PRIMITIVE ORE EXTENSIONS

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Introduction. Apart from simple Ore extensions such as the Weyl algebras, the best known example of a primitive Ore extension is the universal enveloping algebra $U(g)$ of the 2-dimensional solvable Lie algebra g over a field k of characteristic zero, see [**4**, p. 22]. $U(g)$ is a polynomial algebra over k in two indeterminates x and y with multiplication subject to the relation $xy - yx = y$, and may be regarded either as an Ore extension of $k[x]$ by the k-automorphism which maps x to $x-1$ or as an Ore extension of $k[y]$ by the derivation $y\,d/dy$. The argument suggested in [**4**, p. 22] to prove the primitivity of $U(g)$ can easily be generalised [**6**] to show that, if α is an automorphism of the ring R then the following conditions are sufficient for $R[x, \alpha]$ to be primitive: (i) no power α^s, $s \geqq 1$, of α is inner; (ii) the only ideals of R invariant under α are 0 and R. These conditions are necessary and sufficient for the simplicity of the skew Laurent polynomial ring $R[x, x^{-1}, \alpha]$ but are not necessary for the primitivity of $R[x, \alpha]$ (the ordinary polynomial ring $D[x]$ over a division ring D not algebraic over its centre is easily seen to be primitive).

In the case of Ore extensions by derivations the arguments given in [**3**, p. 353] can be adapted to prove that, if the ring R contains the rationals and if δ is a derivation on R, then $R[x, \delta]$ is simple if and only if δ is outer and the only ideals of R invariant under δ are 0 and R [**6**]. The object of this paper is to find conditions which are satisfied by the above example $R = k[y]$, $\delta = y\,d/dy$ and which are sufficient for $R[x, \delta]$ to be primitive. After notation and background information are established in §1, two such conditions are found in §2 and these are shown to be logically independent. The results are then applied to two other problems involving Ore extensions, namely those of when an Ore extension is a fully bounded ring (§3) and a Jacobson ring (§4).

1. Throughout this paper R will denote a right noetherian ring with 1, δ will be a derivation on R and S will be the Ore extension $R[x, \delta]$, i.e. S is a ring of polynomials over R in an indeterminate x with multiplication subject to the relation

$$xr = rx + \delta(r) \quad \text{for all} \quad r \in R.$$

R is said to be a *Ritt ring* if R contains the field of rational numbers as a subring and is said to have no **Z**-*torsion* if for all $r \in R$ and positive integers n, $nr = 0$ if and only if $r = 0$.

An ideal I of R is said to be a δ-*ideal* of R if $\delta(I) \subseteq I$. A δ-ideal I of R is δ-*prime* if for all δ-ideals A, B of R, $AB \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$.

PROPOSITION 1. *If R is a Ritt ring then the δ-prime ideals of R are precisely those prime ideals of R which are δ-ideals.*

Proof. See [**5**, Lemma 2.1 and Theorem 2.2].

We denote the nilpotent radicals of R and S by $N(R)$ and $N(S)$ respectively and their

Jacobson radicals by $J(R)$ and $J(S)$. If I is an ideal of R we denote by $N(I)$ (resp. $J(I)$) the ideal of R such that $N(I)/I = N(R/I)$ (resp. $J(I)/I = J(R/I)$). $\mathscr{C}_R(I)$ will denote the set $\{c \in R \mid [c+I]$ is a regular element of $R/I\}$.

PROPOSITION 2. $J(S) = \left(\bigcap_{i=0}^{\infty} \delta^{-i}(N(R))\right) S$. *If R has no $\mathbb{Z}$-torsion then* $J(S) = N(R)S$.

Proof. See [5, Lemma 2.1 and Corollary 2.7].

2. Following [1] δ will be said to be *rigid* on R if the mapping θ from the set of ideals of S to the set of δ-ideals of R, defined by $\theta(I) = I \cap R$ for all ideals I of S, is a bijection. δ will be said to be *stiff* on R if $I \cap R \neq 0$ for all non-zero ideals I of S. In [1, §4.8] it is shown that, if R is a Ritt ring and there exists a central element z of R such that $\delta(z)$ is a unit, then δ is rigid on R. The proof is adapted below to prove the corresponding result on stiff derivations.

LEMMA 1. *Let R be a ring with no $\mathbb{Z}$-torsion. If there exists a central element z of R such that $\delta(z) \in \mathscr{C}_R(0)$ then δ is stiff on R.*

Proof. Let $I\,(\neq 0)$ be an ideal of S and

$$f(x) = a_n x^n + \dots + a_1 x + a_0 \qquad (a_i \in R,\ 0 \leqq i \leqq n,\ a_n \neq 0),$$

be of minimal degree among the non-zero polynomials in I. Suppose that $n > 0$. Then

$$f(x)z - zf(x) = n\delta(z)a_n x^{n-1} + g(x),$$

where the degree of $g(x)$ is less than $n-1$. Hence, by the minimality of n, $n\delta(z)a_n = 0$, which gives a contradiction since R has no $\mathbb{Z}$-torsion, $\delta(z) \in \mathscr{C}_R(0)$ and $a_n \neq 0$. Thus $n = 0$, $I \cap R \neq 0$ and δ is stiff on R.

