AN EXPANSION THEOREM FOR A PAIR OF SINGULAR FIRST ORDER EQUATIONS

S. D. CONTE AND W. C. SANGREN

1. Introduction. Titchmarsh **(4)** has shown how the classical method of complex variables can be used to obtain expansion theorems for the singular cases of the second order equation

$$y''(x) + [\lambda - q(x)]y(x) = 0. \tag{1}$$

The purpose of this paper is to indicate how these results can be generalized to the singular cases of the pair of first order equations

$$\begin{aligned} u'(x) - [\lambda + q_1(x)]v &= 0, \\ v'(x) + [\lambda + q_2(x)]u &= 0. \end{aligned} \tag{2}$$

The system (2) is a special case of the Dirac wave equations for a particle in a central field in the relativistic case, a system which has recently been investigated at the Oak Ridge National Laboratories. The presentation is largely formal to avoid excessive detail (see especially §4), but all omitted proofs are included in a report **(1)** and in any case are direct generalizations of the corresponding proofs given by Titchmarsh for the second order equation (1). The principal result may be summarized in the following

THEOREM I. *Consider the system* (2) *over the semi-infinite interval* $[0 \leqslant x < \infty]$ *and under the boundary condition*

$$u(0)\cos\alpha + v(0)\sin\alpha = 0, \tag{3}$$

where α *is a real constant. Let* $q_1(x)$, $q_2(x)$ *be real-valued continuous functions of* x *which belong to* $L(0, \infty)$. *We define a solution of* (2), (3) *as a pair of functions* $[u(x, \lambda), v(x, \lambda)]$, *with continuous first derivatives, satisfying this system. Then the values of* λ *for which such solutions exist form a continuous spectrum over the real* λ-*axis* $[-\infty < \lambda < \infty]$. *An arbitrary function pair* $f(x) = [f_1(x), f_2(x)]$ *which are continuous, of bounded variation and* $L^2(0, \infty)$, *and which satisfy the condition* (3) *at* $x = 0$ *may be represented by the generalized Fourier integrals*

$$f_1(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} g(\lambda)\, u(x, \lambda)\, d\lambda,$$

$$f_2(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} g(\lambda)\, v(x, \lambda)\, d\lambda,$$

where

$$g(\lambda) = [\mu^2(\lambda) + \nu^2(\lambda)]^{-1}\int_0^{\infty}[u(y, \lambda)f_1(y) + v(y, \lambda)f_2(y)]\, dy,$$

and $\mu(\lambda)$, $\nu(\lambda)$ *are functions of* λ *which do not vanish simultaneously.*

Received November 23, 1953.

2. Preliminaries. Consider the system (2) over the finite interval $[0 \leqslant x \leqslant b)$. Let

$$\phi(x, \lambda) = [\phi_1(x, \lambda), \phi_2(x, \lambda)], \qquad \theta(x, \lambda) = [\theta_1(x, \lambda), \theta_2(x, \lambda)]$$

be two solutions of (2) such that

$$\begin{aligned} \phi_1(0) &= -\sin\alpha, & \phi_2(0) &= \cos\alpha, \\ \theta_1(0) &= -\cos\alpha, & \theta_2(0) &= -\sin\alpha \end{aligned}$$

Let the Wronskian of ϕ, θ be defined as $W_x[\phi, \theta] = \phi_1\theta_2 - \phi_2\theta_1$. Then it is easily shown that $W_x[\phi, \theta]$ is independent of x. Now $W_0[\phi, \theta] = 1$, so $\phi(x, \lambda)$ and $\theta(x, \lambda)$ are linearly independent solutions. A general solution of (2) may be written $\theta(x, \lambda) + l(\lambda)\,\phi(x, \lambda)$. If this general solution is required to satisfy a real boundary condition of Sturmian type at $x = b$, it is known **(3)** that the eigenvalues are real, simple, discrete, and extend from $\lambda = -\infty$ to $\lambda = +\infty$. Moreover, the corresponding eigenfunctions are real. To obtain the spectrum in the singular case we take the limit of the general solution as $b \to \infty$. Then, following Titchmarsh, it is easily shown that, for values of λ other than real values, (2) has a solution $\psi(x, \lambda) = [\psi_1, \psi_2]$, say

