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Lyapunov Theorems for the Asymptotic Behavior of Evolution Families on the Half-Line

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Abstract. Two theorems regarding the asymptotic behavior of evolution families are established in terms of the solutions of a certain Lyapunov operator equation.

1 Introduction and Preliminaries

Let X be a Hilbert space and $B(X)$ the Banach algebra of all bounded linear operators acting on X. Consider the Cauchy Problem

$$\frac{du(t)}{dt} = A(t)u(t),\ \ u(0) = x \in X,\ \ t \geq 0$$

with $A(t) \in B(X)$, for each $t \geq 0$, and $A(\,\cdot\,)$ locally integrable on R_+.

A modern instrument to analyze the asymptotic behaviour of the above system $\dot{u}(t) = A(t)u(t))$ is to associate a two-parameter family of bounded and linear operators, the so-called evolution family. We refer the reader to [4] for details.

Definition 1.1 An operator-valued two variables function

$$\Phi\colon \{(t,s) \in \mathbb{R} \times \mathbb{R} : t \geq s \geq 0\} \mapsto B(X)$$

is called an *evolution family* if the following properties hold:

(e1) $\Phi(t,t) = I$, for all $t \geq 0$;
(e2) $\Phi(t,s)\Phi(s,r) = \Phi(t,r)$, for all $t \geq s \geq r \geq 0$;
(e3) $\Phi(\,\cdot\,,s)x$ is continuous on $[s,\infty)$, for all $s \geq 0,\ x \in X$;
$\Phi(t,\,\cdot\,)x$ is continuous on $[0,t)$, for all $t \geq 0,\ x \in X$;
(e4) $\Phi^*(\,\cdot\,,s)x$ is continuous on $[s,\infty)$, for each $s \geq 0,\ x \in X$;
(e5) there are $M, \omega > 0$ such that

$$\|\Phi(t,s)\| \leq Me^{\omega(t-s)}, \quad \text{for all} \quad t \geq s \geq 0.$$

Remark 1.2 If $\Phi(t,s) = \Phi(t-s,0)$, then the one-parameter family of linear operators $\{T(t)\}_{t\geq 0}$ defined by $T(t) = \Phi(t,0)$ is a C_0-semigroup. For a general presentation of C_0-semigroups theory, see for instance [10].

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Definition 1.3 An evolution family $\Phi = \{\Phi(t,s)\}_{t \geq s \geq 0}$ is called *exponentially unstable* if there exist $N, \nu > 0$ such that

$$\|\Phi(t,s)x\| \geq Ne^{\nu(t-s)}\|x\| \quad \text{for all } t \geq s \geq 0 \text{ and } x \in X.$$

Recall now that the widely known theorem of A. M. Lyapunov establishes that if A is an $n \times n$ complex matrix, then A has all its characteristic roots with real parts negative if and only if for any positive definite Hermitian matrix H there exists a unique positive definite Hermitian matrix W satisfying the equation (L_H) $A^*W + WA = -H$ (where * denotes the conjugate transpose of a matrix) (see [1]). Daletskiĭ and Krein extend this result for one-parameter semigroups e^{tA} with bounded generators (see [4]), and later R. Datko approaches the general case of linear time-invariant systems $\dot{u}(t) = Au(t)$, where A is an unbounded linear operator that generates a C_0-semigroup (see [5]).

Theorem (R. Datko, 1970) *A C_0-semigroup $\mathbf{T} = \{T(t)\}_{t \geq 0}$ is exponentially stable if and only if there exists $W \in B(X)$, $W^* = W$, $W \geq 0$ such that*

$$\langle Ax, Wx \rangle + \langle Wx, Ax \rangle = -\|x\|^2 \tag{1.1}$$

for all $x \in D(A)$, where A denotes the generator of $\{T(t)\}_{t \geq 0}$.

