8

Periodic magnetic channels

In the previous chapters, we considered magnets that for the most part had continuous transverse or longitudinal fields along some system axis. In this chapter, we look instead at magnetic channels where the on-axis field is periodic. Periodic field configurations are used for focusing charged particle beams and for production of radiation at light sources. We begin by considering the field produced by helical conductor windings. Then we examine several examples where we demand some desired field configuration along the axis and then find off-axis field components that satisfy Maxwell's equations.

8.1 Field from a helical conductor

A helically wound conductor can produce a periodic field. The parametric equations of a helix are

$$x = a \cos \phi$$
$$y = a \sin \phi$$
$$z = \frac{\lambda}{2\pi} \phi,$$

where a is the radius and λ is the period of the helix. We define the axial wavenumber as $k = 2\pi/\lambda$. We parameterize the nature of the helix by the angle α in Figure 8.1.[1] We have

$$\tan \alpha = \frac{\Delta z}{a \, \Delta\phi} = \frac{\lambda}{2\pi a} = \frac{1}{ka}$$

since z progresses by a distance λ as the azimuthal angle goes once around the circumference. With this definition, $\alpha = 0$ corresponds to the limiting case when the helix reduces to a circular loop. It is convenient to write λ as a function of a.

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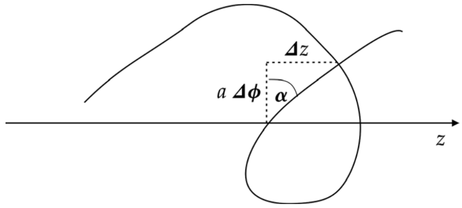

Figure 8.1 Definition of the helical angle α.

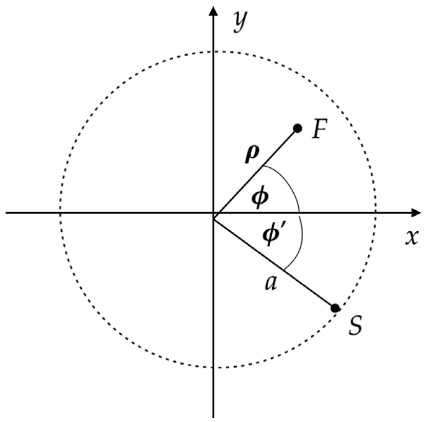

Figure 8.2 Helical conductor geometry at a fixed value of z.

$$\lambda = 2\pi a \tan \alpha$$

The azimuthal angle and axial distance are connected through the helix constraint

$$z = a\,\phi\ \tan \alpha. \tag{8.1}$$

Consider the cross-section through the helix shown in Figure 8.2. A single conductor follows a helical path in a current sheet of radius a. Let the observation point F have cylindrical coordinates $(\rho,\ \phi,\ z)$ and the current element at the location S on the conductor have coordinates (a, ϕ', z'). The current element is given by

$$\overrightarrow{dl} = -a \sin \phi'\ d\phi'\ \hat{x} + a \cos \phi'\ d\phi'\ \hat{y} + a \tan \phi'\ d\phi'\ \hat{z}$$

and the distance vector is

$$\overrightarrow{R} = (\rho \cos \phi - a \cos \phi')\,\hat{x} + (\rho \sin \phi - a \sin \phi')\,\hat{y} + (z - a\,\phi' \tan \alpha)\,\hat{z}.$$

Taking into account the constraint between z' and ϕ', the direct evaluation of $\boldsymbol{B}$ using the Biot-Savart equation only requires an integration over ϕ'. The

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integration limits for a winding of finite length can be found using Equation 8.1. Although the resulting integral for the general case is complicated, the solution for observation points along the axis of the helix is fairly straightforward.[1]

The vector potential for an infinitely long helix is given by

$$\vec{A}(\rho, \phi, z) = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(a, \phi', z')}{R} \, dS'. \tag{8.2}$$

The sheet current density only has components in the ϕ' and z' directions. The pitch angle α for the helical winding can be written as

$$\tan\, \alpha = \frac{K_{z'}}{K_{\phi'}},$$

so the components of the current density are related by

$$K_{\phi'} = ka\, K_{z'}.$$

Thus the current density is [2]

$$\vec{K}(a, \phi', z') = \frac{I}{a} \, (\hat{z}' + ka\hat{\phi}') \, \delta(\phi' - kz' - \varepsilon), \tag{8.3}$$

where ε is the azimuthal angle of the winding at $z' = 0$. The Dirac delta function enforces the constraint between changes in z' and changes in ϕ'.