DEFINITION. R is said to be *δ-primitive* if there exists a maximal right ideal M of R containing no non-zero δ-ideals of R.

THEOREM 1. *If R is δ-primitive and δ is stiff on R then $R[x, \delta]$ is primitive.*

Proof. Let M be a maximal right ideal of R containing no non-zero δ-ideals. Let $I\,(\neq 0)$ be an ideal of S. Then $I \cap R$ is a non-zero δ-ideal of R and hence $(I \cap R) + M = R$. It follows that $I + MS = S$, so that MS is comaximal with every non-zero ideal of S. Consequently S is primitive.

As observed in the introduction, ordinary polynomial rings over certain division rings are primitive. Thus the condition of Theorem 1 is not necessary for $R[x, \delta]$ to be primitive. We now obtain an alternative sufficient condition for the primitivity of $R[x, \delta]$.

We recall that a prime ring is said to be a G-ring if the intersection of its non-zero prime ideals is non-zero.

DEFINITION. R is said to be a *δG-ring* if it is δ-prime and the intersection of its non-zero δ-prime ideals is non-zero.

THEOREM 2. *If R is a δG-ring and δ is stiff on R then $R[x, \delta]$ is primitive.*

Proof. Let I denote the intersection of the non-zero δ-prime ideals of R and let P be a non-zero prime ideal of S. Then $P \cap R$ is a non-zero δ-prime ideal of R by [**5**, Lemma 1.3] and since δ is stiff on R. Thus $I \subseteq P \cap R$. In particular I is contained in every non-zero primitive ideal of S. Hence if S is not primitive then $0 \neq I \subseteq J(S)$. But $J(S) = 0$ by Proposition 2 and hence S is primitive.

Examples 1 and 2 below show that the conditions of Theorems 1 and 2 are logically independent, even in the case where R is commutative with no $\mathbf{Z}$-torsion, and hence that neither is necessary for an Ore extension of a commutative noetherian ring with no $\mathbf{Z}$-torsion to be primitive.

EXAMPLE 1. Let R be the power series ring $k[[y]]$ over a field k of characteristic 0 in an indeterminate y and δ be the derivation $y\,d/dy$. Then the only non-zero prime ideal of R is the maximal ideal yR which is a δ-ideal. Hence, by Proposition 1, yR is the only non-zero δ-prime ideal of R and R is a δG-ring. Furthermore δ is stiff on R by Lemma 1, since $\delta(y) = y \in \mathscr{C}_R(0)$. Thus $R[x, \delta]$ is primitive by Theorem 2. However R is not δ-primitive since the only maximal ideal of the commutative ring R is a δ-ideal. Thus the condition of Theorem 1 is not satisfied.

EXAMPLE 2. Let $R = k(t)[y]$ be the polynomial ring in an indeterminate y over the field of rational functions in an indeterminate t over a field k of characteristic 0. Let $\delta = t\,\partial/\partial t + y\,\partial/\partial y$. Consider the maximal ideal $M = (y-1)R$ and suppose that there exists a non-zero δ-ideal I of R contained in M. I must be of the form

$$(y-1)^n h(y)R, \quad \text{where} \quad h(y) \in k(t)[y],\, n \geqq 1 \quad \text{and} \quad (y-1) \nmid h(y).$$

But

$$\delta((y-1)^n h(y)) = (y-1)^n \delta(h(y)) + ny(y-1)^{n-1} h(y),$$

so that, since I is a δ-ideal, $ny(y-1)^{n-1}h(y) \in (y-1)^n R$, which contradicts $(y-1) \nmid h(y)$. Thus M contains no non-zero δ-ideals of R and R is δ-primitive. Since $\delta(y) = y$ it follows, by Lemma 1 and Theorem 1, that $R[x, \delta]$ is primitive. However R is not δG, since for each $\lambda \in k$ the maximal ideal $(y - \lambda t)R$ is a δ-ideal of R, so that the intersection of the non-zero δ-prime ideals of R must be zero. Thus the condition of Theorem 2 is not satisfied.

EXAMPLE 3. It is easily seen that the polynomial ring $R = k[y]$ over a field k of characteristic 0 and the derivation $\delta = y\,d/dy$ satisfy the conditions of both Theorem 1 and Theorem 2. Clearly, δ is stiff by Lemma 1 and R is δ-primitive since the maximal ideal $(y-1)R$ contains no non-zero δ-ideals. Thus the condition of Theorem 1 is satisfied. That of Theorem 2 is also satisfied since δ is stiff and the only non-zero δ-prime ideal of R is yR.