See also C. Chicone and Y. Latushkin [2], A. Pazy [9], J. Goldstein [6], and L. Pandolfi [8] for the Lyapunov equation with unbounded A. Let us go back for a moment to the finite-dimensional case. Taking into account the spectral mapping theorem we can deduce that an $n \times n$ complex matrix A has all its characteristic roots with real parts contained in the open *right* half-plane if and only if the above Lyapunov equation (1.1) holds for a unique *negative* definite Hermitian matrix W. From here we are led to a first model to obtain a Lyapunov equation for the exponential instability of $\{e^{tA}\}_{t \geq 0}$ with A a $n \times n$ complex matrix. Naturally, this model can be extended to the infinite-dimensional case when the matrix A is replaced by a linear and bounded operator A. Unfortunately, the spectral mapping theorem fails for the general case of C_0-semigroups generated by an unbounded linear operator A and thus to get a similar Lyapunov equation for the exponential *instability* is no longer straightforward, as in the case of $\{e^{tA}\}_{t \geq 0}$ with $A \in B(X)$. For the time-varying finite-dimensional systems we have firstly the work of Coppel [3] where it is stated that if the non-stationary Lyapunov equation $W'(t) + A^*(t)W(t) + W(t)A(t) = -I$ has a bounded self-adjoint solution, then the system $\dot{u}(t) = A(t)u(t))$ has an exponential dichotomy. Note that Coppel [3] involves essentially the finite-dimensional assumption, since his proof uses the compactness of the unit ball. Also, this subject is touched on by Massera and Schaffer in [7] even in the infinite-dimensional context but with the additional assumption that the subspace which induces dichotomy has finite codimension. Thus, to identify a corresponding Lyapunov operator equation for the exponential instability is not at all trivial even for the particular case of time-invariant systems $\dot{u}(t) = Au(t)$ where A is an unbounded linear operator that generates a C_0-semigroup. In order to solve this problem in the most general setting, we will deal with the abstract evolution families (see Definition 1.1) *not* necessarily arising from a differential system, and we attempt to propose a Lyapunov approach for their exponential instability. Beside the intrinsic interest of this topic in the abstract Cauchy problems theory, we should mention that instability for steady states for partial differential equations is a topic of current interest. Usually one looks for linear instability (by computing the spectrum of an appropriate generator) and then tries to prove instability in the nonlinear PDE. We think that this approach can be of some interest in this topic also, if the PDE can be cast as an abstract Cauchy problem and if we take into account the connection between the time-dependent linear problems and the time-invariant nonlinear problems.

2 Results

Since our approach is developed in the most general setting of the abstract evolution families, the main issue here is to deal with the absence of the differentiability assumption (which is heavily exploited in the proofs of the Lyapunov equation in the case of C_0-semigroups or time-varying differential systems). The best way to explain our approach is to go back to the C_0-semigroups particular case and to look at Datko's Theorem again (see the Introduction). Since we do not have a "generator" for two-parameters evolution families and since we want to avoid any differentiability assumption, we want now to derive an equivalent form of equation (1.1) from Datko's Theorem which does not contain the generator A and from here to attempt to transfer a "Lyapunov-type" equation for the general case of evolution families. For this reason we will assume for the moment that (1.1) holds and let $f_x\colon \mathbb{R} \to \mathbb{C}$, the function defined by $f_x(t) = \langle WT(t)x, T(t)x\rangle$, for $x \in D(A)$. One can easily see that f_x is differentiable and we have that

$$f_x'(t) = \langle WAT(t)x, T(t)x\rangle + \langle WT(t)x, AT(t)x\rangle = -\|T(t)x\|^2$$

(as it is known, $T(t)x \in D(A)$ whenever $x \in D(A)$ (see for instance [10])). From here we get that

$$\langle WT(t)x, T(t)x\rangle - \langle Wx, x\rangle = -\int_0^t \|T(s)x\|^2 ds,$$

which is equivalent to

$$\left\langle T^*(t)WT(t)x + \int_0^t T^*(s)T(s)xds, x\right\rangle = \langle Wx, x\rangle,$$

for all $x \in D(A)$. Using the fact that the generator has dense domain we obtain that

$$\left\langle T^*(t)WT(t)x + \int_0^t T^*(s)T(s)xds - Wx, x\right\rangle = 0,$$

for all $x \in$ X which implies that

$$T^*(t)WT(t)x + \int_0^t T^*(s)T(s)xds = Wx \tag{2.1}$$

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for all $t \geq 0, x \in X$. It is easy to check that if (2.1) holds, then (1.1) is also true. Keeping this in mind and taking into account Remark 1.2, we define

$$\mathcal{L}(t,t_0)x = \int_{t_0}^{t} \Phi^*(\tau,t_0)\Phi(\tau,t_0)x\,d\tau$$

(from Definition 1.1 we have that $\mathcal{L}$ is well defined), and we propose a Lyapunov-type approach for the abstract evolution families, as shown in Theorems 2.1 and 2.5.