We can write the periodic delta function in Equation 8.3 as a Fourier series. Let $\tau = \varepsilon + kz'$. Then

$$f(\phi') = \delta(\phi' - \tau) = a_0 + \sum_{n=1}^{\infty} [a_n \cos\, n\phi' + b_n \sin\, n\phi'].$$

The coefficients are

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(\phi' - \tau) \, d\phi' = \frac{1}{2\pi}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \delta(\phi' - \tau) \cos\, n\phi' \, d\phi' = \frac{1}{\pi} \cos\, n\tau, \qquad n > 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \delta(\phi' - \tau) \, \sin\, n\phi' \, d\phi' = \frac{1}{\pi} \sin\, n\tau.$$

Thus the delta function can be expressed as

$$\begin{aligned}\delta(\phi' - \tau) &= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}[\cos n\tau \cos n\phi' + \sin n\tau \sin n\phi'] \\ &= \frac{1}{2\pi}\left[1 + 2\sum_{n=1}^{\infty}\cos\left(n(\phi' - \tau)\right)\right].\end{aligned} \tag{8.4}$$

The distance from the current element to the observation point can be written as

$$R = \{a^2 + \rho^2 - 2\ a\rho \cos(\phi - \phi') + (z - z')^2\}^{1/2}. \tag{8.5}$$

We need to express the unit vectors in Equation 8.3 in terms of unit vectors in the coordinate system for the observation point. The axial unit vectors are identical, $\hat{z'} = \hat{z}$. The azimuthal unit vector is given by

$$\hat{\phi}' = -\hat{\rho}\ \ \sin(\phi - \phi') + \hat{\phi}\ \cos(\phi - \phi'). \tag{8.6}$$

Thus there are in general nonvanishing components of the vector potential in all three dimensions.

Substituting Equations 8.3–8.6 into Equation 8.2, the *axial* component of the vector potential is given by

$$A_z(\rho, \phi, z) = \frac{\mu_0}{4\pi}\ \frac{I}{2\pi a}\iint \frac{\left[1 + 2\ \sum_{n=1}^{\infty}\cos\left(n(\phi' - kz' - \varepsilon)\right)\right]}{\{a^2 + \rho^2 - 2\ a\rho\ \cos(\phi - \phi') + (z - z')^2\}^{1/2}}\ a\, d\phi'\, dz'. \tag{8.7}$$

For $\rho < a$, evaluation of the integrals give [2]

$$A_z(\rho, \phi, z) = -\frac{\mu_0 I}{2\pi}\ \ln a + \frac{\mu_0 I}{\pi}\sum_{n=1}^{\infty} K_n(nka)\, I_n(nk\rho)\, \cos\left(n(\phi - kz - \varepsilon)\right) \tag{8.8}$$

and for $\rho > a$

$$A_z(\rho, \phi, z) = -\frac{\mu_0 I}{2\pi}\ \ln \rho + \frac{\mu_0 I}{\pi}\sum_{n=1}^{\infty} K_n(nk\rho)\, I_n(nka)\, \cos\left(n(\phi - kz - \varepsilon)\right). \tag{8.9}$$

The functions K_n and I_n are modified Bessel functions of order n.

