just $K\langle u_{J^+}\rangle/(p_{J^+}) \simeq L$. Thus, after the extension of scalars from K to L, we can lift the idempotent elements in $L \otimes_K L \simeq \prod_{g \in G_{L/K}} L_g$ to idempotent elements in $\mathcal{O}^{a'}_{\mathrm{TS},L/K,R_{J^+}} \otimes_K L$ (respectively, $\mathcal{O}^{a'}_{\mathrm{TS},L/K,\log,R_{J^+}} \otimes_K L$). This proves (2). □

Corollary 3.5.4. *Given the differential module $\mathcal{E}$ over $\mathrm{ET}_{L/K,R_{J^+}}$ with respect to some admissible subset $(R_{J^+}) \subset (\delta_{J^+}) \cdot \mathcal{S}_K$, we have*

$$b(L/K) = \min\{s \mid A_K^{m+1}[0, \theta^s] \subseteq \mathrm{ET}_{L/K,R_{J^+}} \text{ and } \mathrm{IR}(\mathcal{E}; \underline{s}) = 1\}$$

and

$$b_{\log}(L/K) = \min\{s \mid A_K^1[0, \theta^{s+1}] \times A_K^m[0, \theta^s] \subseteq \mathrm{ET}_{L/K,R_{J^+}} \text{ and } \mathrm{IR}(\mathcal{E}; s+1, \underline{s}) = 1\}.$$

In other words, $b(L/K)$ (respectively, $b_{\log}(L/K)$) corresponds to the intersection of the boundary of $Z(\mathcal{E})$ (cf. Proposition 2.1.23(c)) with the line defined by $s_0 = \cdots = s_m$ (respectively, $s_0 - 1 = s_1 = \cdots = s_m$).

Proof. By Theorems 3.4.2 and 3.4.7, $\mathrm{ET}_{L/K,R_{J^+}}$ is large enough to use for pinning down the exact boundary of $Z(\mathcal{E})$. The corollary follows immediately from Propositions 3.5.2 and 2.1.23. □

3.6 Recursive thickening spaces

In this subsection, we introduce a generalization of thickening spaces. This will give us some freedom when changing the base field.

In this subsection, we continue to assume Hypotheses 3.1.2 and 3.2.10.

Construction 3.6.1. This is a variant of Construction 3.1.6. First, filter the (inseparable) extension l/k by elementary p-extensions

$$k = k_0 \subsetneq k_1 \subsetneq \cdots \subsetneq k_r = l$$

where, for each $\lambda = 1, \ldots, r$, $k_\lambda = k_{\lambda-1}(\bar{\mathfrak{c}}_\lambda)$ with $\bar{\mathfrak{c}}_\lambda^p = \bar{\mathfrak{b}}_\lambda \in k_{\lambda-1}$. Write $\Lambda = \{1, \ldots, r\}$. Pick lifts $\mathfrak{c}_\Lambda$ of $\bar{\mathfrak{c}}_\Lambda$ in $\mathcal{O}_L$. Let $e = e_0, \ldots, e_{r_0} = 1$ be a strictly decreasing sequence of integers such that $e_i | e_{i-1}$ for $1 \leqslant i \leqslant r_0$. Set $I = \{1, \ldots, r_0\}$. For each $i \in I$, pick an element $\pi_{L,i}$ in $\mathcal{O}_L$ with valuation e_i; in particular, we take $\pi_{L,r_0} = \pi_L$. It is easy to see that the $(\mathfrak{c}_\Lambda, \pi_{L,I})$ generate $\mathcal{O}_L$ over $\mathcal{O}_K$. So we have an isomorphism

$$\Delta : \mathcal{O}_K\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle/\mathfrak{I} \xrightarrow{\sim} \mathcal{O}_L$$

such that $\mathfrak{u}_{0,i} \mapsto \pi_{L,i}$ for $i \in I$ and $\mathfrak{u}_\lambda \mapsto \mathfrak{c}_\lambda$ for $\lambda \in \Lambda$, where $\mathfrak{I}$ is some proper ideal and we use the same Δ as in Construction 3.1.6. Moreover,

$$\left\{ \mathfrak{u}_{0,I}^{\mathfrak{e}_{0,I}} \mathfrak{u}_\Lambda^{\mathfrak{e}_\Lambda} \;\middle|\; \mathfrak{e}_{0,i} \in \left\{0, \ldots, \frac{e_{i-1}}{e_i} - 1\right\} \text{ for all } i \in I \text{ and } \mathfrak{e}_\lambda \in \{0, \ldots, p-1\} \text{ for all } \lambda \in \Lambda \right\} \tag{3.6.2}$$

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forms a basis of $\mathcal{O}_K\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle/\mathfrak{I}$ as a free $\mathcal{O}_K$-module, which we will refer to later as the *standard basis*.

We endow $\mathcal{O}_K[\mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda]$ with the following norm: for $h = \sum_{\mathfrak{e}_{0,I},\mathfrak{e}_\Lambda} \alpha_{\mathfrak{e}_{0,I},\mathfrak{e}_\Lambda} \mathfrak{u}_{0,I}^{\mathfrak{e}_{0,I}} \mathfrak{u}_\Lambda^{\mathfrak{e}_\Lambda}$ with $\alpha_{\mathfrak{e}_{0,I},\mathfrak{e}_\Lambda} \in \mathcal{O}_K$, set

$$|h| = \max_{\mathfrak{e}_{0,I},\mathfrak{e}_\Lambda} \{|\alpha_{\mathfrak{e}_{0,I},\mathfrak{e}_\Lambda}| \cdot \theta^{(\mathfrak{e}_{0,1}\cdot e_1 + \cdots + \mathfrak{e}_{0,r_0}\cdot e_{r_0})/e}\}.$$

For $a \in (1/e)\mathbb{Z}_{\geqslant 0}$, we use $\mathfrak{N}^a$ to denote the set consisting of elements in $\mathcal{O}_K[\mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda]$ with norm less than or equal to θ^a; this is, in fact, an ideal.

In $\mathcal{O}_K\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle/\mathfrak{I}$, we can write $\mathfrak{u}_{0,i}^{e_{i-1}/e_i}$ for $i \in I$ and $\mathfrak{u}_\Lambda^p$ in terms of the basis (3.6.2). This gives a set of generators of $\mathfrak{I}$:

$$\begin{aligned}
\mathfrak{p}_{0,1} &\in \mathfrak{u}_{0,1}^{e/e_1} - \mathfrak{d}_1\pi_K + \mathfrak{N}^{1+1/e}\cdot \mathcal{O}_K[\mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda],\\
\mathfrak{p}_{0,i} &\in \mathfrak{u}_{0,i}^{e_{i-1}/e_i} - \mathfrak{d}_i\mathfrak{u}_{0,i-1} + \mathfrak{N}^{(e_{i-1}+1)/e}\cdot \mathcal{O}_K[\mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda] \quad \text{for } i \in I\backslash\{1\},\\
\mathfrak{p}_\lambda &\in \mathfrak{u}_\lambda^p - \tilde{\mathfrak{b}}_\lambda + \mathfrak{N}^{1/e}\cdot \mathcal{O}_K[\mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda],
\end{aligned}$$

where $\mathfrak{d}_I$ are some elements in $\mathcal{O}_K[\mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda]$ whose images under Δ are invertible in $\mathcal{O}_L$ and, for each λ, $\tilde{\mathfrak{b}}_\lambda$ is some element in $\mathcal{O}_K[\mathfrak{u}_1, \dots, \mathfrak{u}_{\lambda-1}]$ whose image under Δ reduces to $\bar{\mathfrak{b}}_\lambda \in k_{\lambda-1}$ modulo π_L.

We say that $\mathfrak{p}_\lambda$ *corresponds* to the extension $k_\lambda/k_{\lambda-1}$.

Definition 3.6.3. As in Definition 3.2.12, we define

$$\mathfrak{S}_K = \mathcal{R}_K\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle = \mathcal{O}_K[\![\delta_0/\pi_K, \delta_J]\!]\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle.$$

For $\omega \in \frac{1}{e}\mathbb{N}\cap[1, \beta_K]$, we say that a set of elements $(\mathfrak{R}_{0,I}, \mathfrak{R}_\Lambda) \subset (\delta_{J^+})\cdot \mathfrak{S}_K$ *has error gauge* $\geqslant\omega$ if $\mathfrak{R}_{0,i} \in (\mathfrak{N}^{\omega-1+e_i/e}\delta_0, \mathfrak{N}^{\omega+e_i/e}\delta_J)\cdot \mathfrak{S}_K$ for $i \in I$ and $\mathfrak{R}_\lambda \in (\mathfrak{N}^{\omega-1}\delta_0, \mathfrak{N}^{\omega}\delta_J)\cdot \mathfrak{S}_K$ for $\lambda \in \Lambda$. The subset $(\mathfrak{R}_{0,I}, \mathfrak{R}_\Lambda) \subset (\delta_{J^+})\cdot \mathfrak{S}_K$ is said to be *admissible* if it has error gauge $\geqslant$1.

Let $(\mathfrak{R}_{0,I}, \mathfrak{R}_\Lambda) \subset (\delta_{J^+})\cdot \mathfrak{S}_K$ be admissible. For $a \in \mathbb{Q}_{>1}$, we define the *(non-logarithmic) recursive thickening space (of level a)* $\mathrm{TS}^a_{L/K,\mathfrak{R}_{0,I},\mathfrak{R}_\Lambda}$ to be the rigid space associated to

$$\mathcal{O}^a_{\mathrm{TS},L/K,\mathfrak{R}_{0,I},\mathfrak{R}_\Lambda} = K\langle \pi_K^{-a}\delta_{J^+}\rangle\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle/(\psi(\mathfrak{p}_{0,I}) + \mathfrak{R}_{0,I}, \psi(\mathfrak{p}_\Lambda) + \mathfrak{R}_\Lambda).$$

For $a \in \mathbb{Q}_{>0}$, we define the *logarithmic recursive thickening space (of level a)* $\mathrm{TS}^a_{L/K,\log,\mathfrak{R}_{0,I},\mathfrak{R}_\Lambda}$ to be the rigid space associated to

$$\mathcal{O}^a_{\mathrm{TS},L/K,\log,\mathfrak{R}_{0,I},\mathfrak{R}_\Lambda} = K\langle \pi_K^{-a-1}\delta_0, \pi_K^{-a}\delta_J\rangle\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle/(\psi(\mathfrak{p}_{0,I}) + \mathfrak{R}_{0,I}, \psi(\mathfrak{p}_\Lambda) + \mathfrak{R}_\Lambda).$$

We still use Δ to denote the natural homomorphism

$$\mathfrak{S}_K/(\psi(\mathfrak{p}_{0,I}) + \mathfrak{R}_{0,I}, \psi(\mathfrak{p}_\Lambda) + \mathfrak{R}_\Lambda) \xrightarrow{\mathrm{mod}(\delta_0/\pi_K,\delta_J)} \mathcal{O}_K\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle/(\mathfrak{p}_{0,I}, \mathfrak{p}_\Lambda) \xrightarrow[\simeq]{\Delta} \mathcal{O}_L;$$

we use $\overline{\Delta}$ to denote the composition with the reduction $\mathcal{O}_L \to l$.

Lemma 3.6.4. *Let* $(\mathfrak{R}_{0,I}, \mathfrak{R}_\Lambda) \subset (\delta_{J^+})\cdot \mathfrak{S}_K$ *be admissible. Then (3.6.2) forms a basis of* $\mathfrak{S}_K/(\psi(\mathfrak{p}_{0,I}) + \mathfrak{R}_{0,I}, \psi(\mathfrak{p}_\Lambda) + \mathfrak{R}_\Lambda)$ *as a free* $\mathcal{R}_K$*-module, which will be referred to later as the* standard basis. *As a consequence, it constitutes a basis of* $\mathcal{O}^a_{\mathrm{TS},L/K,\mathfrak{R}_{0,I},\mathfrak{R}_\Lambda}$ *(respectively,* $\mathcal{O}^a_{\mathrm{TS},L/K,\log,\mathfrak{R}_{0,I},\mathfrak{R}_\Lambda}$*) as a free module over* $K\langle \pi_K^{-a}\delta_{J^+}\rangle$ *(respectively* $K\langle \pi_K^{-a-1}\delta_0, \pi_K^{-a}\delta_J\rangle$*).*

Proof. The proof is the same as that of Lemma 3.2.17. □

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Example 3.6.5. The construction of the thickening spaces in Definition 3.2.13 is a special case of the above construction. If we start with a uniformizer π_L, a p-basis c_J, and relations p_{J^+} in Construction 3.1.6, the following dictionary translates the information to fit Construction 3.6.1.

$$\begin{aligned}\pi_{L,I} &\longleftrightarrow \pi_L \quad (I=\{1\}),\\ \mathfrak{c}_\Lambda &\longleftrightarrow c_1, c_1^p, \dots, c_1^{p^{r_1-1}}, c_2, c_2^p, \dots, c_m^{p^{r_m-1}},\\ \mathfrak{p}_{0,I}, \mathfrak{p}_\Lambda &\longleftrightarrow \text{the ones determined by } \mathfrak{c}_\Lambda \text{ and } \pi_{L,I},\\ \mathfrak{R}_{0,I} &\longleftrightarrow R_0,\\ \mathfrak{R}_\lambda &\longleftrightarrow R_j \text{ when } \lambda \text{ corresponds to some } c_j^{p^{r_j-1}}, \text{ and 0 otherwise.}\end{aligned}$$

Moreover, this construction preserves the error gauge.

Conversely, we have the following.

Proposition 3.6.6. *Let $(\mathfrak{R}_{0,I}, \mathfrak{R}_\Lambda) \subset (\delta_{J^+}) \cdot \mathfrak{S}_K$ be admissible with error gauge $\geqslant \omega \in (1/e)\mathbb{N} \cap [1, \beta_K]$. Then, for any choice of c_J and π_L as in Construction 3.1.6, there exists an $\mathcal{R}_K$-isomorphism*

$$\Theta : \mathcal{S}_K/(\psi(p_{J^+}) + R_{J^+}) \xrightarrow{\sim} \mathfrak{S}_K/(\psi(\mathfrak{p}_{0,I}) + \mathfrak{R}_{0,I}, \psi(\mathfrak{p}_\Lambda) + \mathfrak{R}_\Lambda), \tag{3.6.7}$$

for some admissible R_{J^+} with error gauge $\geqslant \omega$, such that Θ mod $(\delta_0/\pi_K, \delta_J)$ induces the identity map if we identify both sides (modulo $(\delta_0/\pi_K, \delta_J)$) with $\mathcal{O}_L$ via Δ. This gives rise to isomorphisms between the recursive thickening spaces and the thickening spaces:

$$\mathrm{TS}^a_{L/K,\mathfrak{R}_{0,I},\mathfrak{R}_\Lambda} \simeq \mathrm{TS}^a_{L/K,R_{J^+}} (a \in \mathbb{Q}_{>1}) \quad \text{and} \quad \mathrm{TS}^a_{L/K,\log,\mathfrak{R}_{0,I},\mathfrak{R}_\Lambda} \simeq \mathrm{TS}^a_{L/K,\log,R_{J^+}} \quad (a \in \mathbb{Q}_{>0}).$$

Proof. For each $j \in J$, we express c_j as a polynomial $\tilde{\mathfrak{c}}_j$ in $\mathfrak{u}_{0,I}$ and $\mathfrak{u}_\Lambda$ with coefficients in $\mathcal{O}_K$ via $\Delta^{-1} : \mathcal{O}_L \xrightarrow{\sim} \mathcal{O}_K\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle/(\mathfrak{p}_{0,I}, \mathfrak{p}_\Lambda)$. We define a continuous homomorphism $\widetilde{\Theta} : \mathcal{S}_K \to \mathfrak{S}_K/(\psi(\mathfrak{p}_{0,I}) + \mathfrak{R}_{0,I}, \psi(\mathfrak{p}_\Lambda) + \mathfrak{R}_\Lambda)$ by setting $\widetilde{\Theta}(u_j) = \psi(\tilde{\mathfrak{c}}_j)$ for $j \in J$ and $\widetilde{\Theta}(u_0) = \mathfrak{u}_{0,r_0}$. It is then obvious that for $a \in \frac{1}{e}\mathbb{Z}_{\geqslant 0}$, $\widetilde{\Theta}(N^a \cdot \mathcal{S}_K) \subset \mathfrak{N}^a \cdot \mathfrak{S}_K$.

