Glasgow Math. J. **48** (2006) 37–40. © 2006 Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089505002843. Printed in the United Kingdom

SPHERE THEOREM FOR MANIFOLDS WITH POSITIVE CURVATURE

BAZANFARÉ MAHAMAN
Chef du Département de Mathématiques and Informatique Faculté des Sciences, BP 10662 Niamey-Niger
e-mail: bmahaman @yahoo.fr

(Received 16 February, 2005; accepted 10 October, 2005)

Abstract. In this paper, we prove that, for any integer $n \geq 2$, and any $\delta > 0$ there exists an $\epsilon(n, \delta) \geq 0$ such that if M is an n-dimensional complete manifold with sectional curvature $K_M \geq 1$ and if M has conjugate radius $\rho \geq \frac{\pi}{2} + \delta$ and contains a geodesic loop of length $2(\pi - \epsilon(n, \delta))$ then M is diffeomorphic to the Euclidian unit sphere $\mathbb{S}^n$.

2002 *Mathematics Subject Classification.* 53C20, 53C21.

1. Introduction. One of the fundamental problems in Riemannian geometry is to determine the relation between the topology and the geometry of a Riemannian manifold. In this way the Toponogov's theorem and the critical point theory play an important role. Let M be a complete Riemannian manifold and fix a point p in M and define $d_p(x) = d(p, x)$. A point $q \neq p$ is called a critical point of d_p or simply of the point p if, for any nonzero vector $v \in T_qM$, there exists a minimal geodesic γ joining q to p such that the angle $\measuredangle(v, \gamma'(0)) \leq \frac{\pi}{2}$. Suppose M is an n-dimensional complete Riemannian manifold with sectional curvature $K_M \geq 1$. By Myers' theorem the diameter of M is bounded from above by π. In **[4]** Cheng showed that the maximal value π is attained if and only if M is isometric to the standard sphere. It was proved by Grove and Shiohama **[5]** that if $K_M \geq 1$ and the diameter of M diam$(M) > \frac{\pi}{2}$ then M is homeomorphic to a sphere.

Hence the problem of removing homeomorphism to diffeomorphism or finding conditions to guarantee the diffeomorphism is of particular interest. In **[13]** C. Xia showed that if $K_M \geq 1$ and the conjugate radius $\rho(M)$ of M is greater than $\pi/2$ and if M contains a geodesic loop of length 2π, then M is isometric to $\mathbb{S}^n$.

Definition 1.1. Let M be an n-dimensional Riemannian manifold and p be a point in M. Let Conj(p) denote the set of first conjugate points to p on all geodesics issuing from p. The *conjugate radius* $\rho(p)$ of M at p in the sense of Xia **[13]** is defined as

$$\rho(p) = d(p, \mathrm{Conj}(p)) \qquad \text{if } \mathrm{Conj}(p) \neq \emptyset$$

and

$$\rho(p) = +\infty \qquad \text{if } \mathrm{Conj}(p) = \emptyset$$

Then the conjugate radius of M is given by

$$\rho(M) = \inf_{x \in M} \rho(x).$$

Many interesting results have been proved by using the critical points theory and Toponogov's theorem **[3]**, **[5]**, **[7]**, **[8]**, **[10]**, **[11]**, **[12]**, **[13]**. etc...

The purpose of this paper is to prove the following result.

THEOREM 1.2. *For any $n \geq 2$ and any $\delta > 0$, there exists a positive constant $\epsilon(n, \delta)$ depending only on n and δ such that for any $\epsilon \leq \epsilon(n)$, if M is an n-dimensional complete connected Riemannian manifold with sectional curvature $K_M \geq 1$ and conjugate radius $\rho(M) > \frac{\pi}{2} + \delta$ and if M contains a geodesic loop of length $2(\pi - \epsilon)$ then M is diffeomorphic to an n-dimensional unit sphere $\mathbb{S}^n$ and the metric g of M is $\epsilon' = \epsilon'(\epsilon, n, \delta, \alpha)$ close in the C^α topology to the canonical metric of curvature 1 of $\mathbb{S}^n$ for any $\alpha \in]0, 1[$.*

Proof. Let $i(M)$ denote the injectivity radius of M. By definition we have

$$i(M) = \inf_{x \in M} d(x, C(x)),$$

where $C(x)$ is the set of cut points of x. □

A classical result due to Klingenberg (see for instance corollary 4.14 of **[9]**) asserts that if M is compact then $i(M) = \min\{t_0, \frac{l_0}{2}\}$, where l_0 is the minimum of the length of non trivial closed geodesics of M and t_0 is the minimum over unit vector u of TM of the first conjugate value $t_0(u)$ along the geodesic $\gamma_u(t) = \exp(tu)$.

LEMMA 2.1. *Let M be an n-dimensional complete, connected Riemannian manifold with sectional curvature $K_M \geq 1$. With Xia's convention on the conjugate radius we have $i(M) \geq \rho(M)$.*

The proof is a direct application of the Klingenberg's result: by the definition above of the conjugate radius we have $t_0 \geq \rho(M)$ and, since $K_M \geq 1$, every geodesic γ issued from a point p hits Conj(p) at a point q (by the Rauch comparison theorem). Consequently, the length of every non trivial closed geodesic issued from p is bounded below by $2d(p, q) \geq 2\rho(M)$.