3. A prime ring R is said to be *right bounded* if every essential right ideal of R contains a non-zero ideal of R. A ring R is said to be *fully right bounded* if every prime factor ring of R is right bounded. The results of §2 are applied below to show that few Ore extensions of commutative rings are fully bounded.

THEOREM 3. *For a commutative noetherian Ritt ring R with derivation δ the following are equivalent:*

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(i) *$R[x, \delta]$ is fully right bounded;*

(ii) *every primitive factor ring of $R[x, \delta]$ is simple artinian;*

(iii) *$\delta(R) \subseteq N(R)$;*

(iv) *every prime factor ring of $R[x, \delta]$ is commutative;*

(v) *every primitive factor ring of $R[x, \delta]$ is a field.*

Proof. It is sufficient to prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i). The proof that (iv) $\Rightarrow$ (v) $\Rightarrow$ (ii) is trivial.

(i) $\Rightarrow$ (ii). Let T be a primitive factor ring of R. T, being primitive bounded, has a maximal right ideal M which is not essential. Then there exists a right ideal N of T such that $T = M \oplus N$. Since $N \simeq T/M$, N is a minimal right ideal of T. It follows by [**2**, Lemma 1.1] that T is simple artinian.

(ii) $\Rightarrow$ (iii). Suppose that $\delta(R) \nsubseteq N(R)$. Then, by Proposition 2, $\delta(R) \nsubseteq J(S)$, so there exist a primitive ideal P of S and $r \in R$ such that $\delta(r) \notin P \cap R$. By [**5**, Lemma 1.3] and Proposition 1, $R/(P \cap R)$ is a domain and hence, by Lemma 1, the derivation $\bar{\delta}$ induced on $R/(P \cap R)$ is stiff. It follows that $P = (P \cap R)S$ and that $\dfrac{S}{P} \simeq \dfrac{R}{P \cap R}[x, \bar{\delta}]$. But $\dfrac{R}{P \cap R}[x, \bar{\delta}]$ cannot be artinian, which contradicts the hypothesis of (ii).

(iii) $\Rightarrow$ (iv). Let P be a prime ideal of S. Then, by [**5**, Lemma 1.3] and Proposition 1, $P \cap R$ is a prime ideal of R, so that $\delta(R) \subseteq N(R) \subseteq P \cap R$ and the derivation $\bar{\delta}$ induced on $R/(P \cap R)$ by δ is zero. Hence $\dfrac{S}{(P \cap R)S} \simeq \dfrac{R}{P \cap R}[x]$ is commutative and S/P, being a factor ring of $S/(P \cap R)S$, is also commutative.

(iv) $\Rightarrow$ (i). This is immediate.

4. A ring R is said to be *Jacobson* if every prime ideal of R is the intersection of primitives or, equivalently, if $N(I) = J(I)$ for every ideal I of R. In [**5**] it is shown that, for an Ore extension $R[x, \delta]$ of a right noetherian ring R to be a Jacobson ring it is sufficient, but not necessary, for R to be Jacobson. The condition that R be Jacobson may be weakened to the following (see [**5**, proof of Theorem 4.1(ii)]):

$$N(I) = J(I) \quad \text{for every } \delta\text{-ideal } I \text{ of } R. \tag{*}$$

The ring of Example 1 shows that even condition (*) on R and δ is not necessary for $R[x, \delta]$ to be Jacobson. Retaining the notation of Example 1, let P be a prime ideal of S. If $P \neq 0$ then $P \cap R \neq 0$ since δ is stiff on R. It follows that $P \cap R = M = yR$. But

$$\frac{S}{P} \simeq \frac{S/(P \cap R)S}{P/(P \cap R)S} \quad \text{and} \quad \frac{S}{(P \cap R)S} \simeq \frac{R}{P \cap R}[x] = k[x]$$

since the derivation induced on $R/(P \cap R) = R/M = k$ by δ is zero. Thus S/P is a prime factor ring of the Jacobson ring $k[x]$ and hence is semiprimitive. If $P = 0$ then P is primitive by Example 1. It follows that S is Jacobson. To see that (*) is not satisfied by R and δ consider the case $I = 0$.

Finally we record the following generalisation of [**5**, Theorem 4.1(ii)].

THEOREM 4. *Let R be a right noetherian Ritt ring and δ be a derivation on R such that $\delta(r)$ is a unit for some central element r of R. Then $R[x, \delta]$ is a Jacobson ring.*

Proof. From the remarks preceding Lemma 1, every prime ideal P of S satisfies $P = (P \cap R)S$ and hence $\dfrac{S}{P} \simeq \dfrac{R}{P \cap R}[x, \bar{\delta}]$ where $\bar{\delta}$ is the induced derivation. By [**5**, Corollary 2.7] $J(S/P) = N(R/P \cap R)\, S/P = 0$. Thus S is Jacobson.

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