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The *radial* component of the vector potential is given by

$$A_\rho(\rho,\phi,z) = \frac{\mu_0}{4\pi}\,\frac{I\,k}{2\pi}\iint \frac{\sin\,(\phi-\phi')\left[1+2\sum_{n=1}^{\infty}\cos\,\Big(n(\phi'-kz'-\varepsilon)\Big)\right]}{\{a^2+\rho^2-2\;a\rho\,\cos\,(\phi-\phi')+(z-z')^2\}^{1/2}}\,a\,d\phi'\,dz'. \tag{8.10}$$

For $\rho < a$, evaluation of the integrals give [2]

$$A_\rho(\rho,\phi,z) = -\frac{\mu_0 Ika}{2\pi}\sum_{n=1}^{\infty}[K_{n+1}(nka)\,I_{n+1}(nk\rho) - K_{n-1}(nka)\,I_{n-1}(nk\rho)]\,\sin\,\Big(n(\phi-kz-\varepsilon)\Big) \tag{8.11}$$

and for $\rho > a$

$$A_\rho(\rho,\phi,z) = -\frac{\mu_0 Ika}{2\pi}\sum_{n=1}^{\infty}[K_{n+1}(nk\rho)\,I_{n+1}(nka) - K_{n-1}(nk\rho)\,I_{n-1}(nka)]\,\sin\,\Big(n(\phi-kz-\varepsilon)\Big). \tag{8.12}$$

The *azimuthal* component of the vector potential is given by

$$A_\phi(\rho,\phi,z) = \frac{\mu_0}{4\pi}\,\frac{I\,k}{2\pi}\iint \frac{\cos(\phi-\phi')\left[1+2\sum_{n=1}^{\infty}\cos\Big(n(\phi'-kz'-\varepsilon)\Big)\right]}{\{a^2+\rho^2-2\;a\rho\,\cos(\phi-\phi')+(z-z')^2\}^{1/2}}\,a\,d\phi'\,dz'. \tag{8.13}$$

For $\rho < a$ this has the solution [2]

$$A_\phi(\rho,\phi,z) = \frac{\mu_0 I\,k\rho}{4\pi} + \frac{\mu_0 I\,ka}{2\pi}\sum_{n=1}^{\infty}[K_{n+1}(nka)\,I_{n+1}(nk\rho) + K_{n-1}(nka)\,I_{n-1}(nk\rho)]\cos\Big(n(\phi-kz-\varepsilon)\Big) \tag{8.14}$$

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and for $\rho > a$

$$A_\phi(\rho,\phi,z) = \frac{\mu_0 I\, ka^2}{4\pi\rho} + \frac{\mu_0 I\, ka}{2\pi}\sum_{n=1}^{\infty}[K_{n+1}(nk\rho)\, I_{n+1}(nka) + K_{n-1}(nk\rho)\, I_{n-1}(nka)]\cos\Big(n(\phi - kz - \varepsilon)\Big). \tag{8.15}$$

The solution for the magnetic field components can be found by taking the curl of A. For the case $\rho < a$ the field components are [2, 3, 4]

$$\begin{aligned} B_\rho(\rho,\phi,z) &= -\frac{\mu_0 I k^2 a}{\pi}\sum_{n=1}^{\infty} nK_n'(nka)\, I_n'(nk\rho)\, \sin(n(\phi - kz - \varepsilon)) \\ B_\phi(\rho,\phi,z) &= \frac{\mu_0 I k a}{\pi}\sum_{n=1}^{\infty} nK_n'(nka)\, \frac{I_n(nk\rho)}{\rho}\, \cos(n(\phi - kz - \varepsilon)) \\ B_z(\rho,\phi,z) &= \frac{\mu_0 I k}{2\pi} - \frac{\mu_0 I k^2 a}{\pi}\sum_{n=1}^{\infty} nK_n'(nka)\, I_n(nk\rho)\, \cos(n(\phi - kz - \varepsilon)), \end{aligned} \tag{8.16}$$

while the solution for the case $\rho > a$ is

$$\begin{aligned} B_\rho(\rho,\phi,z) &= -\frac{\mu_0 I k^2 a}{\pi}\sum_{n=1}^{\infty} nK_n'(nk\rho)\, I_n'(nka)\, \sin(n(\phi - kz - \varepsilon)) \\ B_\phi(\rho,\phi,z) &= \frac{\mu_0 I}{2\pi\rho} + \frac{\mu_0 I k a}{\pi}\sum_{n=1}^{\infty} nI_n'(nka)\, \frac{K_n(nk\rho)}{\rho}\, \cos(n(\phi - kz - \varepsilon)) \\ B_z(\rho,\phi,z) &= -\frac{\mu_0 I k^2 a}{\pi}\sum_{n=1}^{\infty} nI_n'(nka)\, K_n(nk\rho)\, \cos(n(\phi - kz - \varepsilon)). \end{aligned} \tag{8.17}$$

Primes on the Bessel functions indicate derivatives with respect to the arguments.