We need to determine R_{J^+}. For each fixed $j_0 \in J^+$, since $\Delta(p_{j_0}(u_{J^+})) = 0$ we can write

$$p_{j_0}(\mathfrak{u}_{0,r_0}, \tilde{\mathfrak{c}}_J) = \sum_{i \in I} \mathfrak{h}_{0,i}\mathfrak{p}_{0,i} + \sum_{\lambda \in \Lambda} \mathfrak{h}_\lambda \mathfrak{p}_\lambda \quad \text{in } \mathcal{O}_K\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle,$$

for some $\mathfrak{h}_{0,i}, \mathfrak{h}_\lambda \in \mathcal{O}_K\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle$, for $i \in I$ and $\lambda \in \Lambda$. Moreover, when $j_0 = 0$, we can require that $\mathfrak{h}_{0,i} \in \mathfrak{N}^{1-e_{i-1}/e} \cdot \mathcal{O}_K\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle$ and $\mathfrak{h}_\lambda \in \mathfrak{N}^1 \cdot \mathcal{O}_K\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle$ for $i \in I$ and $\lambda \in \Lambda$. Thus, we expect to define R_{j_0} so that, under $\widetilde{\Theta}$, it is mapped to

$$\begin{aligned}-\psi(p_{j_0})(\widetilde{\Theta}(u_{J^+})) &= -\sum_{i \in I} \psi(\mathfrak{h}_{0,i})\psi(\mathfrak{p}_{0,i}) - \sum_{\lambda \in \Lambda} \psi(\mathfrak{h}_\lambda)\psi(\mathfrak{p}_\lambda) + \mathfrak{E}\\ &= -\sum_{i \in I} \psi(\mathfrak{h}_{0,i})(-\mathfrak{R}_{0,i}) - \sum_{\lambda \in \Lambda} \psi(\mathfrak{h}_\lambda)(-\mathfrak{R}_\lambda) + \mathfrak{E}\\ &\in \begin{cases} (\mathfrak{N}^\omega \delta_0, \mathfrak{N}^{\omega+1}\delta_J) \cdot \mathfrak{S}_K & j_0 = 0,\\ (\mathfrak{N}^{\omega-1}\delta_0, \mathfrak{N}^\omega \delta_J) \cdot \mathfrak{S}_K & j_0 \in J,\end{cases}\end{aligned}$$

where $\mathfrak{E} \in (\mathfrak{N}^{\beta_K}\delta_0, \mathfrak{N}^{(\beta_K+1)}\delta_J) \cdot \mathfrak{S}_K$ if $j_0 = 0$ and $\mathfrak{E} \in (\mathfrak{N}^{(\beta_K-1)}\delta_0, \mathfrak{N}^{\beta_K}\delta_J) \cdot \mathfrak{S}_K$ if $j_0 \in J$; these correspond to the error terms that come from ψ failing to be a homomorphism (see Proposition 3.2.8).

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Thus, we can find polynomials $q_0, \dots, q_m \in \mathcal{O}_K[u_{J^+}]$ such that

$$q_0 \in \begin{cases} N^\omega \cdot \mathcal{S}_K & j_0 = 0, \\ N^{\omega-1} \cdot \mathcal{S}_K & j_0 \in J, \end{cases} \qquad q_1, \dots, q_m \in \begin{cases} N^{\omega+1} \cdot \mathcal{S}_K & j_0 = 0, \\ N^{\omega} \cdot \mathcal{S}_K & j_0 \in J, \end{cases}$$

and

$$-\psi(p_{j_0})(\widetilde{\Theta}(u_{J^+})) - \widetilde{\Theta}(q_0\delta_0 + \cdots + q_m\delta_m) \in \begin{cases} (\delta_0/\pi_K, \delta_J)(\mathfrak{N}^\omega\delta_0, \mathfrak{N}^{\omega+1}\delta_J) \cdot \mathfrak{S}_K & j_0 = 0, \\ (\delta_0/\pi_K, \delta_J)(\mathfrak{N}^{\omega-1}\delta_0, \mathfrak{N}^{\omega}\delta_J) \cdot \mathfrak{S}_K & j_0 \in J. \end{cases}$$

Further, we can similarly find approximations of the coefficients of $\delta_j\delta_{j'}$, for $j, j' \in J^+$. Iterating this approximation gives the expressions for R_{J^+}; they clearly have error gauge $\geqslant \omega$.

By construction, $\widetilde{\Theta}$ factors through the quotient by $\psi(p_{J^+}) + R_{J^+}$; we then obtain the homomorphism Θ as in (3.6.7). The surjectivity of Θ follows from the surjectivity modulo $(\delta_0/\pi_K, \delta_J)$, which is the identity via Δ. Moreover, a surjective morphism between two finite free modules of the same rank over a noetherian base ring is automatically an isomorphism. The theorem is thus proved. □

Remark 3.6.8. The isomorphism Θ is not unique. Basically, $\Theta(u_0)$ mod $(\mathfrak{N}^\omega\delta_0, \mathfrak{N}^{\omega+1}\delta_J) \cdot \mathfrak{S}_K$ and $\Theta(u_j)$ mod $(\mathfrak{N}^{\omega-1}\delta_0, \mathfrak{N}^{\omega}\delta_J) \cdot \mathfrak{S}_K$ for $j \in J$ are fixed; any lifts of them will give a desired isomorphism (with different (R_{J^+})).

LEMMA 3.6.9. *Let $(\mathfrak{R}_{0,I}, \mathfrak{R}_\Lambda) \subset (\delta_{J^+}) \cdot \mathfrak{S}_K$ be admissible. Then an element*

$$h \in \mathfrak{S}_K/(\psi(\mathfrak{p}_{0,I}) + \mathfrak{R}_{0,I}, \psi(\mathfrak{p}_\Lambda) + \mathfrak{R}_\Lambda)$$

is invertible if and only if $\Delta(h) \in \mathcal{O}_L^\times$. In particular, $\mathfrak{u}_{0,r_0}^e/\pi_K$ is invertible.

Proof. The necessity is obvious. To see the sufficiency, we construct the inverse of h directly. Let $h^{(-1)}$ be a lift of $\Delta(h^{-1}) \in \mathcal{O}_L^\times$ in $\mathcal{O}_K\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle$. We have $\Delta(1 - h^{(-1)}h) = 0$ and hence $1 - h^{(-1)}h = g \in (\delta_{J^+}) \cdot \mathfrak{S}_K$. Thus,

$$\frac{1}{h} = \frac{h^{(-1)}}{1-g} = h^{(-1)} \cdot (1 + g + g^2 + \cdots).$$

The series converges to the inverse of h. □

4. Hasse–Arf theorems

4.1 Generic pth roots

The notion of generic pth roots was first (implicitly) introduced by Borger in [Bor04]. Kedlaya [Ked07] realized that in the equal characteristic case, adding generic pth roots to the field extension will not change the (differential) non-logarithmic ramification filtration; hence, one can prove the non-logarithmic Hasse–Arf theorem by reducing to the perfect residue field case.

In this subsection, we continue to assume Hypotheses 3.1.2 and 3.2.10, except in Proposition 4.1.8.

Notation 4.1.1. Let x be transcendental over K. Define $K(x)^\wedge$ to be the completion of $K(x)$ with respect to the 1-Gauss norm, and define K' to be the completion of the maximal unramified extension of $K(x)^\wedge$. Set $L' = K'L$.

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Lemma 4.1.2. *Let $L(x)^{\wedge}$ be the completion with respect to the* 1*-Gauss norm. Then L' is the completion of the maximal unramified extension of $L(x)^{\wedge}$. In particular, the residue field of L' is $l' = k(x)^{\mathrm{sep}} \cdot l$, which is separably closed.*

Proof. First, $L(x)^{\wedge} = LK(x)^{\wedge}$ because the latter is complete and is dense in the former. So, it suffices to prove that L' is complete and has separable residue field. Since L'/K' is finite, L' is complete. Moreover, the residue field l' of L' is separably closed because it is a finite extension of a separably closed field $k(x)^{\mathrm{sep}}$. □

Proposition 4.1.3. *The highest ramification breaks do not change if we make a base change from K to K'. In other words, $b(L/K) = b(L'/K')$ and $b_{\log}(L/K) = b_{\log}(L'/K')$.*

Proof. Since π_L is a uniformizer of L' and $\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$ surjects onto l' by the previous lemma, we have $\mathcal{O}_{L'} = \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$. The result follows from Proposition 2.2.5(4′). □

Definition 4.1.4. Let b_{j_0} be an element in a p-basis of K. We will often need to make a base change $K \hookrightarrow \widetilde{K} = K'((b_{j_0} + x\pi_K)^{1/p})$, a process which we shall refer to as *adding a generic pth root (of b_{j_0})*. It is clear that the absolute ramification degree $\beta_{\widetilde{K}}$ equals β_K. If we begin with a finite field extension L/K, adding a generic pth root will mean considering the extension $\widetilde{L} = L\widetilde{K}/\widetilde{K}$. We have $G_{\widetilde{L}/\widetilde{K}} = G_{L/K}$, as $\widetilde{K}$ is linearly independent of L over K. By convention, we take $\pi_{\widetilde{K}} = \pi_K$ as $\widetilde{K}/K$ is unramified. We provide $\widetilde{K}$ with a p-basis $\{b_{J\setminus\{j_0\}}, (b_{j_0} + x\pi_K)^{1/p}, x\}$, which has one more element than the original p-basis.

Proposition 4.1.5. *Let L/K be as in Hypothesis 3.1.2. Then, after finitely many operations of adding generic pth roots, the field extension we began with becomes a non-fiercely ramified extension; that is, the residue field extension is trivial.*

Proof. The proof is almost identical to that of [Xia10, Proposition 5.2.3], which is stated in terms of an equal-characteristic complete discrete valuation field and adding p^{∞}th roots (see [Xia10, Definition 5.2.2]).

First, the tamely ramified part is always preserved under these operations. So we can assume that L/K is totally wildly ramified and hence that the Galois group $G_{L/K}$ is a p-group. We can filter the extension L/K as $K = K_0 \subset \cdots \subset K_n = L$, where K_i/K_{i-1} is a (wildly ramified) $\mathbb{Z}/p\mathbb{Z}$-Galois extension and K_i/K is Galois for each $i = 1, \ldots, n$. Each of these subextensions has:

(a) either inseparable residue field extension (and hence naïve ramification degree one);

(b) or trivial residue field extension (and hence naïve ramification degree p).

Let i_0 be the maximal number such that K_i/K_{i-1} has trivial residual extension for $i = 1, \ldots, i_0$. Obviously, adding a generic pth root does not decrease i_0, because after adding a generic pth root the naïve ramification degree of $\widetilde{K}_{i_0}/\widetilde{K}$ still equals the degree p^{i_0}. Now, it suffices to show that after finitely many operations of adding generic pth roots, K_{i_0+1}/K_{i_0} has trivial residue field extension (if $i_0 < n$); this would be enough to deduce the proposition. Suppose the contrary.

Let $g \in G_{K_{i_0+1}/K_{i_0}} \simeq \mathbb{Z}/p\mathbb{Z}$ be a generator. We claim that $\gamma = \min_{x \in \mathcal{O}_{K_{i_0+1}}} (v_{K_{i_0+1}}(g(x) - x))$ decreases by at least 1 after adding generic pth roots of each of the elements in the p-basis. This would suffice to conclude the argument, because γ is always a non-negative integer.

Let z be a generator of $\mathcal{O}_{K_{i_0+1}}$ as an $\mathcal{O}_{K_{i_0}}$-algebra. It satisfies the equation

$$z^p + a_1 z^{p-1} + \cdots + a_p = 0 \tag{4.1.6}$$

where $a_1, \ldots, a_{p-1} \in \mathfrak{m}_{K_{i_0}}$ and $a_p \in \mathcal{O}_{K_{i_0}}^\times$ with $\bar{a}_p \in k_{i_0}^\times \backslash (k_{i_0}^\times)^p = k^\times \backslash (k^\times)^p$. It is easy to see that $\gamma = v_{K_{i_0}}(g(z) - z)$.

Adding generic pth roots of each of the elements in the p-basis gives us a field $\widehat{K}$. Now, the field extension $\widehat{K}K_{i_0+1}/\widehat{K}K_{i_0}$ is also generated by z as above. But we can write $a_p = \alpha^p + \beta$ for $\alpha \in \mathcal{O}_{\widehat{K}K_{i_0}}$ and $\beta \in \mathfrak{m}_{\widehat{K}K_{i_0}}$. Hence, if we substitute $z' = z + \alpha$ into (4.1.6), we get $z'^p + a_1' z'^{p-1} + \cdots + a_p' = 0$, with $a_1', \ldots, a_p' \in \mathfrak{m}_{\widehat{K}K_{i_0}}$. Thus, $v_{\widehat{K}K_{i_0+1}}(z') > 0$. By the assumption that the extension $\widehat{K}K_{i_0+1}/\widehat{K}K_{i_0}$ has naïve ramification degree one, $\pi_{K_{i_0}}$ is a uniformizer for $\widehat{K}K_{i_0+1}$, and hence $z'/\pi_{K_{i_0}}$ lies in $\mathcal{O}_{\widehat{K}K_{i_0+1}}$. Thus,

$$\begin{aligned}\gamma' = \min_{x \in \mathcal{O}_{\widehat{K}K_{i_0+1}}} (v_{\widehat{K}K_{i_0+1}}(g(x) - x)) &\leqslant v_{\widehat{K}K_{i_0+1}}(g(z'/\pi_{K_{i_0}}) - z'/\pi_{K_{i_0}}) \\ &= v_{K_{i_0+1}}(g(z) - z) - 1 = \gamma - 1.\end{aligned}$$

This proves the claim and hence the proposition. □

Remark 4.1.7. It is worth pointing out that after these operations, the number of elements in the p-basis of the resulting field will be greater than that of the original field.

For the following theorem, we do not assume either Hypothesis 3.1.2 or Hypothesis 3.2.10.

Proposition 4.1.8. *Fix $\beta_K \in \mathbb{N}_{>1}$. Assume that for any complete discrete valuation field K of mixed characteristic with absolute ramification degree β_K and for any field extension L/K satisfying Hypothesis 3.1.2, the highest non-logarithmic ramification break is invariant under the operation of adding a generic pth root. Then, for all complete discrete valuation fields K of mixed characteristic and with absolute ramification degree β_K, we have that:*

(1) $\mathrm{Art}(\rho)$ *is a non-negative integer for any representation* $\rho : G_K \to \mathrm{GL}(V_\rho)$ *with finite monodromy;*

(2) *the subquotients* $\mathrm{Fil}^a G_K / \mathrm{Fil}^{a+} G_K$ *are trivial if* $a \notin \mathbb{Q}$ *and are abelian groups killed by* p *if* $a \in \mathbb{Q}_{>1}$.

Proof. (1) Since the conductor is additive and is invariant upon base change to the completion of the maximal unramified extension of K (Proposition 2.2.5(4)), we may assume that ρ is irreducible and factors exactly through the Galois group of a totally ramified Galois extension L/K. We may also assume that the residue field k is imperfect and that the extension is wildly ramified since the classical case is well-known (Propositions 2.2.5(7) and 2.2.14). We need only show that $\mathrm{Art}(\rho) = b(L/K) \cdot \dim \rho \in \mathbb{Z}$.

