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STALKING THE SOUSLIN TREE—A TOPOLOGICAL GUIDE

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1. **Introduction.** It has long been known that the existence of a Souslin line entails (and is entailed by) the existence of a Souslin tree; indeed such a tree can be built from the open subsets of the line in a natural way. It will be shown that less onerous restrictions on a topological space than orderability allow the construction to proceed. For example, to the expected requirements that the space satisfy the countable chain condition and not be separable, one can add the hypothesis of local connectivity, and that either first category sets be nowhere dense or that nowhere dense sets be separable.

A *Souslin line* is a connected linearly ordered topological space, without first or last elements, in which every collection of disjoint open intervals is countable, but which is not homeomorphic to the real line. A *tree* is a partial order in which for each element, the set of its predecessors is well ordered. A *Souslin tree* is a tree of cardinality $\aleph_1$ in which every chain (totally ordered subset) and antichain (totally unordered subset) is countable. A proof of the equivalence of the existence of Souslin lines and Souslin trees can be found in [11], along with some historical background. The existence of a Souslin line cannot be proved or refuted from the usual axioms of set theory. See [11] for references.

I do not know interesting, necessary and sufficient conditions on a topological space for its topology to include a Souslin tree; however, by considering what suffices to grow a tree, we get topological equivalents of the existence of a Souslin line which do not mention order. Our work simultaneously generalizes the "Souslin line yields Souslin tree" construction and the work of Rudin [10].

2. **Forestry.** Let us attempt to build a Souslin tree using the "includes" relation on the non-empty open sets in a space. Each level (set of elements of the tree whose sets of predecessors are order-isomorphic) will consist of disjoint open sets. At successor stages, each member of the previous level will have at least two successors. To ensure this, we must avoid isolated points. We shall be considering *CCC* (every collection of disjoint open sets is countable) non-separable spaces; every such space has an open set that includes no separable open set, and hence no isolated points. By starting our tree with such

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open set and by confining ourselves to Hausdorff spaces, we may guarantee that each open set splits into infinitely many disjoint open sets. Assuming the tree continues with levels for each countable ordinal, it is not hard to see that if the space is *CCC*, the tree is Souslin. What can go wrong at limit levels? First, it is possible that the intersection of the unions of previous levels has empty interior, in which case the tree cannot continue. Second, even if the interior is not empty, it is possible that the intersection of any branch going up to the limit level has no interior, so again the tree cannot continue. We first deal with the first problem.

DEFINITION. *A small class* (for X) is a collection of nowhere dense sets such that any countable union of members of the class is nowhere dense. Its members are called small sets. For example, if every first category set is nowhere dense, the nowhere dense sets form a small class. Similarly if the nowhere dense sets coincide with the separable ones.

If at successor stages we always remove a small set from each open set as we split it into at least two pieces, then, since levels are countable (for *CCC* X) and limits are countable ordinals, we will always have a non-empty interior at limit levels.

To take care of the second problem, we make another definition.

DEFINITION. Let X be a topological space. A *π-pseudo-base* for X is a collection of sets with non-empty interiors such that each non-empty open set includes one. A π-pseudobase $\mathcal{P}$ for X is called *generic* if, whenever a member of $\mathcal{P}$ is included in a disjoint union of interiors of members of $\mathcal{P}$, it is included in the interior of one of them.

For example,

LEMMA. *Each of the following classes of spaces possesses a generic π-pseudo-base.*

(1) *Connected linearly ordered spaces;*

(2) *Locally connected spaces;*

(3) *Spaces in which every open set has a component with non-empty interior;*

(4) *Spaces in which no open set is first category, but every open set has only countably many components.*

Proof. (1) is a special case of (2). (2) and (4) are special cases of (3). For (3) let $\mathcal{P}$ be the collection of such components of open sets. Note that if a connected set is included in a disjoint union of open sets, it is included in one of them.

It is clear how to continue the tree at limit levels, given a generic π-pseudobase. Namely take an infinite disjoint collection of members of the generic π-pseudobase included within the interior of the intersection of the unions of previous levels. Then (the interior of) each member is included in precisely one element of each previous level, and thus determines a branch up to the limit level. (We assume without loss of generality that previous levels are made up of interiors of members of the generic π-pseudobase.) Thus we have

THEOREM 1. *Let X be CCC Hausdorff with a generic π-pseudobase $\mathcal{P}$ and a small class $\mathcal{S}$, such that each member of $\mathcal{P}$ is the disjoint union of at least two distinct members of $\mathcal{P}$ and a small set. Then there is a Souslin tree.*

COROLLARY 2. *The existence of any of the following entails the existence of a Souslin tree.*

(1) A CCC non-separable linearly ordered space;

(2) A locally connected CCC non-separable Hausdorff space in which first category sets are nowhere dense;

(3) A locally connected CCC Hausdorff space in which nowhere dense sets are separable, but which is not hereditarily separable.

(4) A locally connected CCC non-separable Hausdorff space X such that for any distinct $x, y \in X$ there is a separable $Y \subseteq X$ such that $X - Y = Z_1 \cup Z_2$, $Z_1 \cap \bar{Z}_2 = Z_2 \cap \bar{Z}_1 = \emptyset$, and $x \in Z_1$, $y \in Z_2$.