$$\psi(x, \lambda) = \theta(x, \lambda) + m(\lambda)\,\phi(x, \lambda), \tag{4}$$

which belongs to $L^2(0, \infty)$. The definition of the function $m(\lambda)$ depends upon a limit of circles in the complex λ-plane which may be either a limit point or a limit circle. In the limit circle case all solutions are $L^2(0, \lambda)$. In addition $m(\lambda)$ is analytic in either the upper or lower half plane, it has the property $m(\bar{\lambda}) = \overline{m(\lambda)}$, and its imaginary part determines the spectrum. We proceed to determine $m(\lambda)$ for our system.

3. Nature of the spectrum. In this section we investigate order properties of the solution of (2) for large values of x, and apply these properties to the determination of the spectrum. It can be verified directly that a solution of (2) where $\phi_1(0) = -\sin\alpha$, $\phi_2(0) = \cos\alpha$ satisfies

$$\phi_1(x, \lambda) = \sin(\lambda x - \alpha) + \int_0^x \phi_2 q_1 \cos\lambda(x-s)\,ds - \int_0^x \phi_1 q_2 \sin\lambda(x-s)\,ds, \tag{5}$$

$$\phi_2(x, \lambda) = \cos(\lambda x - \alpha) - \int_0^x \phi_2 q_1 \sin\lambda(x-s)\,ds - \int_0^x \phi_1 q_2 \cos\lambda(x-s)\,ds. \tag{6}$$

Let $\lambda = \sigma + it$, $t > 0$, and let $\phi_1(x, \lambda) = e^{tx} h_1(x)$, $\phi_2(x, \lambda) = e^{tx} h_2(x)$, substitute in (5), (6) and take absolute values. We obtain

$$\begin{aligned} |h_1(x)| &\leqslant M + \int_0^x \{|h_2|\cdot|q_1| + |h_1|\cdot|q_2|\}\,ds, \\ |h_2(x)| &\leqslant M + \int_0^x \{|h_2|\cdot|q_1| + |h_1|\cdot|q_2|\}\,ds, \end{aligned} \tag{7}$$

where $M = O(1)$ for large x. At this point we need the following lemma which is proved in another paper by the authors **(2)**:

LEMMA. *Let h_1, h_2, g_1, g_2 be non-negative functions of x over the interval $[0 \leqslant x \leqslant x_1]$; let h_1, h_2 be continuous and g_1, g_2 integrable over this interval. If h_1, h_2 satisfy the inequalities*

$$h_1(x), h_2(x) \leqslant M + \int_0^x \{h_1(s)\, g_1(s) + h_2(s)\, g_2(s)\}\, ds,$$

then

$$h_1(x), h_2(x) \leqslant C_1 \exp\left\{ \int_0^x (g_1 + g_2)\, ds \right\}, \qquad [0 \leqslant x \leqslant x_1].$$

This lemma may be applied to the inequalities (7) to yield the result

$$|h_1|, |h_2| \leqslant M \exp\left\{ \int_0^x [|q_1| + |q_2|]\, ds \right\},$$

and since $q_1(x)$, $q_2(x)$ are $L(0, \infty)$ it follows that $h_1(x)$, $h_2(x)$ are bounded for all x. Hence for large x, $\phi_1(x, \lambda) = O(e^{tx})$, $\phi_2(x, \lambda) = O(e^{tx})$.

