Glasgow Math. J. **60** (2018) 481–486. © Glasgow Mathematical Journal Trust 2017.
doi:10.1017/S0017089517000210.

K-THEORY FOR THE C^*-ALGEBRAS OF THE SOLVABLE BAUMSLAG–SOLITAR GROUPS

SANAZ POOYA AND ALAIN VALETTE
Institut de Mathéematiques, Université de Neuchâtel Unimail,
Rue Emile Argand 11 CH-2000 Neuchatel, Switzerland
e-mail: sanaz.pooya@unine.ch, alain.valette@unine.ch

(Received 8 April 2016; revised 13 April 2017; accepted 16 June 2017;
first published online 18 October 2017)

Abstract. We provide a new computation of the K-theory of the group C^*-algebra of the solvable Baumslag–Solitar group $BS(1, n)$ $(n \neq 1)$; our computation is based on the Pimsner–Voiculescu 6-terms exact sequence, by viewing $BS(1, n)$ as a semi-direct product $\mathbb{Z}[1/n] \rtimes \mathbb{Z}$. We deduce from it a new proof of the Baum–Connes conjecture with trivial coefficients for $BS(1, n)$.

2010 *Mathematics Subject Classification.* 46L80.

1. Introduction. The Baum–Connes conjecture with coefficients for G, BC_{coef}, proposes that for every C^*-algebra A on which G acts by automorphisms, the analytical assembly map

$$\mu_{G,A} : KK_i^G(\underline{\mathrm{E}G}, A) \to K_i(A \rtimes_r G) \quad i = 0, 1$$

is an isomorphism; where $KK_i^G(\underline{\mathrm{E}G}, A)$ denotes the G-equivariant K-homology with G-compact supports and coefficients in A, of the classifying space $\underline{\mathrm{E}G}$ for G-proper actions; and $K_i(A \rtimes_r G)$ denotes the analytical K-theory of the reduced crossed product $A \rtimes_r G$. Although BC_{coef} failed to be true in general, it has been proved for several classes of groups. Among them are one-relator groups, see [**1, 11, 12**]. Furthermore, Higson–Kasparov [**7**] established BC_{coef} for the class of amenable groups. Let $A = \mathbb{C}$, we come up with the original Baum–Connes conjecture [**2**] that was strengthened by Baum, Connes and Higson [**3**] to the above formulation.

Our interest is to understand better the Baum–Connes conjecture through known examples and to go beyond the abstract isomorphism of Higson–Kasparov [**7**]. From this perspective, we consider the solvable Baumslag–Solitar groups $BS(1, n) = \langle a, b | aba^{-1} = b^n \rangle$ for $n \in \mathbb{Z}\backslash\{0\}$.

These groups are both one-relator and amenable hence the conjecture is known for them. Moreover, computations related to the K-theory of $C^*(BS(1, n))$ appeared in [**5**]; however, they are situated in the more general setting of solenoid algebras and are less explicit at the same time. Our (direct) approach toward proving the conjecture for these groups provides us with an elementary proof for which we do not need to use KK-theory or any advanced theory. Furthermore, not only we compute the K-groups but also we specify their generators and show their relevance for the Baum–Connes assembly map.

Precisely, as a result of our computations we get the following description for the K-theory and the Baum–Connes conjecture of the solvable Baumslag–Solitar groups.

$$K_0(C^*(BS(1,n))) = \mathbb{Z}.[1]$$

and

$$K_1(C^*(BS(1,n))) = \mathbb{Z} \oplus \mathbb{Z}/|n-1|.\mathbb{Z}, \quad n \neq 1$$

with the generators $[a]$ (of infinite order) and $[b]$ (of order $|n-1|$). The assembly map $\mu_{G,\mathbb{C}}$ is an isomorphism, identifying the generators of both sides.