An example of the variation of the field components for a helical conductor is shown for one period in Figure 8.3. The calculations were done using Equations 8.16. In this case, the magnitude of the transverse components are small compared to the axial component. At radii large compared to a, the azimuthal field component becomes dominant and the field approaches that of a straight wire.

The on-axis field of the helical conductor can be found by evaluating Equation 8.16 at $\rho = 0$. Using the relations

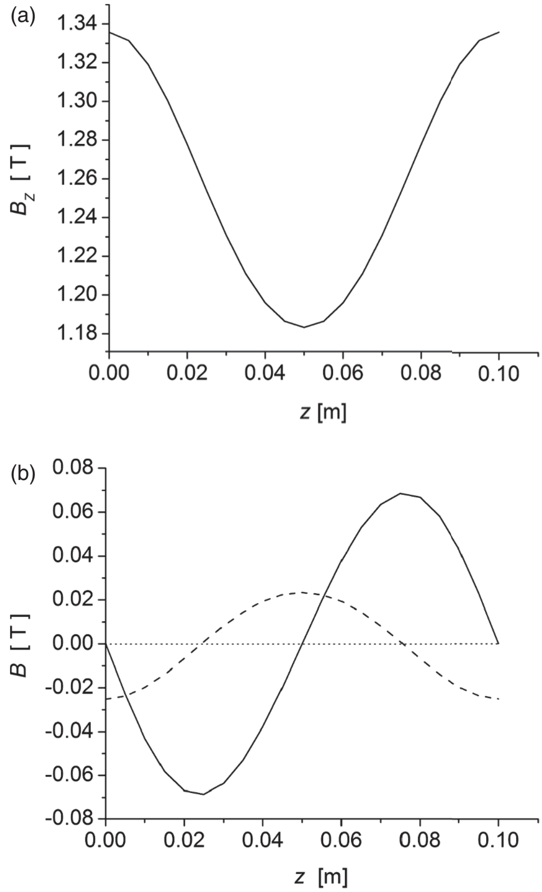

Figure 8.3 (a) The dependence of B_z along one period of the helix; (b) the dependence of B_ρ (solid) and B_ϕ (dashed) versus z. The parameters used here were $\lambda = 10$ cm, $a = 10$ cm, $\rho = 5$ cm, $\phi = 0$, $\varepsilon = 0$, $I = 10^5$ A, and $N = 40$ terms in the sums.

$$
\begin{aligned}
I_n(0) &= 0 \qquad &&\text{for } n > 0 \\
I'_n(0) &= 0 \qquad &&\text{for } n > 1 \\
I'_1(0) &= ½,
\end{aligned}
$$

the on-axis field is [1, 3, 5]

$$
\begin{aligned}
B_\rho(0,0,z) &= \frac{\mu_0 I k^2 a}{2\pi} \; K'_1(ka) \\
B_\phi(0,0,z) &= 0 \\
B_z(0,0,z) &= \frac{\mu_0 I\, k}{2\pi}.
\end{aligned}
$$

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There is a close relationship between these results and our previous results for the field of a solenoid. In practice, the conductor in a solenoid is wound in many helical layers over a cylindrical form. The helical pitch length λ for a solenoid is very small. In the previous chapter, the field for a solenoid was derived by assuming that the field came from a longitudinal distribution of parallel infinitesimal current loops. In the limit that $k \to \infty$, it can be shown by taking the asymptotic limits for the Bessel functions that Equation 8.16 approaches the on-axis field of an infinitely long solenoid

$$B_z(0, 0, z) = \mu_0 n I,$$

where $n = 1/\lambda$ is the number of turns per unit length.[6] In the opposite limit where $k \to 0$, the helical fields approach that of a straight conductor.