Proof. It is well-known that one can "connect" the order in (1), to get a connected *CCC* non-separable linearly ordered space. Then consider the open intervals and the countable sets. For successor stages of (2), take maximal infinite disjoint collections. For (3), note first that the space is not separable, since it is easily seen [2] that separable spaces in which nowhere dense sets are separable are hereditarily separable. Second, note that without loss of generality we may assume separable sets are nowhere dense since, as pointed out earlier, any *CCC* non-separable space has an open set that includes no separable open set. (4), due to M. E. Rudin [10] requires a bit more effort. First note that without loss of generality we may assume every open subset of X satisfies all the conditions of (4). The only one to check is the last. If x and y are distinct points in an open U, then there is a separable $Y \subseteq X$, $X - Y = Z_1 \cup Z_2$, etc. Then $x \in Z_1 \cap U$, $y \in Z_2 \cap U$, $U - Y = (Z_1 \cap U) \cup (Z_2 \cap U)$, and clearly $Z_1 \cap U$ and $Z_2 \cap U$ are separated in U. $U \cap Y$ is separable, since it is an open subset of a separable space. To get a generic π-pseudobase, for each open connected U take distinct $x, y \in U$ and separable $Y \subseteq U$ separating x from y, say $U - Y = Z_1 \cup Z_2$. Note that neither Z_1 nor Z_2 is nowhere dense, for suppose e.g. that Z_2 is. Then Z_1 is dense in U, since Y is nowhere dense. But then the closure of Z_1 in U cannot be disjoint from Z_2. Our generic π-pseudobase is therefore composed of U_{Z_1}'s and U_{Z_2}'s for every non-empty open connected U, while the small sets are the separable ones.

A Souslin line (with separable intervals removed) provides converses to all parts of Corollary 2. The only non-obvious point is taken care of by the proof in [6] that nowhere dense sets are separable in a *CCC* linearly ordered space.

The usual argument would show that if there were a locally connected *CCC* Hausdorff space in which the Lindelöf sets coincided with the nowhere dense ones, then there would be a Souslin tree. However, it is not difficult to prove that no such space exists; indeed a *CCC* space in which nowhere dense sets are Lindelöf is hereditarily Lindelöf.

3. **Logging.** We have given sufficient conditions for a topology to include a Souslin tree; the problem of necessity appears to be much more difficult. All I can say is

THEOREM 3. *If $X\times X$ is CCC, there is no Souslin tree of open subsets of X (under "includes").*

This result is essentially due to Kurepa [7], who proved that the square of a Souslin line is not *CCC*. The idea is to "normalize" [11] the tree so that each node has at least two immediate successors. For every $\alpha<\omega_1$, let $S_{\alpha,0}$, $S_{\alpha,1}$, be successors of a node of level α. Then $\{S_{\alpha,0}\times S_{\alpha,1}\}_{\alpha<\omega_1}$ is an uncountable disjoint collection of open sets in $X\times X$.

Jensen has constructed a model of set theory in which the continuum hypothesis holds but in which there are no Souslin trees [4]. In this model the converse of Theorem 3 fails non-trivially since an unpublished result due independently to R. Laver and F. Galvin states that the continuum hypothesis implies there is a *CCC* space X such that X^2 is not *CCC*.

Using Theorem 3 and noting that if $X\times X$ is *CCC*, so is X, Corollary 2 may be rephrased to get such results as

COROLLARY 4. *If X is Hausdorff and locally connected, nowhere dense subsets of X are separable, and $X\times X$ is CCC, then X is hereditarily separable.*

It should be remarked that a sufficient condition for $X\times X$ to be *CCC* is that X have *property* (K):

every uncountable collection of open sets includes an uncountable subcollection in which no pair is disjoint.

4. **Examples.** The variety of conditions in Corollary 2 may lead one to wonder about their necessity. There are several illuminating examples.

EXAMPLE 1. At first glance one might think that the non-separability in Corollary 2 (2) is superfluous, but the countable discrete space satisfies all the other conditions. One may of course replace "non-separable" by "without isolated points".

EXAMPLE 2. M. E. Rudin [10] proves that if there is a Souslin tree, there is a Moore space satisfying (4). The space is not linearly ordered since linearly ordered Moore spaces are metrizable [3].

EXAMPLE 3. It follows from [12] that if there is a Souslin tree, there is a 0-dimensional non-separable *CCC* linearly ordered space. Such a space cannot be locally connected or even satisfy (3) of the Lemma.

EXAMPLE 4. There exist examples [9], [8] of locally connected non-separable Moore spaces satisfying the countable chain condition. Aarts and Lutzer [1] have shown that if a Moore space satisfying the countable chain condition has the property that the intersection of countably many dense open sets is dense, then it is separable. It follows that in these examples, there exist first category sets which are not nowhere dense.

EXAMPLE 5. The density topology on the real line [5], [13] is an example of a connected, property (K), non-separable Hausdorff space in which first category sets are nowhere dense. It follows from Theorem 3 and the remark after Corollary 4 that the density topology includes no Souslin trees. Thus the local connectivity in Corollary 2 (2) cannot be omitted or even weakened to connectedness.

EXAMPLE 6. To any regular space the topology of which includes a Souslin tree, can canonically be associated a compact Hausdorff extremally disconnected space with the same property: the dual space of its regular open algebra. The only tricky point in verifying this is to make sure that two elements of the original tree do not become identified through passage to the algebra. This can be done by starting with a normalized tree.

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