Now we reduce to the finite p-basis case. Choose a finite subset $J_0 \subset J$ such that $k(\bar{b}_j^{1/p})$ is linearly independent of l for any $j \in J \backslash J_0$. Pick lifts $b_j \in \mathcal{O}_K$ of $\bar{b}_j$ for each $j \in J \backslash J_0$. Define $K_1 = K(b_j^{1/p^n}; j \in J \backslash J_0, n \in \mathbb{N})^\wedge$ and $L_1 = K_1 L$. It is easy to see that $[L_1 : K_1] = [L : K]$, $e_{L_1/K_1} \geqslant e_{L/K}$, and $[l_1 : k_1] \geqslant [l : k]$, where k_1 and l_1 are the residue fields of K_1 and L_1, respectively. Thus, all the inequalities are forced to be equalities. This implies $G_{L_1/K_1} = G_{L/K}$ and $\mathcal{O}_{L_1} = \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K_1}$. By Proposition 2.2.5(4′), $b(L_1/K_1) = b(L/K)$. Therefore, we may reduce to the case where Hypothesis 3.1.2 holds.

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Since adding generic pth roots does not change β_K, the condition of this proposition says that $b(L/K)$ is invariant under the operation of adding generic pth roots. By Proposition 4.1.5, we may assume that L/K is non-fiercely ramified as the base changes do not change the conductor. In this case, Proposition 2.2.5(4′) implies that replacing K by $K(b_j^{1/p^n}; j \in J, n \in \mathbb{N})^\wedge$ does not change the conductor. Hence, we can reduce to the classical case; the statement then follows from Proposition 2.2.14.

Now we prove (2), following the idea of [Ked07, Theorem 3.5.13]. Let L be a finite Galois extension of K with Galois group $G_{L/K}$; then we obtain an induced filtration on $G_{L/K}$. It suffices to check that $\mathrm{Fil}^a G_{L/K}/\mathrm{Fil}^{a+} G_{L/K}$ is abelian and killed by p; moreover, we may quotient further to reduce to the case where $\mathrm{Fil}^{a+} G_{L/K}$ is the trivial group but $\mathrm{Fil}^a G_{L/K}$ is not. As above, we may reduce to the classical case because the ramification break of any intermediate extension between L and K is also preserved under the operations above. The statement follows from Proposition 2.2.14. □

4.2 Base change for generic pth roots

In this subsection, we prove the key technical result, Theorem 4.2.9. We retain Hypotheses 3.1.2 and 3.2.10. When proving the main theorem, we will need a technical assumption, Hypothesis 4.2.8, which is satisfied by any recursive thickening space coming from a thickening space, owing to Example 3.6.5.

Notation 4.2.1. For this subsection, fix $j_0 \in J$ and $n \in \mathbb{N}$ coprime to p. As in Definition 4.1.4, let $K(x)^\wedge$ be the completion of $K(x)$ with respect to the 1-Gauss norm, and let K' be the completion of the maximal unramified extension of $K(x)^\wedge$. Let $\widetilde{K} = K'((b_{j_0} + x\pi_K^n)^{1/p})$ and $\widetilde{L} = L\widetilde{K}$. Write $\beta_{j_0} = (b_{j_0} + x\pi_K^n)^{1/p}$ for simplicity. Denote the residue fields of $\widetilde{K}$ and $\widetilde{L}$ by $\tilde{k}$ and $\tilde{l}$, respectively.

Lemma 4.2.2. *If $\bar{b}_{j_0}^{1/p} \notin l$, we have the ramification break $b(\widetilde{L}/\widetilde{K}) = b(L/K)$.*

Proof. Since $\tilde{l} = \tilde{k}l$, we have $\mathcal{O}_{\widetilde{L}} = \mathcal{O}_{\widetilde{K}} \otimes_{\mathcal{O}_K} \mathcal{O}_L$; the lemma follows from Proposition 2.2.5(4′). □

So we need to deal with the non-trivial case where $\bar{b}_{j_0}^{1/p} \in l$. We present an elementary lemma first.

Lemma 4.2.3. *Assume $s \in \mathbb{Z}_{\geqslant 0}$ and $\beta_K > s/e + 1$. Let $\pi \in \mathcal{O}_L$ be such that $\pi/\pi_L^s \in \mathcal{O}_L^\times$. Then, there exist no $\mu \in \mathcal{O}_{L'}$ and $b \in \mathcal{O}_L$ such that $\mu^p - b - x\pi \in \pi_L^{s+1}\mathcal{O}_{L'}$.*

Proof. We use induction on s. When $s = 0$, this statement is equivalent to $x \notin \tilde{l}^p + l$, which is true. Assume that the statement is true for $s < s_0$ with $s_0 \in \mathbb{Z}_{>0}$. Suppose that for $\pi \in \pi_L^{s_0}\mathcal{O}_L^\times$, we can find $\mu \in \mathcal{O}_{L'}$ and $b \in \mathcal{O}_L$ such that $\mu^p + b - x\pi \in \pi_L^{s_0+1}\mathcal{O}_{L'}$. Then we must have $\mu^p \equiv b \bmod \pi_L$. Since $\tilde{l}^p \cap l = l^p$, we may write $\mu = \mu_0 + \pi_L\mu_1$ with $\mu_0 \in \mathcal{O}_L$ and $\mu_1 \in \pi_L\mathcal{O}_{L'}$ such that $b \equiv \mu_0^p \bmod \pi_L$. Thus,

$$\mu^p - b - x\pi \equiv \pi_L^p\mu_1^p + (\mu_0^p - b) + x\pi \bmod p.$$

Since $\beta_K > s_0/e + 1$ and x is transcendental over L, we must have $\mu_0^p - b \in \pi_L^p\mathcal{O}_L$ and $s_0 \geqslant p$. We would then have

$$\mu_1^p + \frac{\mu_0^p - b}{\pi_L^p} + x\frac{\pi}{\pi_L^p} \in \pi_L^{s-p+1}\mathcal{O}_{L'},$$

which should not exist by the inductive hypothesis. This is a contradiction. □

Notation 4.2.4. From now on, we write ψ_K instead of ψ, as we will be considering the ψ-functions for different fields.

Notation 4.2.5. Write $\mathcal{R}_{\widetilde{K}} = \mathcal{O}_{\widetilde{K}}[\![\eta_0/\pi_K, \eta_J, \eta_{m+1}]\!]$. Applying Construction 3.2.1 to $\widetilde{K}$ gives a function $\psi_{\widetilde{K}} : \mathcal{O}_{\widetilde{K}} \to \mathcal{R}_{\widetilde{K}}$ which is an approximate homomorphism modulo the ideal $I_{\widetilde{K}} = p(\eta_0/\pi_K, \eta_{J\cup\{m+1\}}) \cdot \mathcal{R}_K$.

Lemma 4.2.6. *There exists a unique continuous $\mathcal{O}_K$-homomorphism $f^* : \mathcal{R}_K \to \mathcal{R}_{\widetilde{K}}$ such that $f^*(\delta_j) = \eta_j$ for $j \in J^+\backslash\{j_0\}$ and $f^*(\delta_{j_0}) = (\beta_{j_0} + \eta_{j_0})^p - (x + \eta_{m+1})(\pi_K + \eta_0)^n - b_{j_0}$. It gives the following approximately commutative diagram modulo $I_{\widetilde{K}}$.*

$$\begin{array}{ccc} \mathcal{O}_K & \xrightarrow{\psi_K} & \mathcal{O}_K[\![\delta_0/\pi_K, \delta_J]\!] = \mathcal{R}_K \\ \big\downarrow & & \big\downarrow f^* \\ \mathcal{O}_{\widetilde{K}} & \xrightarrow{\psi_{\widetilde{K}}} & \mathcal{O}_{\widetilde{K}}[\![\eta_0/\pi_K, \eta_{J\cup\{m+1\}}]\!] = \mathcal{R}_{\widetilde{K}} \end{array} \tag{4.2.7}$$

For $a > 1$, f^* gives a morphism $f : A^{m+2}_{\widetilde{K}}[0, \theta^a] \to A^{m+1}_K[0, \theta^a]$.

Proof. This follows immediately from Proposition 3.2.8. □

Hypothesis 4.2.8. For the next theorem, we assume that in Construction 3.6.1 there exists $\lambda_0 \in \Lambda$ such that the field extension $k_{\lambda_0}/k_{\lambda_0-1}$ is given by $k_{\lambda_0} = k_{\lambda_0-1}(\bar{b}_{j_0}^{1/p})$ and $\bar{\mathfrak{c}}_{\lambda_0} = \bar{b}_{j_0}^{1/p}$.

Theorem 4.2.9. *Assume Hypothesis 4.2.8 and keep the notation as above. Moreover, assume that $\beta_K \geqslant n+1$. Let $a \in \mathbb{Q}_{>1}$ and $\omega \geqslant n+1$. Let $\mathrm{TS}^a_{L/K,\mathfrak{R}_{0,I},\mathfrak{R}_\Lambda}$ be a recursive thickening space with error gauge $\geqslant\omega$. Then $\mathrm{TS}^a_{L/K,\mathfrak{R}_{0,I},\mathfrak{R}_\Lambda} \times_{A^{m+1}_K[0,\theta^a],f} A^{m+2}_{\widetilde{K}}[0, \theta^a]$ is a recursive thickening space for $\widetilde{L}/\widetilde{K}$ with error gauge $\geqslant\omega - n$.*

The reader can feel free to skip this proof when reading this paper for the first time; one may get a feel of the proof through understanding Example 4.2.10.

Example 4.2.10. We continue with Example 3.3.4 and use the notation from there. As in Notation 4.2.1, we set K' to be the completion of $K(x) = \mathbb{Q}_p(\zeta_p)(b, x)^\wedge$ with respect to the 1-Gauss norm. (It turns out that K' having separably closed residue field is not important for this example, so we ignore this minor point.) Let $\widetilde{K} = K'((b + x\pi_K)^{1/p})$ and $\widetilde{L} = L\widetilde{K}$. Write $\beta = (b + x\pi_K)^{1/p}$ for simplicity. Denote the residue fields of $\widetilde{K}$ and $\widetilde{L}$ by $\tilde{k} = \mathbb{F}_p(x, b)$ and $\tilde{l}$, respectively.

We first try to understand the extension $\widetilde{L}/\widetilde{K}$ in terms of generators and relations. Recall that the extension $\mathcal{O}_L/\mathcal{O}_K$ is generated by $c = (b + \pi_K)^{1/p}$ and $\pi_L = (b\pi_K)^{1/p}$ with relations $p_0 = u_0^p - b\pi_K$ and $p_1 = u_1^p - b - \pi_K$. These relations do generate $\widetilde{L}/\widetilde{K}$, but they may not generate the extension on the level of rings of integers. In particular, we need to modify p_1 to be

$$u_1^p - \beta^p + x\pi_K - \pi_K = (u_1 - \beta)^p + x\pi_K - \pi_K + p(u_1^{p-1}\beta - \cdots - \beta^{p-1}u_1).$$

So, to get a proper relation, we should use the generator $\mathfrak{c} = (c - \beta)/\pi_L$ with the proxy $\mathfrak{v}$. The relation then becomes

$$\mathfrak{q} = \mathfrak{v}^p + \frac{x-1}{b} + \frac{p}{b\pi_K}((\beta + u_0\mathfrak{v})^{p-1}\beta - \cdots - (\beta + u_0\mathfrak{v})\beta^{p-1}).$$

Hence $\mathfrak{v}$ generates an extension of $\widetilde{K}(\pi_L)$ of degree p with inseparable residue field extension. The upshot here is that *the introduction of the transcendental element x guarantees that we only divide the relation p_1 by an element of norm $|\pi_K|$ but not any further.*

Now, we try to understand the base change $\mathrm{TS}^a_{L/K,\psi} \times_{A^2_K[0,\theta^a],f} A^3_{\widetilde{K}}[0,\theta^a]$. Its ring of functions is just

$$K\langle u_0, u_1, \pi_K^{-a}\delta_0, \pi_K^{-a}\delta_1\rangle/(\psi(p_0),\psi(p_1)) \otimes_{K\langle \pi_K^{-a}\delta_0, \pi_K^{-a}\delta_1\rangle, f^*} K\langle \pi_K^{-a}\eta_0, \pi_K^{-a}\eta_1, \pi_K^{-a}\eta_2\rangle, \tag{4.2.11}$$

where $f^*(\delta_0) = \eta_0$ and $f^*(\delta_1) = (\beta+\eta_1)^p - (x+\eta_2)(\pi_K+\eta_0) - b$.

Upon replacing u_1 by $\beta + \eta_1 + u_0\mathfrak{v}$, we see that (4.2.11) becomes

$$K\langle u_0, \beta+\eta_1+u_0\mathfrak{v}, \pi_K^{-a}\eta_0, \pi_K^{-a}\eta_1, \pi_K^{-a}\eta_2\rangle/(q_1,q_2)$$

where

$$q_1 = u_0^p - (\pi_K+\eta_0)(\beta+\eta_1)^p - (\pi_K+\eta_0)^2(x+\eta_2),$$
$$q_2 = (\beta+\eta_1+u_0\mathfrak{v})^p - (\beta+\eta_1)^p + (\pi_K+\eta_0)(x+\eta_2) - (\pi_K+\eta_0).$$

With the help of q_1, q_2 can be replaced by

$$q_2' = ((\beta+\eta_1)^p - (\pi_K+\eta_0)(x+\eta_2))\mathfrak{v}^p + p(\cdots)/(\pi_K+\eta_0) + x + \eta_2 - 1.$$

It may not be too easy to see immediately that $K\langle u_0, \beta+u_0\mathfrak{v}, \pi_K^{-a}\eta_0, \pi_K^{-a}\eta_1, \pi_K^{-a}\eta_2\rangle/(q_1,q_2')$ gives a thickening space for $\widetilde{L}/\widetilde{K}$ of error gauge $\leqslant \beta_K - 1 = p-2$. But at least q_1 is just $\psi_{\widetilde{K}}(u_0^p - \beta^p\pi_K - x\pi_K^2)$ and the major term $((\beta+\eta_1)^p - (\pi_K+\eta_0)(x+\eta_2))\mathfrak{v}^p + x + \eta_2 - 1$ of q_2' is close to $\psi(b\mathfrak{q})$.

Proof of Theorem 4.2.9. Step 1. Find the generators of $\mathcal{O}_{\widetilde{L}}/\mathcal{O}_{\widetilde{K}}$.

The difficulty comes from the fact that $\pi_{L,I}$ and $\mathfrak{c}_\Lambda$ do not generate $\mathcal{O}_{\widetilde{L}}$ over $\mathcal{O}_{\widetilde{K}}$ (although they do generate $\widetilde{L}$ over $\widetilde{K}$). We need to change the generator $\mathfrak{c}_{\lambda_0}$ to an element which gives either of the following cases.

Case A. The inseparable extension $\tilde{l}$ of $l(\bar{x})^{\mathrm{sep}}$, which happens when $\widetilde{L}/\widetilde{K}$ has naïve ramification degree e.

Case B. A ramified extension of naïve ramification degree p, which happens when $\widetilde{L}/\widetilde{K}$ has naïve ramification degree ep; in this case, the generator is a uniformizer of $\widetilde{L}$.

Write $L' = LK'$, which has residue field $l' = l(\bar{x})^{\mathrm{sep}}$. Then we have $\mathcal{O}_{L'} = \mathcal{O}_{K'} \otimes_{\mathcal{O}_K} \mathcal{O}_L$. Hence, $\mathcal{O}_{\widetilde{K}} \otimes_{\mathcal{O}_K} \mathcal{O}_L \cong \mathcal{O}_{\widetilde{K}} \otimes_{\mathcal{O}_{K'}} \mathcal{O}_{L'} \subseteq \mathcal{O}_{\widetilde{L}}$. We may extend the valuation $v_{L'}(\cdot)$ to $\widetilde{L}$ by allowing rational valuations in case B. Let $\beta_{j_0} - \mu$ for $\mu \in \mathcal{O}_{L'}$ be an element achieving the maximal valuation under $v_{L'}(\cdot)$ among the $\beta_{j_0} + \mathcal{O}_{L'}$.