LEMMA 2.2. *For any $\delta > 0$, there exists a function τ_δ which satifies* $\lim_{\epsilon \mapsto 0} \tau_\delta(\epsilon) = 0$ *and such that if M is a complete manifold with $K_M \geq 1$, injectivity radius $i(M) \geq \frac{\pi}{2} + \delta$ and which contains a geodesic loop of length $2(\pi - \epsilon)$ then we have diam$(M) \geq \pi - \tau_\delta(\epsilon)$.*

Proof. Let γ be a loop with length $2\pi - 2\epsilon$. Let $x = \gamma(0) = \gamma(2\pi - 2\epsilon)$, $y = \gamma(\pi/2 + \delta)$, $m = \gamma(\pi - \epsilon)$ and $z = \gamma(\frac{3(\pi-\epsilon)}{2} - \delta)$

Let

$$\gamma_1 = \gamma \Big/ \left[0, \frac{\pi}{2} + \delta\right], \quad \gamma_2 = \gamma \Big/ \left[\frac{\pi}{2} + \delta, \pi - \epsilon\right], \quad \gamma_3 = \gamma \Big/ \left[\pi - \epsilon, \frac{3(\pi - \epsilon)}{2} - \delta\right],$$

and

$$\gamma_4 = \gamma \Big/ \left[\frac{3(\pi - \epsilon)}{2} - \delta, 2\pi - 2\epsilon\right].$$

Then the geodesics γ_i are minimal. Let σ be a minimal geodesic joining m and x.

Set $\alpha = \measuredangle(\sigma'(0), -\gamma'(\pi - \epsilon))$ and $\beta = \measuredangle(\sigma'(0), \gamma'(\pi - \epsilon))$.

We have $\alpha \leq \pi/2$ or $\beta \leq \pi/2$. Suppose, without loss of generality, that $\alpha \leq \pi/2$. Applying the Toponogov comparison theorem on length to the hinge formed by γ_2

https://doi.org/10.1017/S0017089505002843 Published online by Cambridge University Press

and σ at $\gamma(\pi - \epsilon)$ we have

$$\cos\left(\frac{\pi}{2} + \delta\right) \geq \cos L(\sigma) \cos\left(\frac{\pi}{2} - \epsilon - \delta\right) + \cos\alpha \sin L(\sigma) \sin\left(\frac{\pi}{2} - \epsilon - \delta\right)$$

so that

$$\cos L(\sigma) \leq -\frac{\sin\delta}{\sin(\delta + \epsilon)} \Rightarrow L(\sigma) \geq \pi - \tau_\delta(\epsilon)$$

and the conclusion follows. □

Note that Anderson [**1**] and Otsu [**6**] constructed, for $n \geq 4$ n-dimensional closed manifolds with $\mathrm{Ric} \geq n - 1$ and diameter arbitrarily close to π but whose homotopy type is distinct from that of the sphere. Thus additional assumptions are needed.

In [**2**] G. Pacelli Bessa proved the following theorem from which we deduce Theorem 1.2.

THEOREM 2.3. *Given $n \geq 2$ and $i_0 > 0$ there exists an $\epsilon = \epsilon(n, i_0)$ such that if M admits a metric g satisfying*

$$\mathrm{Ric} \geq n - 1, \qquad i(M) \geq i_0, \qquad \mathrm{Diam}(M) \geq \pi - \epsilon$$

then, for any $\alpha \in]0, 1[$, M is diffeomorphic to $\mathbb{S}^n$ and the metric g of M is $\epsilon' = \epsilon'(\epsilon, n, \alpha)$ close in the C^α topology to the canonical metric of curvature 1 of $\mathbb{S}^n$, where ϵ' tends to 0 with ϵ.

REMARK. The complex projectif space shows that theorem 1.2 is false under the weaker hypothesis $\rho \geq \frac{\pi}{2}$.

ACKNOWLEDGEMENT. I would like to thank the referee for his/her insightful and helpful comments.

REFERENCES

1. M. T. Anderson, Metrics of positive Ricci curvature with large diameters, *Mannscripta Math.* **68** (1990), 405–415.

2. G. P. Bessa, Differentiable sphere theorems for Ricci curvature, *Math. Z.* **214** (1993), 245–259.

3. J. Cheeger, Critical points of distance functions and applications to geometry, in *Geometric topology: recent devlopments* (*Montecafini Terme, 1990*), Lectures Notes in Mathematics, No. 1504 (Springer-Verlag, 1991), 1–38.

4. S. Y. Cheng, Eigenvalue comparison theorem and geometric applications, *Math. Z.* **143** (1975), 289–297.

5. K. Grove and K. Shiohama, A generalized sphere theorem, *Ann. of Math.* (2) **106** (1977), 201–211.

6. Yukio Otsu, On manifolds of positive Ricci cunvature with Carge diameter, *Math. Z.* **206** (1991), 255–264.

7. P. Pertersen, Comparison geometry problem list, in *Riemannian geometry*, Waterloo, ON, 1993 (Amer. Math. Soc., Providence, RI 1996), 87–115.

8. K. Shiohama, A sphere theorem for manifolds of positive Ricci curvature, *Trans. Amer. Math. Soc.* **275**, No. 2 (1983), 811–819.

9. T. Sakai, *Riemannian geometry, Transl. Math. Monographs* No. 149 (Amer. Math. Soc., 1996).

10. Z. Shen, Complete manifolds with nonnegative Ricci curvature and large volume growth, *Invent. Math.* **125** (1996), 393–404.

11. J. Sha and Z. Shen, Complete manifolds with nonnegative curvature and quadratically nonnegative curved infinity, *Amer. J. Math.* **119** (1997), 1399–1404.

12. C. Xia, Complete manifolds with sectional curvature bounded below and large volume growth, *Bull. London Math. Soc.* **34** (2002), 229–235.

13. C. Xia, Some applications of critical point theory of distance functions on Riemannian manifolds, *Composition Math.* **132** (2002), 49–55.

https://doi.org/10.1017/S0017089505002843 Published online by Cambridge University Press