8.2 Planar transverse field

A planar *wiggler* has an on-axis transverse field component that oscillates in a fixed plane. It is called a wiggler because a charged particle beam moves back and forth in this type of field and can be used, for example, to produce electromagnetic radiation for light sources. Assume we want an on-axis field given by

$$\begin{aligned} B_{y0} &= B_0 \cos\,(\gamma z - \varphi) \\ B_{x0} &= B_{z0} = 0, \end{aligned} \tag{8.18}$$

where z is the direction of the system axis and φ is an initial phase. The coefficient γ is related to the wavelength of the field oscillation λ by $\gamma = 2\pi/\lambda$. We saw in Chapter 3 that solutions of Laplace's equation in rectangular coordinates (1) can be written as products of trigonometric and hyperbolic sines and cosines and (2) that these solutions must have at least one trigonometric and one hyperbolic factor. Since B_y is assumed to be non-zero on the axis, we must choose the cosine and hyperbolic cosine functions for the general solution. Once we have specified that the z dependence is a cosine function, there are three possible combinations for the product of the x and y dependences. Let us consider the solution where

$$B_y = B_0 \cos(\alpha x) \cosh(\beta y) \cos(\gamma z - \varphi).$$

In free space, the div $B = 0$ equation gives

$$\partial_x B_x + B_0\, \beta \cos(\alpha x)\ \sinh(\beta y)\ \cos(\gamma z - \varphi) + \partial_z B_z = 0.$$

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In order to satisfy this equation for all x, y, and z, B_x and B_z must have the form

$$B_x = f \sin(\alpha x) \sinh(\beta y) \cos(\gamma z - \varphi)$$
$$B_z = g \cos(\alpha x) \sinh(\beta y) \sin(\gamma z - \varphi),$$

where f and g are unknown factors. Substituting these field expressions back into the divergence equation gives the constraint

$$f\alpha + \beta B_0 + g\gamma = 0. \tag{8.19}$$

The x component of the curl $B = 0$ equation gives the relation

$$g = -\frac{\gamma B_0}{\beta},$$

while either the y or z component of the curl equation gives

$$f = -\frac{\alpha B_0}{\beta}.$$

Substituting these values for f and g into Equation 8.19, we find the wave number constraint

$$\gamma^2 = -\alpha^2 + \beta^2. \tag{8.20}$$

This can be written in terms of the period of the field variation as

$$\lambda^2 = \frac{4\pi^2}{\beta^2 - \alpha^2}.$$

For a periodic solution, we have the additional constraint that $\beta > \alpha$. The solution for the planar transverse field is then [7]

$$\begin{aligned} B_x &= -\frac{\alpha B_0}{\beta} \sin(\alpha x) \sinh(\beta y) \cos(\gamma z - \varphi) \\ B_y &= B_0 \cos(\alpha x) \cosh(\beta y) \cos(\gamma z - \varphi) \\ B_z &= -\frac{\gamma B_0}{\beta} \cos(\alpha x) \sinh(\beta y) \sin(\gamma z - \varphi). \end{aligned} \tag{8.21}$$

The other two solutions consistent with Equation 8.18 have the transverse dependences

$$\cosh(\alpha x) \cos(\beta y)$$

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and

$$\cosh(\alpha x)\cosh(\beta y).$$

These solutions can be derived following the same procedure used above. Each solution has a unique relation among the wavenumbers.[7]

It's important to keep in mind that this type of derivation only represents part of the problem. What we have shown is that our desired on-axis field profile is a valid solution of Maxwell's equations. However, what we have not considered is a configuration of conductors that actually produces that desired field. The obvious trial solution here would be a series of transverse permanent magnet or electrically excited magnetic poles that alternate in direction along the system axis. Oftentimes the field from a realistic coil distribution can only approximate the desired field. We define the problem of finding a current distribution that produces a specified magnetic field as an *inverse problem* to distinguish it from the situation encountered using the Biot-Savart formula, where we find the magnetic field produced by a given current distribution. Finding a suitable current distribution frequently involves using numerical optimization methods.