Now for real λ as $x \to \infty$ (5) and (6) may be written

$$\begin{aligned} \phi_1(x, \lambda) &= \mu(\lambda) \cos \lambda x + \nu(\lambda) \sin \lambda x + o(1), \\ \phi_2(x, \lambda) &= \mu(\lambda) \cos \lambda x - \nu(\lambda) \sin \lambda x + o(1), \end{aligned} \tag{8}$$

where

$$\begin{aligned} \mu(\lambda) &= -\sin \alpha + \int_0^\infty [q_1 \phi_2 \cos \lambda s + q_2 \phi_1 \sin \lambda s]\, ds, \\ \nu(\lambda) &= \cos \alpha + \int_0^\infty [q_1 \phi_2 \sin \lambda s - q_2 \phi_1 \cos \lambda s]\, ds, \end{aligned} \tag{9}$$

and $o(1)$ indicates terms which approach zero as $x \to \infty$. The integrals in (9) converge uniformly in λ and hence μ, ν are continuous and bounded functions of λ. If $\theta(x, \lambda)$ is that solution of (2) satisfying $\theta_1(0) = -\cos \alpha$, $\theta_2(0) = -\sin \alpha$, we may similarly write

$$\begin{aligned} \theta_1(x, \lambda) &= \xi(\lambda) \cos \lambda x + \eta(\lambda) \sin \lambda x + o(1), \\ \theta_2(x, \lambda) &= \eta(\lambda) \cos \lambda x - \xi(\lambda) \sin \lambda x + o(1), \end{aligned} \tag{10}$$

where

$$\begin{aligned} \xi(\lambda) &= -\cos \alpha + \int_0^\infty [q_1 \theta_2 \cos \lambda\, s + q_2 \theta_1 \sin \lambda\, s]\, ds, \\ \eta(\lambda) &= -\sin \alpha + \int_0^\infty [q_1 \theta_2 \sin \lambda\, s - q_2 \theta_1 \cos \lambda\, s]\, ds. \end{aligned} \tag{11}$$

Hence we have

$$W[\phi, \theta] = \phi_1 \theta_2 - \phi_2 \theta_1 = \mu\eta - \nu\xi + o(1).$$

But from the boundary conditions at $x = 0$ we know that $W[\phi, \theta] = 1$, so that for real λ as $x \to \infty$

$$\mu\eta - \nu\xi = 1. \tag{12}$$

We deduce from (12) that μ, ν cannot both vanish for the same λ.

Now for complex λ we can show by making use of the order properties on the solutions ϕ, θ and by the fact that $q_1(x)$, $q_2(x)$ are $L(0, \infty)$ that

$$\phi_1(x, \lambda) = e^{-i\lambda x}[M_1(\lambda) + o(1)], \tag{13}$$

$$\phi_2(x, \lambda) = e^{-i\lambda x}[M_2(\lambda) + o(1)], \tag{14}$$

$$\theta_1(x, \lambda) = e^{-i\lambda x}[N_1(\lambda) + o(1)], \tag{15}$$

$$\theta_2(x, \lambda) = e^{-i\lambda x}[N_2(\lambda) + o(1)], \tag{16}$$

where

$$\begin{aligned} M_2(\lambda) &= -\frac{\sin\alpha}{2i} + \frac{\cos\alpha}{2} - \tfrac{1}{2}\int_0^\infty e^{i\lambda s}[iq_1\phi_2 + q_2\phi_1]\,ds, \\ M_1(\lambda) &= -\frac{\sin\alpha}{2} - \frac{\cos\alpha}{2i} + \tfrac{1}{2}\int_0^\infty e^{i\lambda s}[q_1\phi_2 - iq_2\phi_1]\,ds, \\ N_1(\lambda) &= -\frac{\cos\alpha}{2} + \frac{\sin\alpha}{2i} + \tfrac{1}{2}\int_0^\infty e^{i\lambda s}[q_1\theta_2 - iq_2\theta_1]\,ds, \\ N_2(\lambda) &= -\frac{\sin\alpha}{2} - \frac{\cos\alpha}{2i} - \tfrac{1}{2}\int_0^\infty e^{i\lambda s}[iq_1\theta_2 + q_2\theta_1]\,ds. \end{aligned} \tag{17}$$