Theorem 2.1 *Let Φ be an evolution family acting on the Hilbert space X. If there exist $m > 0$ and a bounded operator-valued function $W\colon \mathbb{R}_+ \to B(X)$ with $W^*(t) = W(t)$ for each $t \geq 0$, such that:*

(i) $\Phi^*(t,t_0)W(t)\Phi(t,t_0) + \mathcal{L}(t,t_0) \leq W(t_0)$, *for each $t \geq t_0 \geq 0$;*
(ii) $m\|x\| \leq \|\Phi(t+1,t)x\|$, *for each $t \geq 0$ and $x \in X$;*
(iii) $\langle W(t)x, x\rangle \leq 0$, *for each $x \in X$ and $t \geq 0$;*

then Φ is exponentially unstable.

Proof Let $x \in X$ and $t \geq t_0 \geq 0$. Then

$$\begin{aligned}\int_{t_0}^{t} \|\Phi(\tau,t_0)x\|^2 d\tau &\leq \langle W(t_0)x,x\rangle - \langle W(t)\Phi(t,t_0)x, \Phi(t,t_0)x\rangle \\ &\leq |\langle W(t)\Phi(t,t_0)x, \Phi(t,t_0)x\rangle| \leq k\|\Phi(t,t_0)x\|^2,\end{aligned}$$

for all $t \geq t_0 \geq 0$, where $k = \sup_{t\geq 0}\|W(t)\|$. Denoting $\varphi(t) = \int_{t_0}^{t}\|\Phi(\tau,t_0)x\|^2 d\tau$, we get that $\varphi(t) \leq k\dot{\varphi}(t)$, for all $t \geq t_0 \geq 0$, and thus

$$\varphi(t_0+1)e^{\frac{1}{k}(t-t_0-1)} \leq \varphi(t) \leq k\|\Phi(t,t_0)x\|^2,\ \forall\, t \geq t_0+1,\ \forall\, x \in X. \tag{2.2}$$

But

$$m\|x\| \leq \|\Phi(t_0+1,t_0)x\| \leq \|\Phi(t_0+1,\tau)\|\,\|\Phi(\tau,t_0)x\| \leq Me^{\omega}\|\Phi(\tau,t_0)x\|,$$

for all $\tau \in [t_0, t_0+1]$.

Thus

$$\frac{m^2}{M^2e^{2\omega}}\|x\|^2 \leq \int_{t_0}^{t_0+1}\|\Phi(\tau,t_0)x\|^2 d\tau = \varphi(t_0+1).$$

Using now (2.2) we obtain

$$\frac{m^2}{M^2e^{2\omega}k}e^{-\frac{1}{k}}e^{\frac{1}{k}(t-t_0)}\|x\|^2 \leq \|\Phi(t,t_0)x\|^2,\ \text{for all } t \geq t_0+1,\ \text{for all } x \in X.$$

Denoting $N' = \frac{m}{Me^{\omega}\sqrt{k}}e^{-\frac{1}{2k}}$ we get

$$N'e^{\frac{1}{2k}(t-t_0)}\|x\| \leq \|\Phi(t,t_0)x\|,\ \text{for all } t \geq t_0+1,\ \text{for all } x \in X.$$

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If $t_0 \le t < t_0 + 1$, then

$$m\|x\| \le \|\Phi(t_0+1,t_0)x\| \le \|\Phi(t_0+1,t)\|\,\|\Phi(t,t_0)x\| \le Me^{\omega}\|\Phi(t,t_0)x\|$$
$$= Me^{\omega}e^{-\frac{1}{2k}(t-t_0)}e^{\frac{1}{2k}(t-t_0)}\|\Phi(t,t_0)x\| \le Me^{\omega}e^{-\frac{1}{2k}(t-t_0)}e^{\frac{1}{2k}}\|\Phi(t,t_0)x\|.$$

From here it follows that $\|\Phi(t,t_0)x\| \ge \frac{m}{Me^{\omega}}e^{-\frac{1}{2k}}e^{\frac{1}{2k}(t-t_0)}\|x\|$. Denoting by $N = \min\{1, \frac{1}{\sqrt{k}}\}\frac{m}{Me^{\omega}}e^{-\frac{1}{2k}}$ and by $\nu = \frac{1}{2k}$, we get

$$\|\Phi(t,t_0)x\| \ge Ne^{\nu(t-t_0)}\|x\|, \text{ for all } t \ge t_0 \ge 0,\ x \in X. \qquad \blacksquare$$

Remark 2.2 Although condition (ii) from the above theorem is automatically satisfied for the evolution families arising from differential systems (see for instance [4, Lemma 2.4, p. 111]), it still imposes a certain restriction. For example, in the particular case of a C_0-semigroup $\{T(t)\}_{t\ge 0}$, this assumption means that $\|T(1)x\| \ge m\|x\|$. Since $m > 0$, this means that $T(1)$ has a continuous inverse on its range, which is not necessarily the whole space X, as it can be seen in the example below (see Remark 2.4). Thus assumption (ii) is verified not only by C_0-groups.