8.3 Helical transverse field

Consider an on-axis transverse field that rotates around the system axis, analogously to the magnetic field vector in circularly polarized light. It is convenient to look for a solution in a cylindrical coordinate system that rotates around the system axis, as shown in Figure 8.4. In this system, z and ϕ are coupled and the on-axis field is

$$\begin{aligned} B_{\rho 0} &= B_0 \cos(\gamma z - \phi - \varepsilon) \\ B_{\phi 0} &= B_{z0} = 0, \end{aligned} \tag{8.22}$$

where ε is an initial phase shift. For points off the axis, we look for a solution for B_ρ with the form

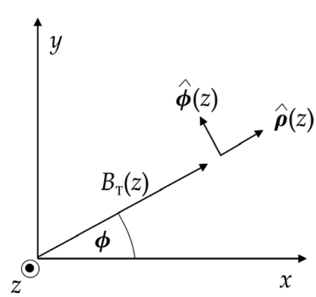

Figure 8.4 Rotating cylindrical coordinate system.

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$$B_\rho = C\,F(\rho)\,\cos(\gamma z - \phi - \varepsilon), \tag{8.23}$$

where $C\,F(0) = B_0$ and F is an undetermined function of ρ. Since the ρ and ϕ coordinates are separated by 90°, we expect the solution for B_ϕ to have the form

$$B_\phi = C\,G(\rho)\,\sin(\gamma z - \phi - \varepsilon), \tag{8.24}$$

where G is another undetermined function. We know from Chapter 3 that the solution of Laplace's equation in cylindrical coordinates must involve Bessel functions. Calculating the ρ component of the curl $B = 0$ equation allows us to obtain an expression for B_z.

$$B_z = -C\,\gamma\,\rho\,G(\rho)\,\sin(\gamma z - \phi - \varepsilon). \tag{8.25}$$

Calculating the ϕ component of the curl equation lets us determine a relation between the unknown functions F and G.

$$F(\rho) = \partial_\rho[\rho\,G(\rho)]. \tag{8.26}$$

Using Equations 8.23–8.26, we can write the div $\boldsymbol{B} = 0$ equation directly in terms of G.

$$\frac{1}{\rho}[\rho\,\partial_\rho^2(\rho G) + \partial_\rho(\rho G)] - \frac{1}{\rho}\,G - \gamma^2\rho\,G = 0. \tag{8.27}$$

Rearranging terms, this can be written as

$$\gamma^2\rho^2\,\frac{\partial^2(\gamma\,\rho\,G)}{\partial(\gamma\,\rho)^2} + \gamma\,\rho\,\frac{\partial(\gamma\,\rho\,G)}{\partial(\gamma\,\rho)} - [1 + (\gamma\,\rho)^2](\gamma\,\rho\,G) = 0. \tag{8.28}$$

This is the differential equation for the modified Bessel[1] function I_1, where the unknown variable is $\gamma\,\rho G$ and the argument of the Bessel function is $\gamma\,\rho$. Thus we have

$$\gamma\,\rho\,G(\rho) = I_1(\gamma\,\rho)$$

and the unknown function G is

$$G(\rho) = \frac{I_1(\gamma\,\rho)}{\gamma\,\rho}. \tag{8.29}$$

[1] Some properties of the modified Bessel functions are described in Appendix C.

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We can now find F from Equation 8.26.

$$F(\rho) = \frac{\partial I_1(\gamma\rho)}{\partial(\gamma\rho)}$$

Using a recursion relation,[2] we can write this as

$$F(\rho) = I_0(\gamma\rho) - \frac{1}{\gamma\rho} I_1(\gamma\rho), \tag{8.30}$$

where I_0 is the modified Bessel function of order 0. Substituting this back into Equation 8.23, we find

$$B_\rho = C\left[I_0(\gamma\rho) - \frac{1}{\gamma\rho} I_1(\gamma\rho)\right] \cos(\gamma z - \phi - \varepsilon).$$