Now let $\psi(x, \lambda) = \theta(x, \lambda) + m(\lambda)\,\phi(x, \lambda)$ be that solution of (2) which for complex λ is $L^2(0, \infty)$. Using (13)-(16) we have

$$\begin{aligned} \psi_1 &= \theta_1 + m\phi_1 = e^{-i\lambda x}[N_1 + mM_1 + o(1)], \\ \psi_2 &= \theta_2 + m\phi_2 = e^{-i\lambda x}[N_2 + mM_2 + o(1)]. \end{aligned}$$

Now ϕ, θ certainly do not belong to $L^2(0, \infty)$, and if ψ is to be $L^2(0, \infty)$ we must have

$$m(\lambda) = -\frac{N_1}{M_1} = -\frac{N_2}{M_2}.$$

In (17) let λ tend to a real limit formally, i.e., let $t \to 0$. We obtain

$$\begin{aligned} N_1 &\to \tfrac{1}{2}(\xi + i\eta), & M_1 &\to \tfrac{1}{2}(\mu + i\nu), \\ N_2 &\to \tfrac{1}{2}(\eta - i\xi), & M_2 &\to \tfrac{1}{2}(\nu - i\mu), \end{aligned}$$

where ξ, η, μ, ν are defined by (9), (11). Hence

$$\lim_{t\to 0} m(\lambda) = -\frac{\xi(\lambda) + i\eta(\lambda)}{\mu(\lambda) + i\nu(\lambda)}.$$

The imaginary part of $m(\lambda)$ for real λ is therefore

$$\Im\{m(\lambda)\} = -(\mu^2 + \nu^2)^{-1}, \tag{18}$$

and from (12), (18) it is apparent that $\Im\{m(\lambda)\}$ is a non-positive, non-vanishing continuous function bounded for all λ over the range $[-\infty < \lambda < \infty]$.

4. The expansion theorem. We define a function pair $\Phi(x, \lambda) = [\Phi_1, \Phi_2]$ by the equations

$$\begin{aligned} \Phi_1 &= \psi_1(x, \lambda)\int_0^x \phi(y, \lambda)\cdot f(y)\,dy + \phi_1(x, \lambda)\int_x^\infty \psi(y, \lambda)\cdot f(y)\,dy, \\ \Phi_2 &= \psi_2(x, \lambda)\int_0^x \phi(y, \lambda)\cdot f(y)\,dy + \phi_2(x, \lambda)\int_x^\infty \psi(y, \lambda)\cdot f(y)\,dy, \end{aligned}$$

where λ is a complex parameter, $\phi(x, \lambda)$, $\theta(x, \lambda)$ are those solutions of (2) discussed in §3, $f(x) = [f_1(x), f_2(x)]$ is a pair of functions which are continuous and of bounded variation and which belong to $L^2(0, \infty)$, and, for example, $\phi \cdot f \equiv \phi_1 f_1 + \phi_2 f_2$. It may be verified directly that Φ_1, Φ_2 satisfy the inhomogeneous equations

$$\Phi_1' - [\lambda + q_1(x)]\Phi_2 = f_2(x),$$
$$\Phi_2' + [\lambda + q_2(x)]\Phi_1 = f_1(x).$$

It can be shown by deforming the straight line joining $-R + i\delta$ to $R + i\delta$ into the semicircle on that base and lying in the upper half plane that

$$f(x) = \lim_{R \to \infty} \left\{ -\frac{1}{i\pi} \int_{-R+i\delta}^{R+i\delta} \Phi(x, \lambda)\, d\lambda \right\}, \tag{19}$$

uniformly in $\delta > 0$. The proof of this is straightforward provided that the following theorem on the asymptotic behavior of the solutions of (2) for large λ is available.