CLAIM. We have $\alpha = v_{L'}(\beta_{j_0} - \mu) \leqslant en/p$ and one of the following.

Case A. The reduction of $\tilde{\mathfrak{c}}_{\lambda_0} = \pi_L^{-\alpha}(\beta_{j_0} - \mu)$ in $\tilde{l}$ generates $\tilde{l}$ over l' (we also set $d=1$ by convention).

Case B. We have $v_{\widetilde{L}}(\pi_L^{-[\alpha]}(\beta_{j_0} - \mu)) = d/p$ for some $d \in \{1,\ldots,p-1\}$; in this case we fix a dth root $\pi_{\widetilde{L},r_0+1}$ of $\pi_L^{-[\alpha]}(\beta_{j_0} - \mu)$, which then generates the naïvely ramified extension $\mathcal{O}_{\widetilde{L}}/\mathcal{O}_{L'}$.

Proof of the claim. We have the norm $\mathbf{N}_{\widetilde{L}/L'}(\mu - \beta_{j_0}) = \mu^p - (b_{j_0} + x\pi_K^n)$. By Lemma 4.2.3, there is no $\mu \in \mathcal{O}_{L'}$ whose pth power can cancel with the $x\pi_K^n$ term, $v_{L'}(\mathbf{N}_{\widetilde{L}/L'}(\beta_{j_0} - \mu)) \leqslant en$,

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and the first statement of the claim follows. When $\alpha \notin \mathbb{N}$, we fall in case B and the claim is obvious. Assume, for contradiction, that $\alpha \in \mathbb{N}$ and the reduction of $\tilde{\mathfrak{c}}_{\lambda_0}$ lies in l'. Then there exists $\mu' \in \mathcal{O}_{L'}$ such that $\mu'/\pi_L^\alpha \equiv \tilde{\mathfrak{c}}_{\lambda_0} \pmod{\mathfrak{m}_{\widetilde{L}}}$. But then $\beta_{j_0} - \mu - \mu'$ will have bigger valuation, which contradicts our choice of μ. This proves the claim.

Step 2. Find the generating relations.

By the previous step, we can write

$$\mathcal{O}_{\widetilde{K}}\langle \tilde{\mathfrak{u}}_{0,I}, \tilde{\mathfrak{u}}_{\Lambda\backslash\lambda_0}, \tilde{\mathfrak{v}}\rangle / (\tilde{\mathfrak{p}}_{0,I}, \tilde{\mathfrak{p}}_{\Lambda\backslash\lambda_0}, \tilde{\mathfrak{q}}) \simeq \mathcal{O}_{\widetilde{L}}$$

by sending $\tilde{\mathfrak{u}}_{0,I}$ to $\mathfrak{c}_{0,I}$, $\tilde{\mathfrak{u}}_{\Lambda\backslash\lambda_0}$ to $\mathfrak{c}_{\Lambda\backslash\lambda_0}$, and $\tilde{\mathfrak{v}}$ to $\tilde{\mathfrak{c}}_{\lambda_0}$ in case A and to $\pi_{\widetilde{L},r_0+1}$ in case B, where the relations $\mathfrak{p}_{0,I}, \tilde{\mathfrak{p}}_{\Lambda\backslash\lambda_0}$ and $\tilde{\mathfrak{q}}$ corresponding to $\tilde{\mathfrak{u}}_{0,I}, \tilde{\mathfrak{u}}_{\Lambda\backslash\lambda_0}$ and $\tilde{\mathfrak{v}}$ can be obtained using Construction 3.6.1. Now, we link these relations to the relations $\mathfrak{p}_{0,I}$ and $\mathfrak{p}_\Lambda$ for $\mathcal{O}_L/\mathcal{O}_K$. We first lift the isomorphism

$$\bar{\chi} : \widetilde{K}\langle \tilde{\mathfrak{u}}_{0,I}, \tilde{\mathfrak{u}}_{\Lambda\backslash\lambda_0}, \tilde{\mathfrak{v}}\rangle / (\tilde{\mathfrak{p}}_{0,I}, \tilde{\mathfrak{p}}_{\Lambda\backslash\lambda_0}, \tilde{\mathfrak{q}}) \simeq \widetilde{L} \cong \widetilde{K} \otimes_{\mathcal{O}_K} \mathcal{O}_L \simeq \widetilde{K}\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle / (\mathfrak{p}_{0,I}, \mathfrak{p}_\Lambda)$$

to a homomorphism $\chi : \mathcal{O}_{\widetilde{K}}\langle \tilde{\mathfrak{u}}_{0,I}, \tilde{\mathfrak{u}}_{\Lambda\backslash\lambda_0}, \tilde{\mathfrak{v}}\rangle \to \mathcal{O}_{\widetilde{K}}\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle[1/(\mathfrak{u}_{0,r_0})]$ that sends $\tilde{\mathfrak{u}}_{0,I}$ to $\mathfrak{u}_{0,I}$, $\tilde{\mathfrak{u}}_{\Lambda\backslash\lambda_0}$ to $\mathfrak{u}_{\Lambda\backslash\lambda_0}$ and $\tilde{\mathfrak{u}}_{0,r_0}^{[\alpha]}\tilde{\mathfrak{v}}$ to the lift of $\bar{\chi}(\tilde{\mathfrak{u}}_{0,r_0}^{[\alpha]}\tilde{\mathfrak{v}})$ using the standard basis defined in Construction 3.6.1. Then $\mathfrak{u}_{0,r_0}^{(p-1)[\alpha]}\chi(\tilde{\mathfrak{p}}_{0,I}), \mathfrak{u}_{0,r_0}^{(p-1)[\alpha]}\chi(\tilde{\mathfrak{p}}_{\Lambda\backslash\lambda_0})$ and $\mathfrak{u}_{0,r_0}^{p[\alpha]}\chi(\tilde{\mathfrak{q}})$ are contained in the ideal $(\mathfrak{p}_{0,I}, \mathfrak{p}_\Lambda)\mathcal{O}_K\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle$, because the maximal powers of $\tilde{\mathfrak{v}}$ in the equations are $p-1, p-1$ and p, respectively.

Step 3. Explain the goal.

We are going to establish an $\mathcal{R}_{\widetilde{K}}$-isomorphism $\boldsymbol{\chi} : \widetilde{\mathcal{A}} \xrightarrow{\sim} \mathcal{A}$ where

$$\mathcal{A} = \mathfrak{S}_K / (\psi_K(\mathfrak{p}_{0,I}) + \mathfrak{R}_{0,I}, \psi_K(\mathfrak{p}_\Lambda) + \mathfrak{R}_\Lambda) \otimes_{\mathcal{R}_K, f^*} \mathcal{R}_{\widetilde{K}}\left[\frac{1}{p}\right], \tag{4.2.12}$$

$$\widetilde{\mathcal{A}} = \mathfrak{S}_{\widetilde{K}}\left[\frac{1}{p}\right] \Big/ (\psi_{\widetilde{K}}(\tilde{\mathfrak{p}}_{0,I}) + \widetilde{\mathfrak{R}}_{0,I}, \psi_{\widetilde{K}}(\tilde{\mathfrak{p}}_{\Lambda\backslash\lambda_0}) + \widetilde{\mathfrak{R}}_{\Lambda\backslash\lambda_0}, \psi_{\widetilde{K}}(\tilde{\mathfrak{q}}) + \widetilde{\mathfrak{R}}_{\tilde{\mathfrak{q}}}). \tag{4.2.13}$$

Here, $\mathfrak{S}_{\widetilde{K}} = \mathcal{R}_{\widetilde{K}}\langle \tilde{\mathfrak{u}}_{0,I}, \tilde{\mathfrak{u}}_{\Lambda\backslash\lambda_0}, \tilde{\mathfrak{v}}\rangle$ and we can define $\mathfrak{N}_{\widetilde{K}}^a$ for $a \in (1/ep)\mathbb{N}$ similarly to Construction 3.6.1. We first define a ring homomorphism $\widetilde{\boldsymbol{\chi}} : \mathfrak{S}_{\widetilde{K}}[1/p] \to \mathcal{A}$ by $\widetilde{\boldsymbol{\chi}}(\tilde{\mathfrak{u}}_{0,I}) = \mathfrak{u}_{0,I}$, $\widetilde{\boldsymbol{\chi}}(\tilde{\mathfrak{u}}_{\Lambda\backslash\lambda_0}) = \mathfrak{u}_{\Lambda\backslash\lambda_0}$ and $\widetilde{\boldsymbol{\chi}}(\tilde{\mathfrak{v}}) = \psi_{\widetilde{K}}(\chi(\tilde{\mathfrak{v}}))$; the set $\widetilde{\mathfrak{R}}_{0,I}, \widetilde{\mathfrak{R}}_{\Lambda\backslash\lambda}, \widetilde{\mathfrak{R}}_{\tilde{\mathfrak{q}}}$ will be admissible with error gauge $\geqslant \omega - n$, so that $\widetilde{\boldsymbol{\chi}}$ factors through $\widetilde{\mathcal{A}}$.

Step 4. Bound the error gauge. We first determine $\widetilde{\mathfrak{R}}_{0,I}, \widetilde{\mathfrak{R}}_{\Lambda\backslash\lambda_0}, \widetilde{\mathfrak{R}}_{\tilde{\mathfrak{q}}}$. We proceed similarly to Proposition 3.6.6. To write this argument uniformly, we divide into the following four cases.

Case (a). Let $\tilde{\mathfrak{p}} = \mathfrak{u}_{0,r_0}^{(p-1)[\alpha]}\tilde{\mathfrak{p}}_{0,i_0}$ for some $i_0 \in I$ and $\widetilde{\mathfrak{R}} = \mathfrak{u}_{0,r_0}^{(p-1)[\alpha]}\widetilde{\mathfrak{R}}_{0,I}$.

Case (b). Let $\tilde{\mathfrak{p}} = \mathfrak{u}_{0,r_0}^{(p-1)[\alpha]}\tilde{\mathfrak{p}}_\lambda$ for $\lambda \in \Lambda\backslash\{\lambda_0\}$ and $\widetilde{\mathfrak{R}} = \mathfrak{u}_{0,r_0}^{(p-1)[\alpha]}\widetilde{\mathfrak{R}}_\lambda$.

Case (c). Let $\tilde{\mathfrak{p}} = \mathfrak{u}_{0,r_0}^{p[\alpha]}\tilde{\mathfrak{q}}$ and $\widetilde{\mathfrak{R}} = \mathfrak{u}_{0,r_0}^{p[\alpha]}\widetilde{\mathfrak{R}}_{\tilde{\mathfrak{q}}}$, assuming we are in case A.

Case (d). Let $\tilde{\mathfrak{p}} = \mathfrak{u}_{0,r_0}^{p[\alpha]}\tilde{\mathfrak{q}}$ and $\widetilde{\mathfrak{R}} = \mathfrak{u}_{0,r_0}^{p[\alpha]}\widetilde{\mathfrak{R}}_{\tilde{\mathfrak{q}}}$, assuming we are in case B.

By Step 2,

$$\bar{\chi}(\tilde{\mathfrak{p}}) = \sum_{i\in I} \mathfrak{h}_{0,i}\mathfrak{p}_{0,i} + \sum_{\lambda\in\Lambda} \mathfrak{h}_\lambda \mathfrak{p}_\lambda$$

with some $\mathfrak{h}_{0,i}, \mathfrak{h}_\lambda \in \mathcal{O}_{\widetilde{K}}\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle$, for $i \in I$ and $\lambda \in \Lambda$. Moreover, in case (a), for some $i_0 \in I$ we can require that $\mathfrak{h}_{0,i} \in \mathfrak{N}_K^{\max\{(e_{i_0-1}-e_{i-1})/e,0\}} \cdot \mathcal{O}_{\widetilde{K}}\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle$ and $\mathfrak{h}_\lambda \in \mathfrak{N}_K^{e_{i_0-1}/e} \cdot \mathcal{O}_{\widetilde{K}}\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda\rangle$

for $i \in I$ and $\lambda \in \Lambda$; in case (d), we can require that $\mathfrak{h}_\lambda \in \mathfrak{N}_K^{1/e} \cdot \mathcal{O}_{\widetilde{K}}\langle \mathfrak{u}_{0,I}, \mathfrak{u}_\Lambda \rangle$ for $\lambda \in \Lambda$. Thus, we want to define $\widetilde{\mathfrak{R}} \in \mathfrak{S}_{\widetilde{K}}$ so that $-\widetilde{\boldsymbol{\chi}}(\widetilde{\mathfrak{R}})$ is equal to

$$\begin{aligned}
\widetilde{\boldsymbol{\chi}}(\psi_{\widetilde{K}}(\tilde{\mathfrak{p}})) &= \sum_{i \in I} \psi_{\widetilde{K}}(\mathfrak{h}_{0,i}) \psi_{\widetilde{K}}(\mathfrak{p}_{0,i}) + \sum_{\lambda \in \Lambda} \psi_{\widetilde{K}}(\mathfrak{h}_\lambda) \psi_{\widetilde{K}}(\mathfrak{p}_\lambda) + \mathfrak{E} \\
&= \sum_{i \in I} \psi_{\widetilde{K}}(\mathfrak{h}_{0,i})(-\mathfrak{R}_{0,i}) + \sum_{\lambda \in \Lambda} \psi_{\widetilde{K}}(\mathfrak{h}_\lambda)(-\mathfrak{R}_\lambda) + \mathfrak{E} \\
&\in \begin{cases} (\mathfrak{N}^{\omega-1+e_{i_0-1}/e}\eta_0, \mathfrak{N}^{\omega+e_{i_0-1}/e}\eta_{J\cup\{m+1\}}) \cdot \mathfrak{S}_K \otimes_{\mathcal{R}_K} \mathcal{R}_{\widetilde{K}} & \text{in case (a)}, \\ (\mathfrak{N}^{\omega-1}\eta_0, \mathfrak{N}^{\omega}\eta_{J\cup\{m+1\}}) \cdot \mathfrak{S}_K \otimes_{\mathcal{R}_K} \mathcal{R}_{\widetilde{K}} & \text{in case (b) or (c)}, \\ (\mathfrak{N}^{\omega-1+1/e}\eta_0, \mathfrak{N}^{\omega+1/e}\eta_{J\cup\{m+1\}}) \cdot \mathfrak{S}_K \otimes_{\mathcal{R}_K} \mathcal{R}_{\widetilde{K}} & \text{in case (d)}, \end{cases}
\end{aligned}$$

where the error term $\mathfrak{E}$ that comes from ψ failing to be a homomorphism (see Proposition 3.2.8) can be bounded as

$$\mathfrak{E} \in \begin{cases} (\mathfrak{N}^{\beta_K}\eta_0, \mathfrak{N}^{\beta_K+1}\eta_{J\cup\{m+1\}}) \cdot \mathfrak{S}_K \otimes_{\mathcal{R}_K} \mathcal{R}_{\widetilde{K}} & \text{in case (a)}, \\ (\mathfrak{N}^{\beta_K-1}\delta_0, \mathfrak{N}^{\beta_K}\delta_J) \cdot \mathfrak{S}_K \otimes_{\mathcal{R}_K} \mathcal{R}_{\widetilde{K}} & \text{in case (b) or (c)}, \\ (\mathfrak{N}^{\beta_K}\eta_0, \mathfrak{N}^{\beta_K+1}\eta_{J\cup\{m+1\}}) \cdot \mathfrak{S}_K \otimes_{\mathcal{R}_K} \mathcal{R}_{\widetilde{K}} & \text{in case (d)}. \end{cases}$$