To prove this result, we view $BS(1,n)$ as a semi-direct by $\mathbb{Z}$, hence $C^*(BS(1,n))$ as a crossed product by $\mathbb{Z}$, and we compute the analytical K-groups of $BS(1,n)$ thanks to the Pimsner–Voiculescu 6-terms exact sequence **[13]**[1].

Finally to reach the Baum–Connes conjecture with trivial coefficients for $BS(1,n)$, we appeal to two useful facts: on one hand, for G, a torsion-free group, we have $K_i^G(\underline{\mathrm{E}G}) = K_i(BG)$, the K-homology with compact supports of a classifying space BG for G; on the other hand for G one-relator torsion-free, there is a simple two-dimensional model for BG, namely the presentation complex of G, see **[9]**.

2. The C^*-algebra of $BS(1,n)$.

For $n \neq 1$, there is a faithful homomorphism from $BS(1,n)$ to the affine group of the real line, given by

$$BS(1,n) \to Aff_1(\mathbb{R}) : \begin{cases} a \mapsto (x \mapsto nx) & \text{(dilation by } n) \\ b \mapsto (x \mapsto x+1) & \text{(translation by } +1) \end{cases}.$$

It realizes an isomorphism

$$BS(1,n) \simeq \mathbb{Z}[1/n] \rtimes_\alpha \mathbb{Z},$$

where $\mathbb{Z}[1/n] = \{\frac{m}{n^\ell} \in \mathbb{Q} : m \in \mathbb{Z}, \ell \in \mathbb{N}\}$, viewed as an additive group; and α is multiplication by n.

It is well-known that if a discrete group G decomposes as a semi-direct product $G = H \rtimes_\alpha \mathbb{Z}$, with H a normal abelian subgroup, then

$$C^*(G) = C^*(H) \rtimes_\alpha \mathbb{Z} = C(\hat{H}) \rtimes_{\hat{\alpha}} \mathbb{Z},$$

where $\hat{H}$ denotes the Pontryagin dual of H (so $\hat{H}$ is a compact abelian group), and $\hat{\alpha}$ is the dual automorphism.

In the case of $BS(1,n)$, we have $H = \mathbb{Z}[1/n]$, viewed as the inductive limit of

$$\mathbb{Z} \xrightarrow{i_0} \mathbb{Z} \xrightarrow{i_1} \mathbb{Z} \xrightarrow{i_2} \dots,$$

where $i_k : \mathbb{Z} \to \mathbb{Z}$ (for $k \geq 0$) is multiplication by n. So $\widehat{i_k} : \mathbb{T} \to \mathbb{T}$ is raising to the power n, and $\hat{H}$ is the projective limit of

$$\dots \xrightarrow{\widehat{i_2}} \mathbb{T} \xrightarrow{\widehat{i_1}} \mathbb{T} \xrightarrow{\widehat{i_0}} \mathbb{T},$$

[1]For $n = -1$, i.e. the Klein bottle group, a computation based on the Pimsner–Voiculescu sequence appears in Proposition 2.1 of **[14]**, apparently not aware of previous results on the subject.

https://doi.org/10.1017/S0017089517000210 Published online by Cambridge University Press

which we identify with the solenoid[2]

$$X_n = \{z = (z_k)_{k\geq 0} \in \mathbb{T}^{\mathbb{N}} : z_{k+1}^n = z_k, \forall k \geq 0\}.$$

The duality between X_n and $\mathbb{Z}[1/n]$ is given by $(z, m) = z_\ell^m$, where m belongs to the ℓth copy of $\mathbb{Z}$; this is well defined as $(z, i_\ell(m)) = z_{\ell+1}^{n.m} = z_\ell^m = (z, m)$. For $\frac{m}{n^\ell} \in \mathbb{Z}[1/n]$, this corresponds to $(z, \frac{m}{n^\ell}) = z_\ell^m$ for $z = (z_k)_{k\geq 0} \in X_n$.