THEOREM II. *Under the initial condition* $u(0) = -\sin\alpha$, $v(0) = \cos\alpha$ *the system* (2) *has the following asymptotic solution for large* $\lambda = \sigma + it$, $t > 0$,

$$u(x) = \sin(\xi - \alpha) + O(e^{tx}/|\lambda|),$$
$$v(x) = \cos(\xi - \alpha) + O(e^{tx}/|\lambda|),$$

where

$$\xi(x) = \lambda x + \tfrac{1}{2} \int_0^x [q_1(s) + q_2(s)]\, ds.$$

The proof of Theorem II is given in §5. Let us proceed to the limit formally in (19), recalling that $\psi = \theta + m\phi$, that

$$\Im\{m(\lambda)\} = -(\mu^2 + \nu^2)^{-1},$$

and that ϕ, θ are real for real λ. Letting $R \to \infty$, $\delta \to 0$, we obtain

$$\begin{aligned}
\Im\left\{ -\frac{1}{\pi} \int_{-R+i\delta}^{R+i\delta} \Phi_1(x, \lambda)\, d\lambda \right\}
&= \Im\left\{ -\frac{1}{\pi} \int_{-R+i\delta}^{R+i\delta} \psi_1(x, \lambda)\, d\lambda \int_0^x \phi \cdot f\, dy \right\} \\
&\qquad + \Im\left\{ -\frac{1}{\pi} \int_{-R+i\delta}^{R+i\delta} \phi_1(x, \lambda)\, d\lambda \int_x^\infty \psi \cdot f\, dy \right\} \\
&\to \frac{1}{\pi} \int_{-\infty}^{\infty} \phi_1(x, \lambda)(\mu^2 + \nu^2)^{-1} d\lambda \int_0^x \phi \cdot f\, dy \\
&\qquad + \frac{1}{\pi} \int_{-\infty}^{\infty} \phi_1(x, \lambda)\, d\lambda \int_x^\infty (\mu^2 + \nu^2)^{-1} \phi \cdot f\, dy.
\end{aligned}$$

The real part is non-contributing and hence upon combining the last two terms above we have the expansion for $f_1(x)$:

$$f_1(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\phi_1(x, \lambda)}{\mu^2 + \nu^2} d\lambda \int_0^\infty \{\phi_1(y, \lambda) f_1(y) + \phi_2(y, \lambda) f_2(y)\}\, dy.$$

https://doi.org/10.4153/CJM-1954-060-0 Published online by Cambridge University Press

The expansion for $f_2(x)$ is found similarly. The functions $f_1(x), f_2(x)$ are not entirely independent but must satisfy at $x = 0$ the condition

$$\cos\alpha\, f_1(0) + \sin\alpha\, f_2(0) = 0.$$

As a simple example of the expansion theorem consider the system (2) with $q_1(x) \equiv q_2(x) \equiv 0$. The solution of (2) for $u(0) = -\sin\alpha$, $v(0) = \cos\alpha$ is

$$\phi_1(x, \lambda) = \sin(\lambda x - \alpha), \qquad \phi_2(x, \lambda) = \cos(\lambda x - \alpha).$$

From §3 we have $\mu = -\sin\alpha$, $\nu = \cos\alpha$, $\mathfrak{J}\{m(\lambda)\} = -1$. The expansions for $f = [f_1(x), f_2(x)]$ are

$$\text{(20)} \quad f_1(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \sin(\lambda x - \alpha)\, d\lambda \int_0^{\infty} \{\sin(\lambda y - \alpha) f_1(y) + \cos(\lambda y - \alpha) f_2(y)\}\, dy,$$

$$\text{(21)} \quad f_2(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \cos(\lambda x - \alpha)\, d\lambda \int_0^{\infty} \{\sin(\lambda y - \alpha) f_1(y) + \cos(\lambda y - \alpha) f_2(y)\}\, dy$$