Thus, we can find polynomials $\tilde{\mathfrak{r}}_0, \ldots, \tilde{\mathfrak{r}}_{m+1} \in \mathcal{O}_{\widetilde{K}}[\tilde{\mathfrak{u}}_{0,I}, \tilde{\mathfrak{u}}_{\Lambda\backslash\lambda_0}, \tilde{\mathfrak{u}}_{0,r_0}^{[\alpha]}\tilde{\mathfrak{v}}] \twoheadrightarrow \mathcal{O}_{\widetilde{K}} \otimes_{\mathcal{O}_K} \mathcal{O}_L$ such that

$$\tilde{\mathfrak{r}}_0 \in \begin{cases} \tilde{\mathfrak{u}}_{0,r_0}^{\omega e-e+e_{i_0-1}} \cdot \mathcal{O}_{\widetilde{K}}[\tilde{\mathfrak{u}}_{0,I}, \tilde{\mathfrak{u}}_{\Lambda\backslash\lambda_0}, \tilde{\mathfrak{u}}_{0,r_0}^{[\alpha]}\tilde{\mathfrak{v}}] & \text{in case (a)}, \\ \tilde{\mathfrak{u}}_{0,r_0}^{\omega e-e} \cdot \mathcal{O}_{\widetilde{K}}[\tilde{\mathfrak{u}}_{0,I}, \tilde{\mathfrak{u}}_{\Lambda\backslash\lambda_0}, \tilde{\mathfrak{u}}_{0,r_0}^{[\alpha]}\tilde{\mathfrak{v}}] & \text{in case (b) or (c)}, \\ \tilde{\mathfrak{u}}_{0,r_0}^{\omega e-e+1} \cdot \mathcal{O}_{\widetilde{K}}[\tilde{\mathfrak{u}}_{0,I}, \tilde{\mathfrak{u}}_{\Lambda\backslash\lambda_0}, \tilde{\mathfrak{u}}_{0,r_0}^{[\alpha]}\tilde{\mathfrak{v}}] & \text{in case (d)}; \end{cases}$$

$$\tilde{\mathfrak{r}}_1, \ldots, \tilde{\mathfrak{r}}_{m+1} \in \begin{cases} \tilde{\mathfrak{u}}_{0,r_0}^{\omega e+e_{i_0-1}} \cdot \mathcal{O}_{\widetilde{K}}[\tilde{\mathfrak{u}}_{0,I}, \tilde{\mathfrak{u}}_{\Lambda\backslash\lambda_0}, \tilde{\mathfrak{u}}_{0,r_0}^{[\alpha]}\tilde{\mathfrak{v}}] & \text{in case (a)}, \\ \tilde{\mathfrak{u}}_{0,r_0}^{\omega e} \cdot \mathcal{O}_{\widetilde{K}}[\tilde{\mathfrak{u}}_{0,I}, \tilde{\mathfrak{u}}_{\Lambda\backslash\lambda_0}, \tilde{\mathfrak{u}}_{0,r_0}^{[\alpha]}\tilde{\mathfrak{v}}] & \text{in case (b) or (c)}, \\ \tilde{\mathfrak{u}}_{0,r_0}^{\omega e+1} \cdot \mathcal{O}_{\widetilde{K}}[\tilde{\mathfrak{u}}_{0,I}, \tilde{\mathfrak{u}}_{\Lambda\backslash\lambda_0}, \tilde{\mathfrak{u}}_{0,r_0}^{[\alpha]}\tilde{\mathfrak{v}}] & \text{in case (d)}; \end{cases}$$

and

$$\begin{aligned}
&-\widetilde{\boldsymbol{\chi}}(\psi_{\widetilde{K}}(\tilde{\mathfrak{p}})) - \widetilde{\boldsymbol{\chi}}(\tilde{\mathfrak{r}}_0\eta_0 + \cdots + \tilde{\mathfrak{r}}_{m+1}\eta_{m+1}) \\
&\in \begin{cases} (\eta_0/\pi_K, \eta_{J\cup\{m+1\}})(\mathfrak{N}^{\omega-1+e_{i_0-1}/e}\eta_0, \mathfrak{N}^{\omega+e_{i_0-1}/e}\eta_{J\cup\{m+1\}}) & \text{in case (a)}, \\ \quad \cdot (\mathfrak{S}_K \otimes_{\mathcal{R}_K} \mathcal{R}_{\widetilde{K}}) & \\ (\eta_0/\pi_K, \eta_{J\cup\{m+1\}})(\mathfrak{N}^{\omega-1}\eta_0, \mathfrak{N}^{\omega}\eta_{J\cup\{m+1\}}) & \text{in case (b) or (c)}, \\ \quad \cdot (\mathfrak{S}_K \otimes_{\mathcal{R}_K} \mathcal{R}_{\widetilde{K}}) & \\ (\eta_0/\pi_K, \eta_{J\cup\{m+1\}})(\mathfrak{N}^{\omega-1+1/e}\eta_0, \mathfrak{N}^{\omega+1/e}\eta_{J\cup\{m+1\}}) & \text{in case (d)}. \\ \quad \cdot (\mathfrak{S}_K \otimes_{\mathcal{R}_K} \mathcal{R}_{\widetilde{K}}) & \end{cases}
\end{aligned}$$

Further, we can similarly approximate the coefficients of $\eta_j\eta_{j'}$ for $j, j' \in J^+ \cup \{m+1\}$. Repeating this approximation gives the expression of $\widetilde{\mathfrak{R}} \in \mathfrak{S}_{\widetilde{K}}$. From this and $\alpha \leqslant en/p$,

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we can obtain $\widetilde{\mathfrak{R}}_{0,I}, \widetilde{\mathfrak{R}}_{\Lambda\backslash\lambda_0}, \widetilde{\mathfrak{R}}_{\tilde{\mathfrak{q}}} \in (\eta_{J^+\cup\{m+1\}}) \cdot \mathfrak{S}_{\widetilde{K}}$ such that

$$\widetilde{\mathfrak{R}}_{0,i_0} \in (\tilde{\mathfrak{u}}_{0,r_0}^{\omega e-e+e_{i_0-1}-en}\eta_0, \tilde{\mathfrak{u}}_{0,r_0}^{\omega e+e_{i_0-1}-en}\eta_{J\cup\{m+1\}}) \cdot \mathfrak{S}_{\widetilde{K}} \quad \text{for } i_0 \in I,$$
$$\widetilde{\mathfrak{R}}_{\lambda} \in (\tilde{\mathfrak{u}}_{0,r_0}^{\omega e-e-en}\eta_0, \tilde{\mathfrak{u}}_{0,r_0}^{\omega e-en}\eta_{J\cup\{m+1\}}) \cdot \mathfrak{S}_{\widetilde{K}} \quad \text{for } \lambda \in \Lambda\backslash\lambda_0;$$
$$\widetilde{\mathfrak{R}}_{\tilde{\mathfrak{q}}} \in \begin{cases} (\tilde{\mathfrak{u}}_{0,r_0}^{\omega e-e-en}\eta_0, \tilde{\mathfrak{u}}_{0,r_0}^{\omega e-en}\eta_{J\cup\{m+1\}}) \cdot \mathfrak{S}_{\widetilde{K}} & \text{in case A,} \\ (\tilde{\mathfrak{u}}_{0,r_0}^{\omega e-e-en+1}\eta_0, \tilde{\mathfrak{u}}_{0,r_0}^{\omega e-en+1}\eta_{J\cup\{m+1\}}) \cdot \mathfrak{S}_{\widetilde{K}} & \text{in case B.} \end{cases}$$

These have error gauge $\geqslant \omega - n$. Moreover, $\widetilde{\boldsymbol{\chi}}$ induces a continuous homomorphism $\boldsymbol{\chi} : \widetilde{\mathcal{A}} \to \mathcal{A}$.

Step 5. Prove that $\boldsymbol{\chi}$ is an isomorphism.

To prove that $\boldsymbol{\chi}$ is an isomorphism, it suffices to show the surjectivity, as both $\widetilde{\mathcal{A}}$ and $\mathcal{A}$ are finite free modules over $\mathcal{R}_{\widetilde{K}}[1/p]$ of the same rank. Since (3.6.2) forms a basis of $\mathcal{A}$ over $\mathcal{R}_{\widetilde{K}}[1/p]$, we only need to show that $\mathfrak{u}_{0,I}$ and $\mathfrak{u}_{\Lambda}$ are in the image of $\boldsymbol{\chi}$. This is obvious for $\mathfrak{u}_{0,I}$ and $\mathfrak{u}_{\Lambda\backslash\lambda_0}$. For $\mathfrak{u}_{\lambda_0}$, we first find an element in $\mathcal{O}_{\widetilde{K}}[\tilde{\mathfrak{u}}_{0,I}, \tilde{\mathfrak{u}}_{\Lambda\backslash\lambda_0}, \tilde{\mathfrak{u}}_{0,r_0}^{[\alpha]}\tilde{\mathfrak{v}}] \twoheadrightarrow \mathcal{O}_{\widetilde{K}} \otimes_{\mathcal{O}_K} \mathcal{O}_L$ whose image under $\bar{\chi}$ is $\mathfrak{u}_{\lambda_0}$. Then we use the similar approximation in Step 4 to find an element in $\widetilde{\mathcal{A}}$ whose image under $\boldsymbol{\chi}$ is exactly $\mathfrak{u}_{\lambda_0}$. This finishes the proof. □

Remark 4.2.14. We expect that when ω and hence β_K is 'large' compared to $[L:K]$, Theorem 4.2.9 will also be valid if we add a generic p^∞th root (defined in [Xia10, Definition 5.2.2]); this amounts to controlling the discrepancy between $\mathcal{O}_{\widetilde{L}}$ and $\mathcal{O}_{\widetilde{K}} \otimes_{\mathcal{O}_K} \mathcal{O}_L$. Hence, in this case, one can obtain a comparison theorem between the arithmetic Artin conductor and Borger's Artin conductor [Bor04] as in [Xia10, § 5.4].

4.3 Non-logarithmic Hasse–Arf theorem

In this subsection, we apply Theorem 4.2.9 to obtain Theorem 4.3.5, the Hasse–Arf theorem for non-logarithmic ramification filtrations.

We assume Hypothesis 3.1.2 until the last theorem. As a reminder, up to the end of the paper Hypothesis 4.2.8 will no longer be assumed.

Notation 4.3.1. Keep the notation as in Construction 3.1.6. Fix $j_0 \in J$ and $n \in \mathbb{N}$. Let $\widetilde{K} = K'((b_{j_0} + x\pi_K^n)^{1/p})$ as in Notation 4.2.1. Write $\beta_{j_0} = (b_{j_0} + x\pi_K^n)^{1/p}$ for simplicity.

Lemma 4.3.2. *Assume $p \nmid n$ and $\beta_K \geqslant n$. Let $a_{J^+} \subset \mathbb{R}_{>0}$ and $a_0 = a_{j_0} = a_{m+1} > \max\{(n-1)/(p-1), 1\}$. Define $a'_j = a_j$ for $j \in J^+\backslash\{j_0\}$ and $a'_{j_0} = a_{j_0} + n - 1$. The morphism f^* defined in Lemma 4.2.6 restricts to a morphism*

$$f : A^1_{\widetilde{K}}[\theta^{a_0}, \theta^{a_0}] \times \cdots \times A^1_{\widetilde{K}}[\theta^{a_{m+1}}, \theta^{a_{m+1}}] \to A^1_K[\theta^{a'_0}, \theta^{a'_0}] \times \cdots \times A^1_K[\theta^{a'_m}, \theta^{a'_m}].$$

In other words, we change the j_0th radius from a_{j_0} to $a_{j_0} + n - 1$.

Proof. It suffices to verify that if $|\eta_0| = |\eta_{j_0}| = |\eta_{m+1}| = \theta^{a_0}$, then $|\delta_j| = \theta^{a_0+n-1}$; indeed,

$$\delta_{j_0} = ((\beta_{j_0} + \eta_{j_0})^p - \beta_{j_0}^p) - x((\pi_K + \eta_0)^n - \pi_K^n) + \eta_{m+1}(\pi_K + \eta_0)^n,$$

which has norm θ^{a_0+n-1} because the second term does and the other terms have bigger norms. □

Lemma 4.3.3. *Keep the notation and assumptions as in the previous lemma. Let $\mathcal{E}$ be a differential module over $A^1_K[0, \theta^{a'_0}] \times \cdots \times A^1_K[0, \theta^{a'_m}]$. Then $\mathrm{IR}(f^*\mathcal{E}; a_{J^+}) = \mathrm{IR}(\mathcal{E}; a'_{J^+\cup\{m+1\}})$.*

Proof. The morphism f^* induces a homomorphism on the differentials: $d\delta_j \mapsto d\eta_j$ for $j \in J^+\backslash\{j_0\}$ and $d\delta_{j_0} \mapsto p(\beta_{j_0} + \eta_{j_0})^{p-1}d\eta_{j_0} + (\pi_K + \eta_0)^n d\eta_{m+1} + n(x + \eta_{m+1})(\pi_K + \eta_0)^{n-1}d\eta_0$.

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Thus,

$$\begin{aligned}
\partial'_j|_{f^*\mathcal{E}} &= \partial_j|_{\mathcal{E}} \quad \text{for } j \in J\backslash\{j_0\},\\
\partial'_{j_0}|_{f^*\mathcal{E}} &= p(\beta_{j_0}+\eta_{j_0})^{p-1}\partial_{j_0}|_{\mathcal{E}},\\
\partial'_{m+1}|_{f^*\mathcal{E}} &= (\pi_K+\eta_0)^n \cdot \partial_{j_0}|_{\mathcal{E}},\\
\partial'_0|_{f^*\mathcal{E}} &= \partial_0|_{\mathcal{E}} + n(x+\eta_{m+1})(\pi_K+\eta_0)^{n-1}\cdot \partial_{j_0}|_{\mathcal{E}},
\end{aligned}$$

where $\partial'_j = \partial/\partial\eta_j$ for $j=0,\ldots,m+1$. Hence,

$$\begin{aligned}
\mathrm{IR}_j(f^*\mathcal{E}; a_{J^+\cup\{m+1\}}) &= \mathrm{IR}_j(\mathcal{E}; a'_{J^+}) \quad \text{for all } j \in J\backslash\{j_0\},\\
\mathrm{IR}_{j_0}(f^*\mathcal{E}; a_{J^+\cup\{m+1\}}) &\leqslant \mathrm{IR}_{j_0}(\mathcal{E}; a'_{J^+}),\\
\mathrm{IR}_{m+1}(f^*\mathcal{E}; a_{J^+\cup\{m+1\}}) &= \theta^n \cdot \mathrm{IR}_{j_0}(\mathcal{E}; a'_{J^+}),\\
\mathrm{IR}_0(f^*\mathcal{E}; a_{J^+\cup\{m+1\}}) &= \min\{\mathrm{IR}_0(\mathcal{E}, a'_{J^+}), \mathrm{IR}_{j_0}(\mathcal{E}; a'_{J^+})\},
\end{aligned}$$

where the second inequality follows from Proposition 2.1.19 and the last equality holds by Proposition 2.1.17 because x is transcendental over K. It follows that $\mathrm{IR}(\mathcal{E}; a'_{J^+}) = \mathrm{IR}(f^*\mathcal{E}; a_{J^+\cup\{m+1\}})$. □

Theorem 4.3.4. *Let L/K be a finite Galois extension satisfying Hypotheses 3.1.2 and 3.2.10. The highest non-logarithmic ramification break of L/K is invariant under the operation of adding a generic pth root.*

Proof. Adding a generic pth root corresponds to setting $n=1$ in the notation of this subsection. Fix a choice of ψ_K in Construction 3.2.1. Let $\mathrm{TS}^a_{L/K,\psi_K}$ be the standard thickening space for L/K. By Example 3.6.5, we can turn this standard thickening space into a recursive thickening space (with error gauge $\geqslant\beta_K$). By Theorem 4.2.9, $\mathrm{TS}^a_{L/K,\psi_K} \times_{A_K^{m+1}[0,\theta^a],f} A^{m+2}_{\widetilde{K}}[0,\theta^a]$ is a recursive thickening space for $\widetilde{L}/\widetilde{K}$ with error gauge $\geqslant\beta_K-1$, which is isomorphic to some thickening space for $\widetilde{L}/\widetilde{K}$ by Proposition 3.6.6.