The automorphism α is given by $\alpha(m) = i_\ell(m)$, where m lies in the ℓth copy of $\mathbb{Z}$. So $\hat{\alpha}$ is the automorphism of X_n given by the backwards shift: $(\hat{\alpha}(z))_k = z_{k+1}$ for $k \geq 0$.

So $C^*(BS(1, n)) = C(X_n) \rtimes_{\hat{\alpha}} \mathbb{Z}$. This crossed product can be viewed as the universal C^*-algebra generated by two unitaries u and v satisfying the relation $uvu^{-1} = v^n$, where u is the unitary of $C^*(\mathbb{Z})$ corresponding to the generator $+1$ of $\mathbb{Z}$ acting on $C(X_n)$, while $v \in C(X_n)$ is given by the function $z \mapsto z_0$ on X_n. This crossed product description of $C^*(BS(1, n))$ appears already in [**6,8**].

LEMMA 1. $K_0(C(X_n)) = \mathbb{Z}.[1]$ *(the infinite cyclic group generated by the class of* $1 \in C(X_n)$*) and* $K_1(C(X_n)) \simeq \mathbb{Z}[1/n]$.

Proof. We have $C(X_n) = C^*(\mathbb{Z}[1/n]) = \varinjlim(C^*(\mathbb{Z}), i_k)$ (where we also denote by i_k the $*$-homomorphism $C^*(\mathbb{Z}) \to C^*(\mathbb{Z})$ associated with the group homomorphism i_k). Since K-theory commutes with inductive limits, we get $K_i(C(X_n)) = \varinjlim(K_i(C^*(\mathbb{Z})), (i_k)_*)$ $(i = 0, 1)$. Since $K_0(C^*(\mathbb{Z})) = \mathbb{Z}.[1]$ and i_k is a unital $*$-homomorphism, we have $K_0(C(X_n)) = \varinjlim(\mathbb{Z}.[1], Id) = \mathbb{Z}.[1]$. On the other hand, let v be the unitary of $C^*(\mathbb{Z})$ corresponding to the generator $+1$ of $\mathbb{Z}$ (so that $K_1(C^*(\mathbb{Z})) = \mathbb{Z}.[v]$). Then $i_k(v) = v^n$, i.e. $(i_k)_*[v] = n[v]$, and the inductive system $(K_1(C^*(\mathbb{Z})), (i_k)_*)$ is isomorphic to the original system $(\mathbb{Z}, i_k)$, so they have the same limit $\mathbb{Z}[1/n]$. □

3. K-theory for $C^*(BS(1, n))$. Let A be a unital C^*-algebra and $\alpha \in Aut(A)$. We can define the crossed product $A \rtimes_\alpha \mathbb{Z}$ associated with the action α of $\mathbb{Z}$ on A. Let $u \in A \rtimes_\alpha \mathbb{Z}$ be the unitary that implements this action in the construction of crossed product. Abstractly, the crossed product $A \rtimes_\alpha \mathbb{Z}$ is generated by $\{A, u : uau^* = \alpha(a), a \in A\}$. The Pimsner–Voiculescu 6-term exact sequence [**13**] gives us a tool to calculate the K-theory of $A \rtimes_\alpha \mathbb{Z}$ via the following cyclic diagram with 6-terms:

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{Id-\alpha_*} & K_0(A) & \xrightarrow{\iota_*} & K_0(A \rtimes_\alpha \mathbb{Z}) \\ \partial_1 \uparrow & & & & \downarrow \partial_0 \\ K_1(A \rtimes_\alpha \mathbb{Z}) & \xleftarrow{\iota_*} & K_1(A) & \xleftarrow{Id-\alpha_*} & K_1(A) \end{array}$$

Here $\iota : A \to A \rtimes_\alpha \mathbb{Z}$ denotes inclusion. We will need some understanding of the connecting map ∂_1; namely, we observe in the next lemma that $\partial_1([u]) = -[1]$. This will help us in later computations.