To obtain the ordinary Fourier sine integral from these, set $\alpha = 0$, $f_2(x) \equiv 0$. Then using the fact that the integrand in (20) is an even function of λ we obtain

$$f_1(x) = \frac{2}{\pi}\int_0^{\infty} \sin\lambda x\, d\lambda \int_0^{\infty} \sin\lambda y\, f_1(y)\, dy.$$

5. An asymptotic solution for large λ. The proof of Theorem II may be outlined as follows. W. Hurwitz **(3)** has obtained a similar asymptotic solution to the system (2) but for real λ and in the regular case. We introduce functions $U(x, \lambda)$, $V(x, \lambda)$ by the relations

$$\text{(22)} \qquad \begin{aligned} u(x) &= U + (1 + q_1/2\lambda)\sin(\xi - \alpha), \\ v(x) &= V + (1 + q_2/2\lambda)\cos(\xi - \alpha), \end{aligned}$$

where $\xi(x)$ is defined as in Theorem II. We wish to show that U, V are $O(e^{tx}/|\lambda|)$. Substitute into equation (2) and rearrange it to obtain

$$\text{(23)} \qquad \begin{aligned} U'(x) - [\lambda + q_1]V &= P(x, \lambda)/\lambda, \\ V'(x) + [\lambda + q_2]U &= Q(x, \lambda)/\lambda, \end{aligned}$$

where

$$\begin{aligned} P(x, \lambda) &= C_1(x)\sin\xi + C_2(x)\cos\xi, \\ Q(x, \lambda) &= C_3(x)\cos\xi + C_4(x)\sin\xi, \end{aligned}$$

and the coefficients C_1, C_2, C_3, C_4 are continuous functions of x, independent of λ. Equations (23) may be written as integral equations

$$\text{(24)} \qquad \begin{aligned} U(x, \lambda) &= F(x, \lambda)/\lambda + \int_0^x [-q_2 U \sin\lambda(x - s) + q_1 V \cos\lambda(x - s)]\, ds, \\ V(x, \lambda) &= G(x, \lambda)/\lambda + \int_0^x [-q_2 U \cos\lambda(x - s) + q_1 V \sin\lambda(x - s)]\, ds, \end{aligned}$$

where F and G are functions which for large λ can be shown to be $O(e^{tx})$. Now set $U = e^{tx} U_1(x, \lambda)$, $V = e^{tx} V_1(x, \lambda)$, substitute into (24) and take absolute values. We obtain the inequalities

$$|U_1|,\ |V_1| \leqslant O(1/|\lambda|) + \int_0^x [|q_2| \cdot |U_1| + |q_1| \cdot |V_1|]\, ds.$$

Now since U_1, V_1 are continuous functions of x for all x and since q_1, q_2 are $L(0, \infty)$, the lemma in §3 applies and hence

$$|U_1|,\ |V_1| \leqslant O(1/|\lambda|) \exp \int_0^x (|q_1| + |q_2|)\, ds.$$

Thus U_1, V_1 are $O(1/|\lambda|)$ and U, V are $O(e^{tx}/|\lambda|)$ for each x over the interval $[0 \leqslant x < \infty]$. Theorem II follows immediately upon substituting these asymptotic expressions for U, V into relations (22).

References

1. S. D. Conte and W. C. Sangren, *An eigenfunction expansion associated with a pair of first order differential equations*, Report, Oak Ridge National Laboratories, 1952.

2. ———, *An asymptotic solution for a pair of first order equations*, Proc. Amer. Math. Soc., *4* (1953), 696–702.

3. W. Hurwitz, *An expansion theorem for a pair of linear first order differential equations*, Trans. Amer. Math. Soc., *22* (1921), 526–543.

4. E. C. Titchmarsh, *Eigenfunction expansions associated with second order differential equations* (Oxford, 1946).

Wayne University *Oak Ridge National Laboratories*

https://doi.org/10.4153/CJM-1954-060-0 Published online by Cambridge University Press