Let $\mathcal{E}$ be the differential module over $A_K^{m+1}[0,\theta^a]$ coming from $\mathrm{TS}^a_{L/K,\psi_K}$. Then the differential module $f^*\mathcal{E}$ is associated to $\widetilde{L}/\widetilde{K}$. Applying Lemma 4.3.3 (to the case $n=1$) gives $\mathrm{IR}(f^*\mathcal{E};\underline{s}) = \mathrm{IR}(\mathcal{E};\underline{s})$ for $s \geqslant b(L/K)-\epsilon$ with $\epsilon>0$ as in Theorem 3.4.2. The theorem follows from Proposition 3.5.2. □

On combining Theorem 4.3.4 and Proposition 4.1.8, we have the following theorem.

Theorem 4.3.5. *Let K be a complete discrete valuation field of mixed characteristic $(0,p)$ which is not absolutely unramified. Let $\rho: G_K \to \mathrm{GL}(V_\rho)$ be a representation with finite monodromy. Then:*

(1) *$\mathrm{Art}(\rho)$ is a non-negative integer;*

(2) *the subquotients $\mathrm{Fil}^a G_K/\mathrm{Fil}^{a+} G_K$ are trivial if $a \notin \mathbb{Q}$ and are abelian groups killed by p if $a \in \mathbb{Q}_{>1}$.*

4.4 Application to finite flat group schemes

This subsection is an analogue of [Xia10, § 4.1] in the mixed characteristic case.

We first recall the definition of Abbes–Saito ramification filtration on finite flat group schemes [AM04].

Convention 4.4.1. All finite flat group schemes are commutative.

Definition 4.4.2. Let A be a finite flat $\mathcal{O}_K$-algebra. Write $A=\mathcal{O}_K[x_1,\dots,x_n]/\mathcal{I}$ with $\mathcal{I}$ an ideal generated by $f_1,\dots,f_r$. For $a\in\mathbb{Q}_{\geqslant 0}$, define the rigid space

$$X^a=\{(x_1,\dots,x_n)\in A_K^n[0,1]:|f_i(x_1,\dots,x_n)|\leqslant\theta^a \text{ for } i=1,\dots,r\}.$$

The *highest break* $b(A/\mathcal{O}_K)$ of A is the smallest number such that for all $a>b(A/\mathcal{O}_K)$, $\#\pi_0^{\mathrm{geom}}(X^a)=\operatorname{rank}_{\mathcal{O}_K}A$. This is the same as Definition 2.2.3 when $A=\mathcal{O}_L$; but, in notation, we use the ring of integers instead of the fields themselves.

Definition 4.4.3. Now we specialize to the case where $G=\operatorname{Spec}A$ is a finite flat group scheme. We have a natural map of points $G(K^{\mathrm{alg}})\hookrightarrow X^a(K^{\mathrm{alg}})$. Upon composing further with the map for geometric connected components, we obtain

$$\sigma^a:G(K^{\mathrm{alg}})\hookrightarrow X^a(K^{\mathrm{alg}})\to\pi_0^{\mathrm{geom}}(X^a).$$

By functoriality of σ^a, one can see that $\pi_0^{\mathrm{geom}}(X^a)$ has a natural group structure and that σ^a is a homomorphism [AM04, 2.3]. Define G^a to be the Zariski closure of $\ker\sigma^a$. Also, put $G^{a+}=\varinjlim_{b>a}G^b$.

Lemma 4.4.4 [AM04, Lemme 2.1.5]. *Let K'/K be a (not necessarily finite) extension of complete discrete valuation fields of naïve ramification index e. Let A be a finite flat $\mathcal{O}_K$-algebra which is a complete intersection relative to $\mathcal{O}_K$. Put $A'=A\otimes_{\mathcal{O}_K}\mathcal{O}_{K'}$; then $b(A'/\mathcal{O}_{K'})=e\cdot b(A/\mathcal{O}_K)$.*

Definition 4.4.5. We say the finite flat group scheme G is *generically trivial* if $G\times_{\mathcal{O}_k}K$ is the disjoint union of copies of $\operatorname{Spec}K$, with some abelian group structure.

Theorem 4.4.6. *Let $G=\operatorname{Spec}A$ be a generically trivial finite flat group scheme over $\mathcal{O}_K$. Then $b(A/\mathcal{O}_K)$ is a non-negative integer.*

Proof. Let $\gcd(n_1,n_2)=1$, and let K_{n_1} and K_{n_2} be two tamely ramified extensions of K with ramification degrees n_1 and n_2, respectively. By Lemma 4.4.4, it suffices to prove the theorem for $G\times_{\mathcal{O}_K}\mathcal{O}_{K_{n_1}}/\mathcal{O}_{K_{n_1}}$ and $G\times_{\mathcal{O}_K}\mathcal{O}_{K_{n_2}}/\mathcal{O}_{K_{n_2}}$, respectively. Thus, we may assume that $\beta_K\geqslant 2$. The result follows from Theorem 4.3.5 and the same argument as in [Xia10, Proposition 5.1.7]. □

4.5 Integrality for Swan conductors

In this subsection, we will deduce the integrality of Swan conductors from that of Artin conductors (Theorem 4.3.5). We will use the fact that the logarithmic ramification breaks behave well under tame base changes.

We will keep Hypotheses 3.1.2 and 3.2.10 until we state Theorem 4.5.14.

Notation 4.5.1. Let $n\in\mathbb{N}$ be such that $n\equiv 1\pmod{ep}$. Define $K_n=K(\pi_K^{1/n})$ and $L_n=LK_n$. Since K_n and L are linearly independent over K, we have $\operatorname{Gal}(L_n/K_n)=\operatorname{Gal}(L/K)$. We take the uniformizers of K_n and L_n to be $\pi_{K_n}=\pi_K^{1/n}$ and $\pi_{L_n}=\pi_L/\pi_{K_n}^{(n-1)/e}$, respectively.

Notation 4.5.2. Write $\mathcal{R}_{K_n}=\mathcal{O}_{K_n}[\![\eta_0/\pi_{K_n},\eta_J]\!]$. Applying Construction 3.2.1 to K_n gives an approximate homomorphism $\psi_{K_n}:\mathcal{O}_{K_n}\to\mathcal{O}_{K_n}[\![\eta_0/\pi_{K_n},\eta_J]\!]$.

Lemma 4.5.3. *There exists a unique continuous $\mathcal{O}_K$-homomorphism $f_n^*:\mathcal{R}_K\to\mathcal{R}_{K_n}$ sending δ_0 to $(\pi_{K_n}+\eta_0)^n-\pi_K$ and δ_j to η_j for $j\in J$. This gives the following approximately commutative*

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diagram modulo $I_{K_n} = p(\eta_0/\pi_{K_n}, \eta_J) \cdot \mathcal{R}_{K_n}$.

$$\begin{array}{ccc} \mathcal{O}_K & \xrightarrow{\ \psi_K\ } & \mathcal{O}_K[\![\delta_0/\pi_K, \delta_J]\!] \\ \big\downarrow & & \big\downarrow f_n^* \\ \mathcal{O}_{K_n} & \xrightarrow{\ \psi_{K_n}\ } & \mathcal{O}_{K_n}[\![\eta_0/\pi_{K_n}, \eta_J]\!] \end{array}$$

Proof. This result follows from Proposition 3.2.8. In fact, one can carefully choose ψ_K and ψ_{K_n} so that the above diagram *commutes*, but we do not need this here. □

Proposition 4.5.4. *Fix* $a \in \mathbb{Q}_{>0}$. *Let* $\mathrm{TS}^a_{L/K,\log,\psi_K}$ *be the standard logarithmic thickening space. Then the space*

$$X = \mathrm{TS}^a_{L/K,\log,\psi_K} \times_{(A^1_K[0,\theta^{a+1}] \times A^m_K[0,\theta^a]), f_n} (A^1_{K_n}[0, \theta^{a+1/n}] \times A^m_{K_n}[0, \theta^a])$$

is a logarithmic thickening space for L_n/K_n *with error gauge* $\geqslant n\beta_K - (n-1)$; *in particular, it is admissible.*

Proof. First, we have

$$\mathcal{S}_K \otimes_{\mathcal{O}_K} K_n \cong \mathcal{O}_{K_n}[\![\eta_0/\pi_{K_n}, \eta_J]\!]\left[\frac{1}{p}\right]\langle u_{J^+}\rangle \Big/ (f_n^*(\psi_K(p_{J^+}))).$$

Now we consider a construction of the logarithmic thickening space of L_n/K_n, using the same c_J as for L/K and π_{L_n} in Notation 4.5.1. Therefore, the ideal $\mathcal{I}_{L_n/K_n}$ is generated by p'_{J^+} and $p'_0/\pi^{n-1}_{K_n}$, where the prime means to replace u_0 with $\pi^{(n-1)/e}_{K_n} u'_0$.

Lemma 4.5.3 implies that

$$\psi_{K_n}(p'_0/\pi^{n-1}_{K_n}) - f_n^*(\psi_K(p'_0))/(\pi_{K_n} + u'_0)^{n-1} \in \pi^{-n+1}_{K_n}(\pi^{n\beta_K - 1}_{K_n}\eta_0, p\eta_J) \cdot \mathcal{S}_{K_n}, \qquad (4.5.5)$$

where $\mathcal{S}_{K_n} = \mathcal{O}_{K_n}[\![\eta_0/\pi_{K_n}, \eta_J]\!]\langle u'_0, u_J\rangle$. Hence,

$$\begin{aligned} \mathcal{S}_K \otimes_{\mathcal{O}_K} K_n &\cong \mathcal{O}_{K_n}[\![\eta_0/\pi_{K_n}, \eta_J]\!]\left[\frac{1}{p}\right]\langle u'_0, u_J\rangle / (f_n^*(\psi_K(p'_0)), f_n^*(\psi_K(p'_J))) \\ &= \mathcal{S}_{K_n}\left[\frac{1}{p}\right] \Big/ (f_n^*(\psi_K(p'_0))/(\pi_{K_n} + \eta_0)^{n-1}, f_n^*(\psi_K(p'_J))) \end{aligned}$$

gives rise to logarithmic thickening spaces for L_n/K_n with error gauge $\geqslant n\beta_K - (n-1)$; note that K_n/K being tamely ramified of ramification degree n gives a different normalization on the error gauge. □

Proposition 4.5.6. *There exists* $N \in \mathbb{N}$ *and* $\alpha_{L/K} \in [0, 1]$ *such that, for all integers* $n > N$ *congruent to* 1 *modulo ep, we have*

$$n \cdot b_{\log}(L/K) = b(L_n/K_n) - \alpha_{L/K}.$$

Proof. By Construction 2.1.16, f_n^* gives a finite étale morphism $f_n : A^1_{K_n}[0, \theta^{1/n}) \times A^m_{K_n}[0, 1) \to A^1_K[0, \theta) \times A^m_K[0, 1)$ for $a > 0$. Let $\mathcal{E}$ denote the differential module associated to L/K coming from a standard logarithmic thickening space. By Proposition 4.5.4, $f_n^*\mathcal{E}$ is a differential module associated to L_n/K_n given by the thickening space X therein (for some admissible subset of error gauge $\leqslant \beta_K n - (n-1)$). In particular,

$$\mathrm{ET}_{L_n/K_n} \supseteq \mathrm{ET}_{L/K} \times_{A^1_K[0,\theta) \times A^m_K[0,1), f_n} A^1_{K_n}[0, \theta^{1/n}) \times A^m_{K_n}[0, 1) =: f_n^*(\mathrm{ET}_{L/K}),$$

where ET_{L_n/K_n} is the étale locus with respect to this chosen admissible subset.

The morphism f_n is an off-centered tame base change, as discussed in § 2.1. By Proposition 2.1.17, for $s_{J^+} \subset \mathbb{R}$ such that $A_K^1[0, \theta^{s_0}] \times \cdots \times A_K^1[0, \theta^{s_m}] \subset \mathrm{ET}_{L/K}$, we have $\mathrm{IR}(f_n^*\mathcal{E}; s_{J^+}) = \mathrm{IR}(\mathcal{E}; s_0 + (n-1)/n, s_J)$. Thus, by Corollary 3.5.4,

$$\begin{aligned} b(L_n/K_n) &= n \cdot \min\left\{s \mid A_{K_n}^{m+1}[0, \theta^s] \subseteq \mathrm{ET}_{L_n/K_n} \text{ and } \mathrm{IR}(f_n^*\mathcal{E}; \underline{s}) = 1\right\} \\ &= n \cdot \min\{s \mid A_{K_n}^{m+1}[0, \theta^s] \subseteq f_n^*(\mathrm{ET}_{L/K}) \text{ and } \mathrm{IR}(f_n^*\mathcal{E}; \underline{s}) = 1\} \\ &= n \cdot \min\{s \mid A_K^1[0, \theta^{s+(n-1)/n}] \times A_K^m[0, \theta^s] \subseteq \mathrm{ET}_{L/K} \\ &\qquad \text{and } \mathrm{IR}(\mathcal{E}; s + (n-1)/n, \underline{s}) = 1\}, \end{aligned} \tag{4.5.7}$$

where the second equality holds because, as we shall see in a moment, the minimum of s can be achieved inside $\mathrm{ET}_{L/K}$. (Here, we have an extra n in the equation because we are supposed to use $|\pi_{K_n}| = \theta^{1/n}$ as the 'base scale' in Corollary 3.5.4.)

Applying Proposition 2.1.23(c) to $\mathcal{E}$, we know that the locus $Z(\mathcal{E}) = \{(s_{J^+}) \mid \mathrm{IR}(\mathcal{E}; s_{J^+}) = 1\}$ is transrational polyhedral in a neighborhood of $[b_{\log}(L/K), +\infty)^{m+1}$, namely, where $\mathcal{E}$ is defined. Hence, in a neighborhood of $s_1 = b_{\log}(L/K)$, the intersection of the boundary of Z with the surface defined by $s_1 = \cdots = s_m$ is of the form

$$s_0 - \alpha' s_1 = b_{\log}(L/K) + 1 - \alpha' b_{\log}(L/K),$$

where α' is the slope; we have $\alpha' \in [-\infty, 0]$ by the monotonicity property of Proposition 2.1.23(c). When $n \gg 0$, it is clear that the line $s \mapsto (s + (n-1)/n, s, \ldots, s)$ hits the boundary of Z at $s = b_{\log}(L/K) + 1/(n(1-\alpha'))$. This justifies the second equality in (4.5.7). It follows that

$$b(L_n/K_n) = n \cdot b_{\log}(L/K) + 1/(1-\alpha').$$

The different normalizations for ramification filtrations on G_K and G_{K_n} give the extra factor n. □

Remark 4.5.8. With more careful calculations, it is possible to prove the above proposition and Proposition 4.5.11 below for any n that is sufficiently large and coprime to p.

Notation 4.5.9. Assume $p > 2$. Let (b_J) be a p-basis of K; it naturally gives a p-basis of K_n. Let $K_n(x_J)^\wedge$ denote the completion of $K_n(x_J)$ with respect to the $(1, \ldots, 1)$-Gauss norm, and let K'_n denote the completion of the maximal unramified extension of $K_n(x_J)^\wedge$. Set

$$\widetilde{K}_n = K'_n((b_J + x_J \pi_{K_n}^2)^{1/p}), \quad \widetilde{L}_n = \widetilde{K}_n L.$$

Write $\beta_j = (b_j + x_j \pi_{K_n}^2)^{1/p}$ for $j \in J$. By Lemma 4.2.6, we have a continuous $\mathcal{O}_{K_n}$-homomorphism $\tilde{f} : \mathcal{O}_{K_n}[\![\eta_0/\pi_{K_n}, \eta_J]\!] \to \mathcal{O}_{\widetilde{K}_n}[\![\xi_0/\pi_{K_n}, \xi_J, \xi'_J]\!]$ such that $\tilde{f}^*(\eta_0) = \xi_0$ and $\tilde{f}^*(\eta_j) = (\beta_j + \xi_j)^p - (x_j + \xi'_j)(\pi_{K_n} + \xi_0)^2 - b_j$ for $j \in J$. For $a > 1$, it gives rise to $\tilde{f} : A_{\widetilde{K}_n}^{2m+1}[0, \theta^a] \to A_{K_n}^{m+1}[0, \theta^a] \hookrightarrow A_{K_n}^1[0, \theta^a] \times A_{K_n}^m[0, \theta^{a-1/n}]$, where the last morphism is the natural inclusion of an affinoid subdomain.