LEMMA 2. *The connecting map* $\partial_1 : K_1(A \rtimes_\alpha \mathbb{Z}) \to K_0(A)$ *maps* $[u]$ *to* $-[1]$.

Proof. Let $C^*(S)$ be the C^*-algebra generated by a non-unitary isometry S and let $P = I - S^*S$. Now $\mathcal{T}_{A,\alpha}$, the Toeplitz algebra for A and α, is the C^*-subalgebra of

[2]Strictly speaking, it is a solenoid only for $|n| > 1$, while it is $\mathbb{T}$ for $|n| = 1$.

https://doi.org/10.1017/S0017089517000210 Published online by Cambridge University Press

$(A \rtimes_\alpha \mathbb{Z}) \otimes C^*(S)$ generated by $u \otimes S$ and $A \otimes I$. Let $\mathcal{K}$ be the C^*-algebra of compact operators on a separable Hilbert space, with the corresponding system of matrix units $(e_{ij})_{i,j\geq 0}$. Consider the Toeplitz extension associated with $A \rtimes_\alpha \mathbb{Z}$ as in [**13**]:

$$0 \to A \otimes \mathcal{K} \xrightarrow{\varphi} \mathcal{T}_{\mathrm{A},\alpha} \xrightarrow{\psi} \mathrm{A} \rtimes \mathbb{Z} \to 0,$$

with $\varphi(a \otimes e_{ij}) = u^i a u^{*j} \otimes S^i P S^{*j}$ and

$$\psi(u \otimes S) = u, \qquad \psi(a \otimes I) = a$$

for any $a \in A$ and $i, j \in \mathbb{N}$.

The map $\partial_1 : K_1(A \rtimes_\alpha \mathbb{Z}) \to K_0(A \otimes \mathcal{K})$ is then the boundary map associated with the Toeplitz extension, we compute $\partial_1([u])$ following the description given in [**4**], 8.3.1. Consider first $\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$. This matrix can be lifted via ψ to a matrix $M = \begin{pmatrix} u \otimes S & 1 \otimes P \\ 0 & u^* \otimes S^* \end{pmatrix}$, where $M \in \mathrm{U}_2(\mathcal{T}_{A,\alpha})$. For $p_1 := (1 \otimes I) \oplus 0 \in \mathrm{M}_2(\mathcal{T}_{A,\alpha})$, we have

$$Mp_1M^* - p_1 = (1 \otimes SS^* - 1 \otimes I) \oplus 0 = (-1 \otimes P) \oplus 0.$$

The pullback of this element via φ is $z := (-1 \otimes e_{00}) \oplus 0 \in \mathrm{M}_2(A \otimes \mathcal{K})$. So $\partial_1([u]) = -[-z]$. Via the isomorphism $K_0(A \otimes \mathcal{K}) \cong K_0(A)$, the element $[-z] = [1 \otimes e_{00}]$ corresponds to $[1]$. Hence, $\partial_1([u]) = -[1]$. □

THEOREM 1. *$K_0(C^*(BS(1,n))) = \mathbb{Z}.[1]$. For $n \neq 1$:*

$$K_1(C^*(BS(1,n))) = \mathbb{Z} \oplus \mathbb{Z}/|n-1|.\mathbb{Z},$$

with generators $[a]$ (of infinite order) and $[b]$ (of order $|n-1|$).