Remark 2.3 It is obvious to see that the conclusion of the above theorem still holds if instead of condition (ii) we only impose that $m\|x\| \le \|\Phi(t+\delta,t)x\|$, for some $\delta > 0$;

Remark 2.4 We want to note that an exponentially unstable evolution family is not a particular case of a hyperbolic evolution family, not even in the particular case of C_0-semigroups. Recall that an evolution family is called hyperbolic if there exists $P\colon \mathbb{R}_+ \to B(X)$ a continuous and bounded projection-valued function and $N_1, N_2, \nu > 0$ such that

- $\Phi(t,t_0)P(t_0) = P(t)\Phi(t,t_0)$, for all $t \ge t_0 \ge 0$
- $\Phi(t,t_0)\colon KerP(t_0) \to KerP(t)$ is an isomorphism for all $t \ge t_0 \ge 0$;
- $\|\Phi(t,t_0)P(t_0)x\| \le N_1e^{-\nu(t-t_0)}\|P(t_0)x\|$, for all $x \in X$ and $t \ge t_0 \ge 0$;
- $\|\Phi(t,t_0)(I-P)(t_0)x\| \ge N_2e^{\nu(t-t_0)}\|(I-P)(t_0)x\|$, for all $x \in X$ and $t \ge t_0 \ge 0$.

For details, we refer the reader to [11]. Now if we take for instance the right shift semigroup

$$(T(t)f)(s) = \begin{cases} f(s-t), & s \ge t \ge 0, \\ 0, & t > s \ge 0, \end{cases}$$

on $L^2(\mathbb{R}_+,\mathbb{R})$, and if we define the semigroup $S(t) = e^tT(t)$, then $\{S(t)\}_{t\ge 0}$ is an exponentially unstable C_0-semigroup, but $S(t)$ is not onto, for each $t \ge 0$.

Theorem 2.5 *Let Φ be an evolution family acting on the Hilbert space X. If there exist $m > 0$ and a bounded operator-valued function $W\colon \mathbb{R}_+ \to B(X)$ with $W^*(t) = W(t)$ for each $t \ge 0$, such that:*

(i) $\Phi^*(t,t_0)W(t)\Phi(t,t_0) + \mathcal{L}(t,t_0) \le W(t_0)$, *for each $t \ge t_0 \ge 0$;*
(ii) $\langle W(t)x, x\rangle \le -m\|x\|^2$, *for each $t \ge 0$ and $x \in X$,*

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then Φ *is exponentially unstable.*

Proof Let $x \in X$ and $t \geq t_0 \geq 0$. Then

$$\big\langle W(t)\Phi(t,t_0)x, \Phi(t,t_0)x\big\rangle + \int_{t_0}^{t} \|\Phi(\tau,t_0)x\|^2 d\tau \leq \langle W(t_0)x, x\rangle,$$

hence

$$\big\langle W(t)\Phi(t,t_0)x, \Phi(t,t_0)x\big\rangle \leq \big\langle W(t_0)x, x\big\rangle \leq -m\|x\|^2, \quad \text{for all} \quad t \geq t_0 \geq 0.$$

Thus

$$m\|x\|^2 \leq \big|\big\langle W(t_0+1)\Phi(t_0+1,t_0)x, \Phi(t_0+1,t_0)x\big\rangle\big| \leq k\|\Phi(t_0+1,t_0)x\|^2,$$

for all $t_0 \geq 0$ and $x \in X$, where $k = \sup_{t\geq 0} \|W(t)\|$.

Hence we get that

$$\sqrt{\frac{m}{k}}\|x\| \leq \|\Phi(t_0+1,t_0)x\|, \text{ for all } t_0 \geq 0, \text{ for all } x \in X.$$

By Theorem 2.1 we obtain that Φ is exponentially unstable. ■

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