Near the axis, I_0 and I_1 have the series expansions[3]

$$I_0(u) \simeq 1 + \frac{1}{4} u^2 + \cdots \tag{8.31}$$

and[4]

$$I_1(u) \simeq \frac{1}{2} u + \frac{1}{16} u^3 + \cdots \tag{8.32}$$

Thus near the axis, we take the leading terms for I_0 and I_1 and find that

$$B_\rho \simeq \frac{C}{2} \cos(\gamma z - \phi - \varepsilon).$$

Comparing this with Equation 8.23 gives $C = 2B_0$. The solution for the helical transverse field is then [8]

$$\begin{aligned} B_\rho &= 2B_0\left[I_0(\gamma\rho) - \frac{1}{\gamma\rho} I_1(\gamma\rho)\right] \cos(\gamma z - \phi - \varepsilon) \\ B_\phi &= 2B_0 \frac{I_1(\gamma\rho)}{\gamma\rho} \sin(\gamma z - \phi - \varepsilon) \\ B_z &= -2B_0 I_1(\gamma\rho) \sin(\gamma z - \phi - \varepsilon). \end{aligned} \tag{8.33}$$

Note that the argument for the Bessel functions involves the longitudinal wave-number γ.

[2] AS 9.6.26. [3] AS 9.6.12. [4] AS 9.6.10.

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It was suggested that this solution could be produced by winding conductors in a helical manner over a cylindrical bore tube.[8] Between adjacent helical turns, a second helix could be laid down with the current running in the opposite direction. However, exact calculations showed that Equation 8.33 is only a reasonable approximation for the field of this interleaved helical winding configuration when a/λ is larger than ~0.3, where a is the winding radius.[3, 5]

8.4 Axial fields

As a final example of constructing a desired on-axis field, we consider the periodic axial field

$$B_{z0} = B_0 \cos(\gamma z - \varepsilon)$$
$$B_{\phi 0} = B_{\rho 0} = 0$$

in a cylindrical coordinate system. By symmetry, B_ϕ vanishes everywhere. Because of the periodic behavior in z, we suspect that the off-axis solution must contain terms proportional to the modified Bessel functions. Therefore we look for the simplest possible solution

$$B_z = B_0\, I_0(\alpha\rho) \cos(\gamma z - \varepsilon).$$

Since the div $B = 0$ equation involves $\partial_z B_z$, we know that B_ρ must be proportional to sin $(\gamma z - \varepsilon)$. Thus we have

$$\frac{1}{\rho}\, \partial_\rho(\rho B_\rho) = B_0\, \gamma I_0(\alpha\, \rho) \sin(\gamma z - \varepsilon).$$

Multiplying by ρ and integrating both sides, we get

$$\rho B_\rho = B_0 \gamma \sin(\gamma z - \varepsilon) \int_0^\rho \rho\, I_0(\alpha\rho)\, d\rho + C,$$

where C is an integration constant. The remaining integral can be evaluated as[5]

$$\int_0^\rho \rho\, I_0(\alpha\rho)\, d\rho = \frac{\rho}{\alpha}\, I_1(\alpha\rho).$$

Thus

$$B_\rho = B_0\, \frac{\gamma}{\alpha}\, I_1(\alpha\rho) \sin(\gamma z - \varepsilon) + \frac{C}{\rho}.$$

[5] AS 11.3.25.

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Since B_ρ must vanish at $\rho = 0$, we must have $C = 0$. To find the relationship between the wavenumbers α and γ, we look at the ϕ component of the curl $B = 0$ equation, which gives

$$\frac{\gamma^2}{\alpha}\, I_1(\alpha\rho) - \frac{\partial I_0(\alpha\rho)}{\partial\rho} = 0.$$

Using[6]

$$\frac{\partial I_0(\alpha\rho)}{\partial\rho} = \alpha\, I_1(\alpha\rho),$$

we find that $\alpha = \gamma$. Thus the solution for the periodic axial field is

$$\begin{aligned} B_\rho &= B_0\, I_1(\gamma\rho)\, \sin(\gamma z - \varepsilon) \\ B_\phi &= 0 \\ B_z &= B_0\, I_0(\gamma\rho)\, \cos(\gamma z - \varepsilon). \end{aligned} \tag{8.34}$$

A likely conductor configuration for producing this field is a series of solenoids along the axis that alternate in direction. In fact, it is possible to design block solenoids that give an excellent approximation to a sinusoidal axial field along the axis.[9]

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