Proof. We view $C^*(BS(1,n))$ as the crossed product $C^*(BS(1,n)) = C^*(\mathbb{Z}[1/n]) \rtimes_\alpha \mathbb{Z}$, and apply the Pimsner–Voiculescu 6-terms exact sequence to it. Denoting by the $\iota : C^*(\mathbb{Z}[1/n]) \to C^*(BS(1,n))$ the inclusion, and appealing to Lemma 1, we get:

$$\begin{array}{ccccc} \mathbb{Z}.[1] & \xrightarrow{Id-\alpha_*} & \mathbb{Z}.[1] & \xrightarrow{\iota_*} & K_0(C^*(BS(1,n))) \\ \partial_1 \uparrow & & & & \downarrow \\ K_1(C^*(BS(1,n))) & \xleftarrow{\iota_*} & \mathbb{Z}[1/n] & \xleftarrow{Id-\alpha_*} & \mathbb{Z}[1/n] \end{array}$$

Since $\alpha(1) = 1$, the upper-left arrow is the zero map. The bottom-right arrow is given by multiplication by $1-n$ on $\mathbb{Z}[1/n]$, so it is injective; hence, the right vertical arrow is zero. This shows that $\iota_* : \mathbb{Z}.[1] \to K_0(C^*(BS(1,n)))$ is an isomorphism.

Turning to K_1, we observe that the relation $[b] = [aba^{-1}] = [b^n]$ implies $(n-1).[b] = 0$, i.e. the order of $[b]$ divides $|n-1|$. To prove that this is exactly $|n-1|$, we look at the bottom line of the Pimsner–Voiculescu sequence. Since $\mathbb{Z}[1/n]/Im(Id-\alpha_*) = \mathbb{Z}/|n-1|.\mathbb{Z}$, we get a short exact sequence:

$$0 \to \mathbb{Z}/|n-1|.\mathbb{Z} \to K_1(C^*(BS(1,n))) \xrightarrow{\partial_1} \mathbb{Z}.[1] \to 0.$$

which splits to give $K_1(C^*(BS(1,n))) = \mathbb{Z} \oplus \mathbb{Z}/|n-1|.\mathbb{Z}$, with $[b]$ a generator of order $|n-1|$. Since $\partial_1([a]) = -[1]$ by Lemma 2, we see that $[a]$ is a generator of infinite order. $\square$

Corollary 1. *Set $G_n =: BS(1,n)$. For $n \neq 1$, the Baum–Connes conjecture without coefficients holds for G_n, i.e. the Baum–Connes assembly map $\mu_{G_n,\mathbb{C}} : K_i(BG_n) \to K_i(C^*(G_n))$ $(i = 0, 1)$ is an isomorphism.*

Proof. We appeal to a result of Lyndon [**9**]: for a torsion-free one-relator group $G =< S \mid r >$ on m generators, the presentation complex (consisting of one vertex, m edges and one 2-cell) is a two-dimensional model for the classifying space BG. By Lemma 4 in [**1**]:

$$K_0(BG) = H_0(BG,\mathbb{Z}) \oplus H_2(BG,\mathbb{Z}) \text{ and } K_1(BG) = H_1(BG,\mathbb{Z}).$$

Let $F(S)$ denote the free group on the set S of the generators. If r is not in the commutator subgroup of $F(S)$, then we have that $H_2(BG,\mathbb{Z}) = 0$. In order to see this, we consider the boundary operator $\partial_2 : C_2(BG,\mathbb{Z}) = \mathbb{Z}[r] \to C_1(BG,\mathbb{Z}) = \mathbb{Z}[S]$ which takes the 2-cell [r] to its boundary obtained by running along r and summing up the letters appearing in r with a sign equal to the exponent (which corresponds to their orientation as you run along the boundary of the 2-cell). If r is not in the commutator subgroup of $F(S)$, then the sum of all exponents of r is non-zero, so ∂_2 is injective, and therefore $H_2(BG,\mathbb{Z}) = Ker\,\partial_2$ is zero. This argument applies to G_n, as we assume $n \neq 1$. Then, $K_0(BG_n) = H_0(BG_n,\mathbb{Z}) = \mathbb{Z}$, generated by the inclusion of a base point. By Example 2.11 on p.97 of [**10**], the image of this element under $\mu_{G_n,\mathbb{C}}$ is [1], the class of 1 in $K_0(C^*(G_n))$. The result for K_0 then follows from Theorem 1.