Proposition 4.5.10. *Assume $p > 2$, $\beta_K \geqslant (2m+n)/n$, and $a \in \mathbb{Q}_{>1}$. Let X be as in Proposition 4.5.4. Then the space*

$$X \times_{(A_{K_n}^1[0, \theta^{a+1/n}] \times A_{K_n}^m[0, \theta^a]), \tilde{f}} A_{\widetilde{K}_n}^{2m+1}[0, \theta^{a+1/n}]$$

is a thickening space for $\widetilde{L}_n/\widetilde{K}_n$ with error gauge $\geqslant n\beta_K - 2m - n + 1$; in particular, it is admissible.

Proof. The assertion follows immediately from Proposition 4.5.6 and applying Theorem 4.2.9 m times. □

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Proposition 4.5.11. *Assume $p>2$. There exists $N \in \mathbb{N}$ such that, for all integers $n>N$ congruent to* 1 *modulo ep, we have*

$$n \cdot b_{\log}(L/K) - 1 = b(\widetilde{L}_n/\widetilde{K}_n) - 2\alpha_{L/K}, \tag{4.5.12}$$

where $\alpha_{L/K}$ is the same *as in Proposition 4.5.6.*

Proof. We continue with the notation from Proposition 4.5.6. The previous proposition implies that $\tilde{f}^* f_n^* \mathcal{E}$ is a differential module associated to $\widetilde{L}_n/\widetilde{K}_n$ when $n>m$. By applying Lemma 4.3.3 m times, we have $\mathrm{IR}(\tilde{f}^* f_n^* \mathcal{E}; \underline{s}) = \mathrm{IR}(f_n^* \mathcal{E}; s, \underline{s+1/n})$. By Proposition 2.1.17, this further equals $\mathrm{IR}(\mathcal{E}; s+(n-1)/n, \underline{s+1/n})$. By the same argument as in Theorem 4.5.6, we deduce our result with the same $\alpha_{L/K}$. □

Remark 4.5.13. When $p=2$, we study $\widetilde{K}_n = K_n'((b_J + x_J \pi_{K_n}^3)^{1/p})$ instead; the same argument as above proves the proposition with (4.5.12) replaced by

$$n \cdot b_{\log}(L/K) - 2 = b(L_n/K_n) - 3\alpha_{L/K}.$$

For the following theorem, we do not impose any supplementary assumptions on K.

Theorem 4.5.14. *Let K be a complete discrete valuation field of mixed characteristic* $(0,p)$ *and let $\rho: G_K \to \mathrm{GL}(V_\rho)$ be a representation with finite monodromy. Then* $\mathrm{Swan}(\rho)$ *is a non-negative integer if $p \neq 2$ and is in $\frac{1}{2}\mathbb{Z}$ if $p=2$.*

Proof. First, as in the proof of Proposition 4.1.8, we may reduce to the case where ρ is irreducible and factors through a finite Galois extension L/K, for which Hypothesis 3.1.2 holds. In this case, $\mathrm{Swan}(\rho) = b_{\log}(L/K) \cdot \dim \rho$.

By Proposition 2.2.5(4), we have $\mathrm{Swan}(\rho|_{K_n}) = n \cdot \mathrm{Swan}(\rho)$ for any $K_n = K(\pi_K^{1/n})$ with $\gcd(n, ep)=1$. We need only prove that $\mathrm{Swan}(\rho|_{K_n}) \in \mathbb{Z}$ for two coprime values of n satisfying $\gcd(n, ep)=1$, and the statement for $\mathrm{Swan}(\rho)$ will follow immediately. In particular, we may assume that $\beta_K \geqslant 2$.

When $p>2$, we repeat the same argument again. There exist n_1 and n_2 that satisfy the condition of Propositions 4.5.6 and 4.5.11 and are such that $\gcd(n_1, n_2)=1$. Thus, by the non-logarithmic Hasse–Arf theorem, Theorem 4.3.5, we have

$$n_1 \mathrm{Swan}(\rho) + \alpha_{L/K} \dim \rho \in \mathbb{Z}, \quad n_1 \mathrm{Swan}(\rho) + 2\alpha_{L/K} \dim \rho \in \mathbb{Z};$$
$$n_2 \mathrm{Swan}(\rho) + \alpha_{L/K} \dim \rho \in \mathbb{Z}, \quad n_2 \mathrm{Swan}(\rho) + 2\alpha_{L/K} \dim \rho \in \mathbb{Z}.$$

This implies immediately that $\alpha_{L/K} \dim \rho \in \mathbb{Z}$; hence, $\mathrm{Swan}(\rho) \in \mathbb{Z}$.

When $p=2$, a similar argument using Remark 4.5.13 gives $\mathrm{Swan}(\rho) \in \frac{1}{2}\mathbb{Z}$. □

Remark 4.5.15. When $p=2$, we expect integrality of Swan conductors in the case where K is the composition of a discrete completely valued field with perfect residue field and an absolutely unramified complete discrete valuation field. In this case, we can factor ψ_K as $\mathcal{O}_K \to \mathcal{O}_K[\![\delta_0/\pi_K]\!] \to \mathcal{O}_K[\![\delta_0/\pi_K, \delta_J]\!]$ with the second map being a *homomorphism*. This fact may allow us to show that $\alpha_{L/K}$ is either 0 or 1 depending on whether ∂_0 dominates.

We do not know whether the integrality of $\mathrm{Swan}(\rho)$ might fail for $p=2$ in general.

4.6 An example of wildly ramified base change

In this subsection, we explicitly calculate an example, which will be used in the next subsection. This example was first introduced in [Ked07, Proposition 2.7.11]. We retain Hypotheses 3.1.2 and 3.2.10.

Lemma 4.6.1. *Let K_* be the finite extension of K generated by a root of*

$$T^p + \pi_K T^{p-1} = \pi_K. \tag{4.6.2}$$

Then K_* is Galois over K. Moreover, the logarithmic ramification break $b_{\log}(K_*/K)$ equals 1.

Proof. Let $h(T) = T^p + \pi_K T^{p-1} - \pi_K$ and let ϖ be a root of h. It is clear that ϖ is a uniformizer of K_*. We have

$$\begin{aligned} h(\varpi + T) &= (\varpi + T)^p + \pi_K(\varpi + T)^{p-1} - \pi_K \\ &= T^p + \sum_{i=1}^{p-1}\binom{p}{i}\varpi^i T^{p-i} + \pi_K \sum_{i=1}^{p-1}\binom{p-1}{i}\varpi^{p-1-i}T^i, \\ h(\varpi + \varpi^2 T) &= \varpi^{2p}T^p + \pi_K \sum_{i=1}^{p-1}\binom{p-1}{i}\varpi^{p-1+i}T^i + \sum_{i=1}^{p-1}\binom{p}{i}\varpi^{2p-i}T^{p-i} \\ &= \pi_K^2(1-\varpi^{p-1})^2 T^p + \pi_K^2(1-\varpi^{p-1})(p-1)T \\ &\quad + \pi_K^2(1-\varpi^{p-1})\sum_{i=2}^{p-1}\binom{p-1}{i}\varpi^{i-1}T^i + \sum_{i=1}^{p-1}\binom{p}{i}\varpi^{2p-i}T^{p-i}. \end{aligned}$$

Here, the terms are organized so that those written in the summations are small. We see that $h(\varpi + \varpi^2 T)/\pi_K^2$ is congruent to $T^p - T$ modulo ϖ. By Hensel's lemma, it splits completely in K_*. Hence, K_*/K is Galois. Moreover, the valuation of the difference between two distinct roots is 2. This implies that $b_{\log}(K_*/K) = 1$. □

Notation 4.6.3. Denote the roots of $h(T) = T^p + \pi_K T^{p-1} - \pi_K$ by $\varpi = \varpi_1, \ldots, \varpi_p$.

For $a > 0$, the standard logarithmic thickening space $\mathrm{TS}^a_{K_*/K,\log,\psi_K}$ for K_*/K is given by

$$\mathcal{O}^{a+1}_{\mathrm{TS},K_*/K,\log,\psi_K} = K\langle \pi_K^{-a-1}\delta_0, \pi_K^{-a}\delta_J, z\rangle/(z^p + (\pi_K + \delta_0)z^{p-1} - (\pi_K + \delta_0)).$$

Lemma 4.6.4. *Assume $a \in \mathbb{Q}_{>1}$. The standard logarithmic thickening space $\mathrm{TS}^a_{K_*/K,\log,\psi_K} \times_K K_*$ is isomorphic to the product of $A^m_{K_*}[0,\theta^a]$ with the disjoint union of p discs defined by $|z - \varpi_\gamma| \leqslant \theta^{a-(p-2)/p}$ for $\gamma = 1, \ldots, p$.*

Proof. We can rewrite $z^p + (\pi_K + \delta_0)z^{p-1} - (\pi_K + \delta_0) = 0$ as

$$\prod_{\gamma=1}^{p}(z - \varpi_\gamma) = \delta_0(1 - z^{p-1}). \tag{4.6.5}$$

Since $|z| \leqslant 1$, the right-hand side of (4.6.5) has norm less than or equal to θ^{a+1}, which is less than θ^2. On the left-hand side, for $\gamma \neq \gamma' \in \{1, \ldots, p\}$ we have $|\varpi_\gamma - \varpi_{\gamma'}| = \theta^{2/p}$. This forces one of the $|z - \varpi_{\gamma_0}|$ to be strictly smaller than the others, for some $\gamma_0 \in \{1, \ldots, p\}$. Thus, $|z - \varpi_{\gamma_0}| = |\delta_0|/(\theta^{2/p})^{p-1} = \theta^{a-(p-2)/p}$. □

Notation 4.6.6. For $\gamma = 1, \dots, p$, we define the K_*-homomorphism $f_\gamma^* : \mathcal{O}_K[\![\delta_0/\pi_K]\!] \to \mathcal{O}_{K_*}[\![\eta_0/\varpi_\gamma]\!]$ by sending δ_0 to

$$\frac{(\varpi_\gamma + \eta_0)^p}{1 - (\varpi_\gamma + \eta_0)^{p-1}} - \pi_K = \sum_{n=0}^{\infty} ((\varpi_\gamma + \eta_0)^{p+n(p-1)} - \varpi_\gamma^{p+n(p-1)}). \tag{4.6.7}$$

Lemma 4.6.8. *For $a > 1$, f_γ^* induces a K-morphism $f_\gamma : A^1_{K_*}[0, \theta^{a-(p-2)/p}] \to A^1_K[0, \theta^{a+1}]$, which is an isomorphism when we tensor the target with K_* over K. Moreover, if we use F_{a+1} and $F^*_{a-(p-2)/p}$ to denote, respectively, the completions of $K(\delta_0)$ and $K_*(\eta_0)$ with respect to the θ^{a+1}-Gauss norm and $\theta^{a+(p-2)/p}$-Gauss norm, then f_γ^* extends to a homomorphism $F_{a+1} \to F^*_{a-(p-2)/p}$.*

Proof. The statement follows from the fact that the leading term in (4.6.7) is $(2p-1)\varpi_\gamma^{2p-2}\eta_0$. □

Proposition 4.6.9. *Assume $a > 1$. Let $\mathcal{E}$ be a differential module over $A^1_K[0, \theta^{a+1}]$. For each $\gamma \in \{1, \dots, p\}$, this gives a differential module $f_\gamma^*\mathcal{E}$ over $A^1_{K_*}[0, \theta^{a-(p-2)/p}]$. Then we have*

$$\mathrm{IR}_0(f_\gamma^*\mathcal{E}; a - (p-2)/p) = \mathrm{IR}_0(\mathcal{E}; a+1).$$

Proof. The proof is similar to Proposition 2.1.17. By Lemma 4.6.8, we have the commutative diagram

$$\begin{array}{ccc} F_{a+1} & \xrightarrow{f^*_{\mathrm{gen}}} & F_{a+1}[\![\pi_K^{-a-1}T_0]\!]_0 \\ \downarrow f_\gamma^* & & \downarrow f_\gamma^* \\ F^*_{a-(p-2)/p} & \xrightarrow{f^*_{\mathrm{gen}}} & F^*_{a-(p-2)/p}[\![\varpi_\gamma^{-pa+p-2}T'_0]\!]_0 \end{array}$$

where we extend f_γ^* by $f_\gamma^*(T_0) = (\varpi_\gamma + \eta_0 + T'_0)^p/(1 - (\varpi_\gamma + \eta_0 + T'_0)^{p-1}) - (\varpi_\gamma + \eta_0)^p/(1 - (\varpi_\gamma + \eta_0)^{p-1})$.

We claim that for $r \in [0, 1)$, f_γ^* induces an isomorphism

$$F^*_{a-(p-2)/p} \times_{f_\gamma^*, F_{a+1}} (A^1_{F_{a+1}}[0, r\theta^{a+1})) \simeq A^1_{F^*_{a-(p-2)/p}}[0, r\theta^{a-(p-2)/p}).$$

Indeed, if $|T'_0| < r\theta^{a-(p-2)/p}$, then

$$\begin{aligned} T_0 &= \frac{(\varpi_\gamma + \eta_0 + T'_0)^p}{1 - (\varpi_\gamma + \eta_0 + T'_0)^{p-1}} - \frac{(\varpi_\gamma + \eta_0)^p}{1 - (\varpi_\gamma + \eta_0)^{p-1}} \\ &= ((\varpi_\gamma + \eta_0 + T'_0)^p - (\varpi_\gamma + \eta_0)^p) + ((\varpi_i + \eta_0 + T'_0)^{2p-1} - (\varpi_\gamma + \eta_0)^{2p-1}) + \cdots \\ &\in (2p-1)(\varpi_\gamma + \eta_0)^{2p-2}T'_0 + ((\varpi_\gamma + \eta_0)^{2p-1}T'_0, T'^p_0) \cdot \mathcal{O}_{K_*}\langle \varpi_\gamma^{-pa+p-2}\eta_0 \rangle [\![\varpi_\gamma^{-pa+p-2}T'_0]\!]. \end{aligned}$$

Hence, $|T_0| = \theta^{(2p-2)/p} \cdot |T'_0| < r\theta^a$.

Conversely, if $|T_0| < r\theta^a$, we rewrite the above equation as

$$T'_0 \in \frac{1}{(2p-1)(\varpi_\gamma + \eta_0)^{2p-2}} T_0 + (\varpi_\gamma T'_0) \cdot \mathcal{O}_{K_*}\langle \varpi_\gamma^{-pa+p-2}\eta_0 \rangle [\![\varpi_\gamma^{-pa+p-2}T'_0]\!]. \tag{4.6.10}$$

We substitute (4.6.10) back into itself recursively. The equation converges to a T'_0, which is an inverse.