Now, for any group G, identify $H_1(BG,\mathbb{Z})$ with the abelianized group G^{ab}. There is a map $\kappa_G : G^{ab} \to K_1(C^*_r(G))$ obtained by mapping a group element $g \in G$ first to the corresponding unitary in $C^*_r(G)$, then to the class $[g]$ of this unitary in $K_1(C^*_r(G))$. We get this way a homomorphism $G \to K_1(C^*_r(G))$, which descends to $\kappa_G : G^{ab} \to K_1(C^*_r(G))$ as the latter group is abelian. By Theorem 1.4 on p.86 of [**10**], for G torsion-free, the map κ_G coincides with $\mu_{G,\mathbb{C}}$ on the lowest-dimensional part of $K_1(BG)$. Here, $\mu_{G_n,\mathbb{C}} : K_1(BG_n) \to K_1(C^*(G_n))$ coincides with $\kappa_{G_n} : G_n^{ab} = \mathbb{Z} \oplus \mathbb{Z}/|n-1|.\mathbb{Z} \to K_1(C^*(G_n))$, which is an isomorphism by Theorem 1. $\square$

Remark 1. Let $\tau : C^*(BS(1,n)) \to \mathbb{C}$ be the canonical trace on $C^*(BS(1,n))$. This induces the homomorphism $\tau_* : K_0(C^*(BS(1,n))) \to \mathbb{R}$ at the K-theory level. Since τ is unital, by Theorem 1 we have $\tau_*(K_0(C^*(BS(1,n)))) = \mathbb{Z}$.

REFERENCES

1. C. Béguin, H. Bettaieb and A. Valette, K-theory for the C^*-algebras of one-relator groups, *K-Theory* **16** (1999), 277–298.

2. P. Baum and A. Connes, Geometric K-theory for Lie groups and foliations, *Enseign. Math.* **46** (2000), 3–42.

3. P. Baum, A. Connes and N. Higson, Classifying space for proper actions and K-theory of group C^*-algebras, *Contemp. Math.* **164** (1994), 241–292.

4. B. Blackadar, *K-theory for operator algebras*, Mathematical Sciences Research Institute Publications, vol. 5 (Springer-Verlag, New York, 1986).

5. B. Brenken, K-groups of solenoidal algebras I, *Proc. Amer. Math. Soc.* **123**(5) (1995), 1457–1464.

6. B. Brenken and P. E. T. Jørgensen, A family of dilation crossed product algebras, *J. Oper. Theory* **25** (1991), 299–308.

7. N. Higson and G. Kasparov, E-theory and KK-theory for groups which act properly and isometrically on Hilbert space, *Invent. Math.* **144** (2001), 23–74.

8. B. Iochum, T. Masson, T. Schücker and A. Sitarz, κ-Deformation and spectral triples, *Acta Phys. Polon. Supp.* **4** (2011), 305–324.

9. R. C. Lyndon, Cohomology theory of groups with a single defining relation, *Ann. Math.* **52** (1950), 650–665.

10. G. Mislin and A. Valette, *Proper group actions and the Baum–Connes conjecture*. Advanced courses in mathematics - CRM Barcelona (Birkhäuser, Basel, 2003).

11. H. Oyono-Oyono, La conjecture de Baum–Connes pour les groupes agissant sur les arbes, *C. R. Acad. Sci. Paris, Sér. I* **326** (1998), 799–804.

12. J.-L. Tu, The Baum–Connes conjecture and discrete group actions on trees, *K-Theory* **17** (1999), 303–318.

13. M. Pimsner and D. Voiculescu, Exact sequences for K-groups and Ext-groups of certain cross product C^*- algebras, *Oper. Theory* **4** (1980), 93–118.

14. T. Sudo, K-theory for the group C^*-algebras of certain solvable discrete groups, *Hokkaido Math. J.* **43** (2014), 209–260.