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Therefore, Lemma 2.1.15 implies that for $r \in [0, 1)$,

$$\begin{aligned}
&\mathrm{IR}_0(\mathcal{E}; a+1) \leqslant r \\
&\quad \Longleftrightarrow f_{\mathrm{gen}}^*(\mathcal{E} \otimes F_{a+1}) \text{ is trivial on } A^1_{F_{a+1}}[0, r\theta^{a+1}) \\
&\quad \Longleftrightarrow \tilde{f}_\gamma^* f_{\mathrm{gen}}^*(\mathcal{E} \otimes F_{a+1}) = f_{\mathrm{gen}}^*(f_\gamma^*\mathcal{E} \otimes F^*_{a-(p-2)/p}) \text{ is trivial on } A^1_{F^*_{a-(p-2)/p}}[0, r\theta^{a-(p-2)/p}) \\
&\quad \Longleftrightarrow \mathrm{IR}_0(f_\gamma^*\mathcal{E}; a-(p-2)/p) \leqslant r.
\end{aligned}$$

The proposition follows. □

Construction 4.6.11. Fix a p-basis (b_J) of K; it naturally gives a p-basis of K_*. Fix a choice of $\psi_K : \mathcal{O}_K \to \mathcal{O}_K[\![\delta_0/\pi_K, \delta_J]\!]$ as in Construction 3.2.1. We will use the method in Construction 3.2.1 to define $\psi_{K_*,\gamma}$ for $\gamma = 1, \ldots, p$ such that the following diagram *commutes*.

$$\begin{array}{ccc}
\mathcal{O}_K & \xrightarrow{\psi_K} & \mathcal{O}_K[\![\delta_0/\pi_K, \delta_J]\!] \\
\big\downarrow & & \big\downarrow f_\gamma^* \\
\mathcal{O}_{K_*} & \xrightarrow{\psi_{K_*}} & \mathcal{O}_{K_*}[\![\eta_0/\varpi_\gamma, \delta_J]\!]
\end{array} \tag{4.6.12}$$

For any element $h \in \mathcal{O}_{K_*}$, first write $h = \sum_{i=0}^{p-1} h_i \varpi_\gamma^i$ where $h_i \in \mathcal{O}_K$. As in Construction 3.2.1, write each of the h_i as $h_i^\circ \pi_K^{e_i}$ for $e_i = v_K(h_i)$ and $h_i^\circ \in \mathcal{O}_K$; choose a compatible system of rth p-basis decomposition of h_i° as

$$h_i^\circ = \sum_{e_J=0}^{p^r-1} b_J^{e_J} \left(\sum_{n=0}^{\infty} \left(\sum_{n'=0}^{\lambda_{i,(r),e_J,n}} \alpha_{i,(r),e_J,n,n'}^{p^r} \right) \pi_K^n \right)$$

for some $\alpha_{i,(r),e_J,n,n'} \in \mathcal{O}_K^\times \cup \{0\}$ and some $\lambda_{i,(r),e_J,n} \in \mathbb{Z}_{\geqslant 0}$. We choose the system of rth p-basis decomposition of $h/\varpi_\gamma^{v_{K_*}(h)}$ to be

$$\frac{h}{\varpi_\gamma^{v_{K_*}(h)}} = \frac{1}{\varpi_\gamma^{v_{K_*}(h)}} \sum_{i=0}^{p-1} \varpi_\gamma^i \sum_{e_J=0}^{p^r-1} b_J^{e_J} \left(\sum_{n=0}^{\infty} \left(\sum_{n'=0}^{\lambda_{i,(r),e_J,n}} \alpha_{i,(r),e_J,n,n'}^{p^r} \right) (\varpi_\gamma^{p-1} + \varpi_\gamma^{2p-1} + \cdots)^{n+e_i} \right)$$

and define $\psi_{K_*,\gamma}(h)$ to be the limit

$$\begin{aligned}
\lim_{r \to +\infty} \Bigg\{ &\sum_{i=0}^{p-1} (\varpi_\gamma + \eta_0)^i \sum_{e_J=0}^{p^r-1} (b_J + \delta_J)^{e_J} \\
&\cdot \left(\sum_{n=0}^{\infty} \left(\sum_{n'=0}^{\lambda_{i,(r),e_J,n}} \alpha_{i,(r),e_J,n,n'}^{p^r} \right) ((\varpi_\gamma + \eta_0)^{p-1} + (\varpi_\gamma + \eta_0)^{2p-1} + \cdots)^{n+e_i} \right) \Bigg\}.
\end{aligned}$$

This gives a $\psi_{K_*,\gamma}$ defined in the way of Construction 3.2.1; the diagram (4.6.12) is commutative.

Hypothesis 4.6.13. For the rest of this subsection, let L/K_* be a finite Galois extension satisfying Hypotheses 3.1.2 and 3.2.10 and such that L/K is Galois.

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Proposition 4.6.14. *Let $a \in \mathbb{Q}_{>1}$. Then there exists admissible $(R_{J+}) \subset (\delta_{J+}) \cdot \mathcal{S}_K$ such that the logarithmic thickening space for L/K, after extension of scalars from K to K_*, is isomorphic to a disjoint union of p (different) logarithmic thickening spaces for L/K_*:*

$$\mathrm{TS}^a_{L/K,\log,R_{J+}} \times_K K_* \xrightarrow{\sim} \coprod_{\gamma=1}^{p} \mathrm{TS}^{pa-p+1}_{L/K_*,\log,\psi_{K_*,\gamma}}.$$

Proof. Write $\mathcal{O}_{K_*}\langle u_{J+}\rangle/(p_{J+}) = \mathcal{O}_L$ using Construction 3.1.6. Since $\mathcal{O}_K\langle z\rangle/(z^p + \pi_K z^{p-1} - \pi_K) = \mathcal{O}_{K_*}$, we may replace the coefficients in p_{J+} by elements in $\mathcal{O}_K\langle z\rangle$ with degree no greater than $p-1$ in z; we denote the resulting polynomials by p'_{J+}. Thus, by Lemma 4.6.4 and the commutativity of (4.6.12),

$$\begin{aligned}&\prod_{\gamma=1}^{p} K_*\langle \varpi_\gamma^{-pa+p-2}\eta_0, \varpi_\gamma^{-pa+p-1}\eta_J\rangle\langle u_{J+}\rangle/(\psi_{K_*,\gamma}(p_{J+}))\\ &\quad\cong K_*\langle \pi_K^{-a-1}\delta_0, \pi_K^{-a}\delta_J\rangle\langle u_{J+}, z\rangle/(\psi_K(p'_{J+}), z^p + (\pi_K+\delta_0)z^{p-1} - (\pi_K+\delta_0)),\end{aligned}$$

where the latter is a recursive logarithmic thickening space for L/K, base changed to K_*. By Proposition 3.6.6, this recursive logarithmic thickening space is isomorphic to a logarithmic thickening space $\mathrm{TS}^a_{L/K,\log,R_{J+}}$ for L/K for some admissible subset $R_{J+} \subset (\delta_{J+}) \cdot \mathcal{S}_K$. □

Corollary 4.6.15. *Let $\mathcal{E}_{L/K}$ be the differential module over $A^1_K[0,\theta^{a+1}] \times A^m_K[0,\theta^a]$ coming from $\mathrm{TS}^a_{L/K,\log,R_{J+}}$. For $\gamma \in \{1,\dots,p\}$, let $\mathcal{E}_{L/K_*,\gamma}$ be the differential module over $A^1_{K_*}[0,\theta^{a-(p-2)/p}] \times A^m_{K_*}[0,\theta^{a-(p-1)/p}]$ coming from $\mathrm{TS}^{ap-p+1}_{L/K_*,\log,\psi_{K_*,\gamma}}$. Then $\mathcal{E}_{L/K} \otimes_K K_* \simeq \bigoplus_{\gamma=1}^{p} f_{\gamma*}\mathcal{E}_{L/K_*,\gamma}$.*

Proof. This follows from Lemma 4.6.4 and Proposition 4.6.14. □

4.7 Subquotients of logarithmic ramification filtration

In this subsection, we prove Theorem 4.7.3, which says that the subquotients $\mathrm{Fil}^a_{\log} G_K/\mathrm{Fil}^{a+}_{\log} G_K$ of logarithmic ramification filtration are abelian groups killed by p if $a \in \mathbb{Q}_{>0}$ and are trivial if $a \notin \mathbb{Q}$. This uses the tricky base change discussed in the previous subsection.

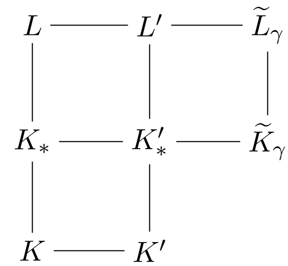
We assume Hypothesis 4.6.13 until we get to stating the main theorem, Theorem 4.7.3.

Notation 4.7.1. Fix $\gamma \in \{1,\dots,p\}$. Let (b_J) be a finite p-basis of K. It naturally gives a p-basis of K_*. Denote by $K(x_J)^\wedge$ the completion of $K(x_J)$ with respect to the $(1,\dots,1)$-Gauss norm and by K' the completion of the maximal unramified extension of $K(x_J)^\wedge$. Write $K'_* = K_*K'$ and $L' = K'_*L$. Set

$$\widetilde{K}_\gamma = K'_*((b_J + x_J\varpi_\gamma^{p-1})^{1/p}).$$

Write $\beta_J = (b_J + x_J\varpi_\gamma^{p-1})^{1/p}$ for simplicity. Denote the residue fields of $\widetilde{K}_\gamma$ and $\widetilde{L}_\gamma = L\widetilde{K}_\gamma$ by $\tilde{k}$ and $\tilde{l}$, respectively. Take the uniformizer and p-basis of $\widetilde{K}_\gamma$ to be ϖ_γ and $\{\beta_J, x_J\}$, respectively.

Situation 4.7.2. We have the following diagram of field extensions.



Note that the $(\widetilde{K}_\gamma)_{\gamma=1,\dots,p}$ are extensions of K'_* conjugate over K'. The ramification filtrations on $G_{\widetilde{K}_\gamma}$ are stable under the conjugate action of $\mathrm{Gal}(K'_*/K')$. To be precise, for any $b \geqslant 0$ and $g \in \mathrm{Gal}(K'_*/K')$, $g\mathrm{Fil}^b_{\log} G_{\widetilde{K}_\gamma} g^{-1} = \mathrm{Fil}^b_{\log} G_{g(\widetilde{K}_\gamma)}$ and $g\mathrm{Fil}^b G_{\widetilde{K}_\gamma} g^{-1} = \mathrm{Fil}^b G_{g(\widetilde{K}_\gamma)}$ inside $G_{K'}$. In particular, since L'/K' and hence $\widetilde{L}_\gamma/\widetilde{K}_\gamma$ is Galois, $b(\widetilde{L}_\gamma/\widetilde{K}_\gamma)$ and $b_{\log}(\widetilde{L}_\gamma/\widetilde{K}_\gamma)$ do not depend on $\gamma = 1, \dots, p$.

For the following theorem, we do not impose any supplementary assumptions on the field K.

Theorem 4.7.3. *Let K be a complete discrete valuation field of mixed characteristic $(0, p)$. Let G_K be its Galois group. Then the subquotients $\mathrm{Fil}^a_{\log} G_K/\mathrm{Fil}^{a+}_{\log} G_K$ of the logarithmic ramification filtration are trivial if $a \notin \mathbb{Q}$ and are abelian groups killed by p if $a \in \mathbb{Q}_{>0}$.*

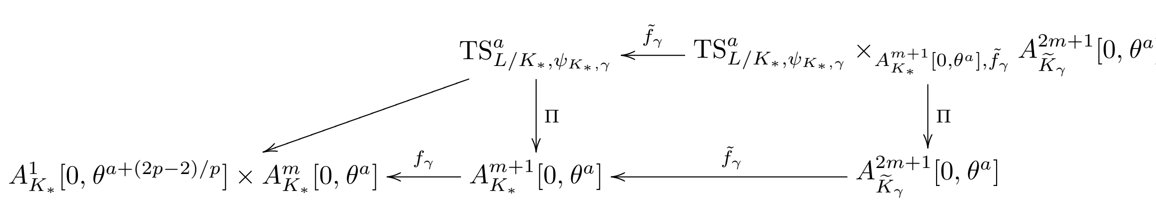
Proof. We proceed as in the proof of Theorem 4.3.5. Fix $a > 0$. Let L be a finite Galois extension of K with Galois group $G_{L/K}$ with an induced ramification filtration. We may assume that $\mathrm{Fil}^{a+}_{\log} G_{L/K}$ is the trivial group but $\mathrm{Fil}^a_{\log} G_{L/K}$ is not. We may also assume Hypothesis 3.1.2. Furthermore, by Proposition 2.2.5(4), we are free to make a tame base change and assume that $a = b_{\log}(L/K) > 1$ and $p\beta_K \geqslant m(p-1)+1$. Finally, we may replace L by LK_* since $b_{\log}(K_*/K) = 1$ by Lemma 4.6.1. We need to show that $\mathrm{Fil}^a_{\log} G_{L/K}$ is an abelian group killed by p if $a \in \mathbb{Q}_{>1}$ and trivial if $a \notin \mathbb{Q}$.

We claim that each of the logarithmic ramification breaks $b > 1$ of L/K will become a non-log ramification break $bp - p + 2$ on $\widetilde{L}_1/\widetilde{K}_1$. In other words, $\mathrm{Fil}^b_{\log} G_{L/K} \subseteq \mathrm{Fil}^{pb-p+2} G_{\widetilde{L}_\gamma/\widetilde{K}_\gamma}$ for any $\gamma \in \{1, \dots, p\}$ and $b > 1$. (It does not matter which γ we choose, as they give the same answer by Situation 4.7.2.) Then the theorem is a direct consequence of the non-logarithmic Hasse–Arf theorem, namely Theorem 4.3.5(2).

To establish the claim, it suffices to prove the highest ramification breaks, as the others will follow from the calculation of the other extensions L.

For each $\gamma \in \{1, \dots, p\}$, there exists a unique continuous $\mathcal{O}_{K_*}[\![\eta_0/\varpi_\gamma]\!]$-homomorphism $\tilde{f}^*_\gamma$: $\mathcal{O}_{K_*}[\![\eta_0/\varpi_\gamma, \delta_J]\!] \to \mathcal{O}_{\widetilde{K}_\gamma}[\![\eta_0/\varpi_\gamma, \eta_J, \eta'_J]\!]$ such that $\tilde{f}^*_\gamma \delta_j = (\beta_j + \eta_j)^p - (x_j + \eta'_j)(\varpi_\gamma + \eta_0)^{p-1} - b_j$ for $j \in J$. For $a > 1$, $\tilde{f}^*_\gamma$ gives a morphism $\tilde{f}_\gamma : A^{2m+1}_{\widetilde{K}_\gamma}[0, \theta^a] \to A^{m+1}_{K_*}[0, \theta^a]$.

Let $\mathrm{TS}^a_{L/K_*, \psi_{K_*,\gamma}}$ be the standard thickening space for L/K_* and $\psi_{K_*,\gamma}$. We have a Cartesian diagram as follows.



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By applying Theorem 4.2.9 m times, $\mathrm{TS}^a_{L/K_*,\psi_{K_*,\gamma}} \times_{A^{m+1}_{K_*}[0,\theta^a],\tilde{f}_\gamma} A^{2m+1}_{\widetilde{K}_\gamma}[0,\theta^a]$ is an admissible recursive non-logarithmic thickening space (of error gauge $\geqslant p\beta_K - m(p-1) \geqslant 1$), which is isomorphic to an admissible non-logarithmic thickening space for $\widetilde{L}_\gamma/\widetilde{K}_\gamma$ by Proposition 3.6.6. Thus $\tilde{f}^*_\gamma \mathcal{E}_{L/K_*,\gamma}$ is a differential module associated to $\widetilde{L}_\gamma/\widetilde{K}_\gamma$.

By Proposition 4.6.9 and Lemma 4.3.3, we have

$$\mathrm{IR}(\tilde{f}^*_\gamma \mathcal{E}_{L/K_*,\gamma}; \underline{s}) = \mathrm{IR}\bigg(\mathcal{E}_{L/K_*,\gamma}; s, \underline{s+\frac{p-2}{p}}\bigg) = \mathrm{IR}\bigg((f_\gamma)_*\mathcal{E}_{L/K_*,\gamma}; s+\frac{2p-2}{p}, \underline{s+\frac{p-2}{p}}\bigg).$$

The claim follows from Corollaries 4.6.15 and 3.5.4. □

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